CLP 4

VECTOR CALCULUS EXERCISES

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CLP-4 Vector Calculus

Exercises

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▲ Legal stuff

How to use this book

▲ Introduction

First of all, welcome back to Calculus!

This book is an early draft of a companion question book for the CLP-4 text. Additional questions are still under active development.

▸▸ How to Work Questions

This book is organized into four sections: Questions, Hints, Answers, and Solutions. As you are working problems, resist the temptation to prematurely peek at the back! It's important to allow yourself to struggle for a time with the material. Even professional mathematicians don't always know right away how to solve a problem. The art is in gathering your thoughts and figuring out a strategy to use what you know to find out what you don't.

If you find yourself at a real impasse, go ahead and look for a hint in the Hints section. Think about it for a while, and don't be afraid to read back in the notes to look for a key idea that will help you proceed. If you still can't solve the problem, well, we included the Solutions section for a reason! As you're reading the solutions, try hard to understand why we took the steps we did, instead of memorizing step-by-step how to solve that one particular problem.

If you struggled with a question quite a lot, it's probably a good idea to return to it in a few days. That might have been enough time for you to internalize the necessary ideas, and you might find it easily conquerable. Pat yourself on the back-sometimes math makes you feel good! If you're still having troubles, read over the solution again, with an emphasis on understanding why each step makes sense.

One of the reasons so many students are required to study calculus is the hope that it will improve their problem-solving skills. In this class, you will learn lots of concepts, and be asked to apply them in a variety of situations. Often, this will involve answering one really big problem by breaking it up into manageable chunks, solving those chunks, then putting the pieces back together. When you see a particularly long question, remain calm and look for a way to break it into pieces you can handle.

▸▸ Working with Friends

Study buddies are fantastic! If you don't already have friends in your class, you can ask your neighbours in lecture to form a group. Often, a question that you might bang your head against for an hour can be easily cleared up by a friend who sees what you've missed. Regular study times make sure you don't procrastinate too much, and friends help you maintain a positive attitude when you might otherwise succumb to frustration. Struggle in mathematics is desirable, but suffering is not.

When working in a group, make sure you try out problems on your own before coming together to discuss with others. Learning is a process, and getting answers to questions that you haven't considered on your own can rob you of the practice you need to master skills and concepts, and the tenacity you need to develop to become a competent problem-solver.

▸▸ Types of Questions

Q[1](∗): In addition to original problems, this book contains problems pulled from quizzes and exams given at UBC for Math 317 (Calculus 4). These problems are marked with a star. The authors would like to acknowledge the contributions of the many people who collaborated to produce these exams over the years.

The questions are organized into Stage 1, Stage 2, and Stage 3.

▸▸ Stage 1

The first category is meant to test and improve your understanding of basic underlying concepts. These often do not involve much calculation. They range in difficulty from very basic reviews of definitions to questions that require you to be thoughtful about the concepts covered in the section.

▸▸ Stage 2

Questions in this category are for practicing skills. It's not enough to understand the philosophical grounding of an idea: you have to be able to apply it in appropriate situations. This takes practice!

▸▸ Stage 3

The last questions in each section go a little farther than Stage 2. Often they will combine more than one idea, incorporate review material, or ask you to apply your understanding of a concept to a new situation.

In exams, as in life, you will encounter questions of varying difficulty. A good skill to practice is recognizing the level of difficulty a problem poses. Exams will have some easy questions, some standard questions, and some harder questions.

CONTENTS

Part I

THE QUESTIONS

Chapter 1

CURVES

1.1 ▲ Derivatives, Velocity, Etc.

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Questions 1 through 5 provide practice with curve parametrization. Being comfortable with the algebra and interpretation of these descriptions are essential ingredients in working effectively with parametrizations.

Q[1]: Find the specified parametrization of the first quadrant part of the circle $x^2 + y^2 = a^2$.

(a) In terms of the y coordinate.

(b) In terms of the angle between the tangent line and the positive x-axis.

(c) In terms of the arc length from $(0, a)$.

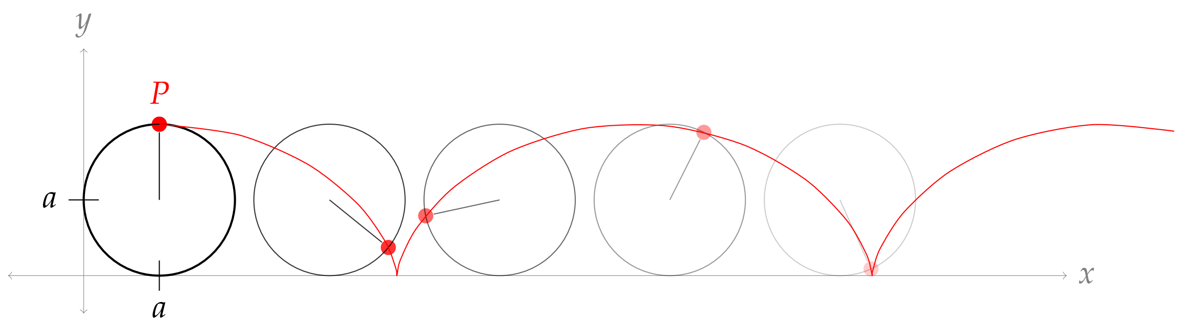
Q[2]: Consider the following time-parametrized curve:

$$\mathbf{r}(t) = \left(\cos\left(\frac{\pi}{4}t\right) \ , \ (t-5)^2\right)$$

List the three points $(-1/\sqrt{2}, 0)$, $(1, 25)$, and $(0, 25)$ in chronological order.

Q[3]: At what points in the xy-plane does the curve $(\sin t, t^2)$ cross itself? What is the difference in t between the first time the curve crosses through a point, and the last?

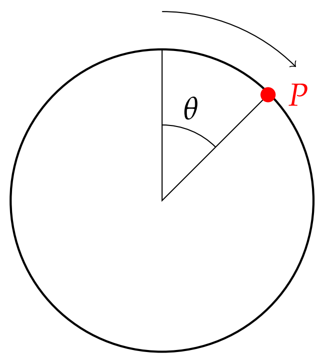
Q[4]:



A circle of radius a rolls along the x-axis in the positive direction, starting with its centre at (a, a). In that position, we mark the topmost point on the circle P. As the circle moves, P moves with it. Let θ be the angle the circle has rolled — see the diagram below.

(a) Give the position of the centre of the circle as a function of θ.

(b) Give the position of P as a function of θ.



Q[5]: The curve C is defined to be the intersection of the ellipsoid

$$x^2 - \frac{1}{4}y^2 + 3z^2 = 1$$

and the plane

$$x + y + z = 0.$$

When y is very close to 0, and z is negative, find an expression giving z in terms of y.

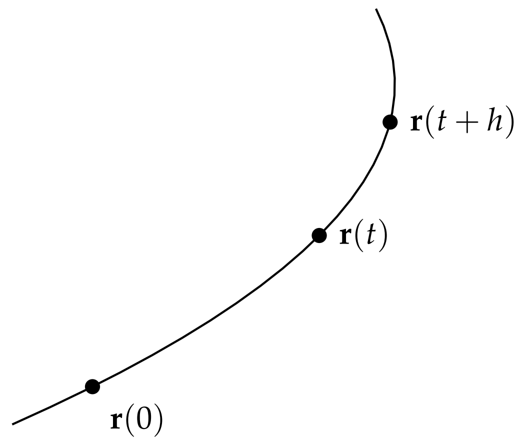
Q[6]: A particle traces out a curve in space, so that its position at time t is

$$\mathbf{r}(t) = e^{-t}\,\hat{\boldsymbol{\imath}} + \frac{1}{t}\,\hat{\boldsymbol{\jmath}} + (t-1)^2(t-3)^2\,\hat{\mathbf{k}}$$

for $t > 0$.

Let the positive z axis point vertically upwards, as usual. When is the particle moving upwards, and when is it moving downwards? Is it moving faster at time $t = 1$ or at time $t = 3$?

Q[7]: Below is the graph of the parametrized function $\mathbf{r}(t)$. Let $s(t)$ be the arclength along the curve from $\mathbf{r}(0)$ to $\mathbf{r}(t)$.



Indicate on the graph $s(t+h) - s(t)$ and $\mathbf{r}(t+h) - \mathbf{r}(t)$. Are the quantities scalars or vectors?

Q[8]: What is the relationship between velocity and speed in a vector-valued function of time?

Q[9](*): Let $\mathbf{r}(t)$ be a vector valued function. Let $\mathbf{r}'$, $\mathbf{r}''$, and $\mathbf{r}'''$ denote $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$, $\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}$ and $\frac{\mathrm{d}^3\mathbf{r}}{\mathrm{d}t^3}$, respectively. Express

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[(\mathbf{r} \times \mathbf{r}') \cdot \mathbf{r}''\big]$$

in terms of $\mathbf{r}$, $\mathbf{r}'$, $\mathbf{r}''$, and $\mathbf{r}'''$. Select the correct answer.

(a) $(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''$

(b) $(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r} + (\mathbf{r} \times \mathbf{r}') \cdot \mathbf{r}'''$

(c) $(\mathbf{r} \times \mathbf{r}') \cdot \mathbf{r}'''$

(d) 0

(e) None of the above.

Q[10]: Show that, if the position and velocity vectors of a moving particle are always perpendicular, then the path of the particle lies on a sphere.

▸▸ Stage 2

Q[11](*): Find the speed of a particle with the given position function

$$\mathbf{r}(t) = 5\sqrt{2}\, t\, \hat{\boldsymbol{\imath}} + e^{5t}\, \hat{\boldsymbol{\jmath}} - e^{-5t}\, \hat{\mathbf{k}}$$

Select the correct answer:

(a) $|\mathbf{v}(t)| = \big(e^{5t} + e^{-5t}\big)$

(b) $|\mathbf{v}(t)| = \sqrt{10 + 5e^{t} + 5e^{-t}}$

(c) $|\mathbf{v}(t)| = \sqrt{10 + e^{10t} + e^{-10t}}$

(d) $|\mathbf{v}(t)| = 5\big(e^{5t} + e^{-5t}\big)$

(e) $|\mathbf{v}(t)| = 5\big(e^t + e^{-t}\big)$

Q[12]: Find the velocity, speed and acceleration at time t of the particle whose position is

$$\mathbf{r}(t) = a\cos t\,\hat{\boldsymbol{\imath}} + a\sin t\,\hat{\boldsymbol{\jmath}} + ct\,\hat{\mathbf{k}}$$

Describe the path of the particle.

Q[13](*):

(a) Let

$$\mathbf{r}(t) = \Big(t^2, 3, \tfrac{1}{3}t^3\Big)$$

Find the unit tangent vector to this parametrized curve at $t = 1$, pointing in the direction of increasing t.

(b) Find the arc length of the curve from (a) between the points $(0, 3, 0)$ and $(1, 3, -\frac{1}{3})$.

Q[14]: Using Lemma 1.1.3 in the CLP-4 text, find the arclength of $\mathbf{r}(t) = \Big(t, \sqrt{\tfrac{3}{2}}t^2, t^3\Big)$ from $t = 0$ to $t = 1$.

Q[15]: Find the length of the parametric curve

$$x = a\cos t\sin t \qquad y = a\sin^2 t \qquad z = bt$$

between $t = 0$ and $t = T > 0$.

Q[16]: A particle's position at time t is given by $\mathbf{r}(t) = (t + \sin t, \cos t)$[1]. What is the magnitude of the acceleration of the particle at time t?

Q[17](*): A curve in $\mathbb{R}^3$ is given by the vector equation $\mathbf{r}(t) = \Big(2t\cos t, 2t\sin t, \frac{t^3}{3}\Big)$

(a) Find the length of the curve between $t = 0$ and $t = 2$.

(b) Find the parametric equations of the tangent line to the curve at $t = \pi$.

Q[18](*): Let $\mathbf{r}(t) = \big(3\cos t, 3\sin t, 4t\big)$ be the position vector of a particle as a function of time $t \geqslant 0$.

(a) Find the velocity of the particle as a function of time t.

(b) Find the arclength of its path between $t = 1$ and $t = 2$.

Q[19]: The plane $z = 2x + 3y$ intersects the cylinder $x^2 + y^2 = 9$ in an ellipse.

(a) Find a parametrization of the ellipse.

(b) Express the circumference of this ellipse as an integral. You need not evaluate the integral[2].

1 The particle traces out a cycloid — see Question 4

2 The indefinite integral involved is one of a class of integrals called elliptic integrals because of their connections to arc lengths of ellipses. In general, elliptic integrals cannot be expressed in terms of elementary functions. You can easily find discussions of elliptic integrals using your favourite search engine.

Q[20](∗): Consider the curve

$$\mathbf{r}(t) = \frac{1}{3}\cos^3 t\,\hat{\imath} + \frac{1}{3}\sin^3 t\,\hat{\jmath} + \sin^3 t\,\hat{\mathbf{k}}$$

(a) Compute the arc length of the curve from $t = 0$ to $t = \frac{\pi}{2}$.

(b) Compute the arc length of the curve from $t = 0$ to $t = \pi$.

Q[21](∗): Let $\mathbf{r}(t) = \left(\frac{1}{3}t^3, \frac{1}{2}t^2, \frac{1}{2}t\right)$, $t \geqslant 0$. Compute $s(t)$, the arclength of the curve at time t.

Q[22](∗): Find the arc length of the curve $\mathbf{r}(t) = \left(t^m\,,\, t^m\,,\, t^{3m/2}\right)$ for $0 \leqslant a \leqslant t \leqslant b$, and where $m > 0$. Express your result in terms of m, a, and b.

Q[23]: Let C be the part of the curve of intersection of the parabolic cylinder $x = y^2$ and the hyperbolic paraboloid $3z = 2xy$ with $y \geqslant 0$.

(a) Write a vector parametric equation for C using x as the parameter.

(b) Find the length of the part of C between the origin and the point $(9, 3, 18)$.

(c) A particle moves along C in the direction for which x is increasing. If the particle moves with constant speed 9, find its velocity vector when it is at the point $(1, 1, \frac{2}{3})$.

(d) Find the acceleration vector of the particle of part (c) when it is at the point $(1, 1, \frac{2}{3})$.

Q[24]: If a particle has constant mass m, position $\mathbf{r}$, and is moving with velocity $\mathbf{v}$, then its angular momentum is $\mathbf{L} = m(\mathbf{r} \times \mathbf{v})$.

For a particle with mass $m = 1$ and position function $\mathbf{r} = (\sin t, \cos t, t)$, find $\left|\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right|$.

▸▸ Stage 3

Q[25](∗): A particle moves along the curve $\mathcal{C}$ of intersection of the surfaces $z^2 = 12y$ and $18x = yz$ in the upward direction. When the particle is at $(1, 3, 6)$ its velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ are given by

$$\mathbf{v} = 6\,\hat{\imath} + 12\,\hat{\jmath} + 12\,\hat{\mathbf{k}} \qquad \mathbf{a} = 27\,\hat{\imath} + 30\,\hat{\jmath} + 6\,\hat{\mathbf{k}}$$

(a) Write a vector parametric equation for $\mathcal{C}$ using $u = \frac{z}{6}$ as a parameter.

(b) Find the length of $\mathcal{C}$ from $(0, 0, 0)$ to $(1, 3, 6)$.

(c) If $u = u(t)$ is the parameter value for the particle's position at time t, find $\frac{\mathrm{d}u}{\mathrm{d}t}$ when the particle is at $(1, 3, 6)$.

(d) Find $\frac{\mathrm{d}^2u}{\mathrm{d}t^2}$ when the particle is at $(1, 3, 6)$.

Q[26](∗): A particle of mass $m = 1$ has position $\mathbf{r}_0 = \frac{1}{2}\,\hat{\mathbf{k}}$ and velocity $\mathbf{v}_0 = \frac{\pi^2}{2}\,\hat{\imath}$ at time 0. It moves under a force

$$\mathbf{F}(t) = -3t\,\hat{\imath} + \sin t\,\hat{\jmath} + 2e^{2t}\,\hat{\mathbf{k}}.$$

(a) Determine the position $\mathbf{r}(t)$ of the particle depending on t.

(b) At what time after time $t = 0$ does the particle cross the plane $x = 0$ for the first time?

(c) What is the velocity of the particle when it crosses the plane $x = 0$ in part (b)?

Q[27](∗): Let C be the curve of intersection of the surfaces $y = x^2$ and $z = \frac{2}{3}x^3$. A particle moves along C with constant speed such that $\frac{dx}{dt} > 0$. The particle is at $(0, 0, 0)$ at time $t = 0$ and is at $(3, 9, 18)$ at time $t = \frac{7}{2}$.

(a) Find the length of the part of C between $(0, 0, 0)$ and $(3, 9, 18)$.

(b) Find the constant speed of the particle.

(c) Find the velocity of the particle when it is at $\left(1, 1, \frac{2}{3}\right)$.

(d) Find the acceleration of the particle when it is at $\left(1, 1, \frac{2}{3}\right)$.

Q[28]: A camera mounted to a pole can swivel around in a full circle. It is tracking an object whose position at time t seconds is $x(t)$ metres east of the pole, and $y(t)$ metres north of the pole.

In order to always be pointing directly at the object, how fast should the camera be programmed to rotate at time t? (Give your answer in terms of $x(t)$ and $y(t)$ and their derivatives, in the units rad/sec.)

Q[29]: A pipe of radius 3 follows the path of the curve $\mathbf{r}(t) = \left(\frac{2\sqrt{2}}{3}t^{3/2}\ ,\ \frac{1}{2}t^2\ ,\ t + 2\right)$, for $0 \leqslant t \leqslant 10$.

What is the volume inside the pipe? What is the surface area of the pipe?

Q[30]: A wire of total length 1000cm is formed into a flexible coil that is a circular helix. If there are 10 turns to each centimetre of height and the radius of the helix is 3 cm, how tall is the coil?

Q[31]: A projectile falling under the influence of gravity and slowed by air resistance proportional to its speed has position satisfying

$$\frac{d^2\mathbf{r}}{dt^2} = -g\hat{\mathbf{k}} - \alpha\frac{d\mathbf{r}}{dt}$$

where α is a positive constant. If $\mathbf{r} = \mathbf{r}_0$ and $\frac{d\mathbf{r}}{dt} = \mathbf{v}_0$ at time $t = 0$, find $\mathbf{r}(t)$.

1.2▲ Reparametrization

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: A curve $\mathbf{r}(s)$ is parametrized in terms of arclength. What is $\displaystyle\int_1^t |\mathbf{r}'(s)|ds$ when $t \geqslant 1$?

Q[2]: The function

$$\mathbf{r}(s) = \sin\left(\frac{s+1}{2}\right)\hat{\imath} + \cos\left(\frac{s+1}{2}\right)\hat{\jmath} + \frac{\sqrt{3}}{2}(s+1)\hat{\mathbf{k}}$$

is parametrized in terms of arclength, starting from the point P. What is P?

Q[3]: A curve $\mathbf{R} = \mathbf{a}(t)$ is reparametrized in terms of arclength as $\mathbf{R} = \mathbf{b}(s) = \mathbf{a}(t(s))$. Of the following options, which best describes the relationship between the vectors $\mathbf{a}'(t_0)$ and $\mathbf{b}'(s_0)$, where $t(s_0) = t_0$?

You may assume $\mathbf{a}'(t)$ and $\mathbf{b}'(s)$ exist and are nonzero for all $t, s \geqslant 0$.

A. they are parallel and point in the same direction

B. they are parallel and point in opposite directions

C. they are perpendicular

D. they have the same magnitude

E. they are equal

▸▸ Stage 2

Q[4](*):

(a) Let

$$\mathbf{r}(t) = (2\sin^3 t, 2\cos^3 t, 3\sin t\cos t)$$

Find the unit tangent vector to this parametrized curve at $t = \pi/3$, pointing in the direction of increasing t.

(b) Reparametrize the vector function $\mathbf{r}(t)$ from (a) with respect to arc length measured from the point $t = 0$ in the direction of increasing t.

Q[5](*): This problem is about the logarithmic spiral in the plane

$$\mathbf{r}(t) = e^t(\cos t, \sin t), \qquad t \in \mathbb{R}$$

(a) Find the arc length of the piece of this spiral which is contained in the unit circle.

(b) Reparametrize the logarithmic spiral with respect to arc length, measured from $t = -\infty$.

▸▸ Stage 3

Q[6]: Define

$$\mathbf{r}(t) = \left(\frac{1}{\sqrt{1+t^2}}, \frac{\arctan t}{\sqrt{1+t^{-2}}}, \arctan t\right)$$

for $0 \leqslant t$. Reparametrize the function using $z = \arctan t$, and describe the curve it defines. What is the geometric interpretation of the new parameter z?

Q[7]: Reparametrize the function $\mathbf{r}(t) = (\frac{1}{2}t^2, \frac{1}{3}t^3)$ in terms of arclength from $t = -1$.

1.3▲ Curvature

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

There are a lot of constants in this chapter that might be new to you. They can take a little getting used to. Questions 1-5 provide practice working with and interpreting these constants and their relations to each other.

Q[1]: Sketch the curve $\mathbf{r}(t) = (3\sin t, 3\cos t)$. At the point $(0,3)$, label $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$. Give the values of κ and ρ at this point as well.

Q[2]: Consider the circle $\mathbf{r}(t) = (3\sin t, 3\cos t)$. Find $\hat{\mathbf{T}}(t)$ and $\hat{\mathbf{T}}(s)$. Then, use parts (b) and (c) of Theorem 1.3.3 to find $\hat{\mathbf{N}}(t)$ and $\hat{\mathbf{N}}(s)$.

Q[3]: The functon $\mathbf{r}(t) = (t\cos t, t\sin t)$, $t \geqslant 0$, defines a spiral centred at the origin. Using only geometric intuition (no calculation), predict $\lim\limits_{t\to\infty} \kappa(t)$.

Q[4]: Let $\mathbf{r}(t) = (e^t, 3t, \sin t)$. What is $\frac{\mathrm{d}s}{\mathrm{d}t}$?

Q[5]: In Question 5 of Section 1.2,we found that the spiral

$$\mathbf{r}(t) = e^t(\cos t, \sin t)$$

parametrized in terms of arclength is

$$\mathbf{R}(s) = \frac{s}{\sqrt{2}}\left(\cos\left(\ln\left(\frac{s}{\sqrt{2}}\right)\right), \sin\left(\ln\left(\frac{s}{\sqrt{2}}\right)\right)\right).$$

Find $\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s}$ and $\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}t}$ for this curve.

Q[6]: In this exercise, we make more precise the sense in which the osculating circle is the circle which best approximates a plane curve at a point.

- By translating and rotating our coordinate system, we can always arrange that the point is $(0,0)$ and that the curve is $y = f(x)$ with $f'(0) = 0$ and $f''(0) > 0$. (We are assuming that the curvature at the point is nonzero.)
- Let $y = g(x)$ be the bottom half of the circle of radius r which is centred at $(0,r)$.

Show that if $f(x)$ and $g(x)$ have the same second order Taylor approximation at $x = 0$, then r is the radius of curvature of $y = f(x)$ at $x = 0$.

▸▸ Stage 2

Q[7]: Given a curve $\mathbf{r}(t) = (e^t, t^2 + t)$, compute the following quantities:

A. $\mathbf{v}(t)$

B. $\mathbf{a}(t)$

C. $\frac{\mathrm{d}s}{\mathrm{d}t}$

D. $\hat{\mathbf{T}}(t)$

E. $\kappa(t)$

Q[8]: Find the curvature $\kappa(t)$ of $\mathbf{r}(t) = (\cos t + \sin t, \sin t - \cos t)$.

Q[9]: Find the minimum and maximum values for the curvature of the ellipse $x(t) = a\cos t, y(t) = b\sin t$. Here $a > b > 0$.

Q[10](*):

(a) Find the curvature of $y = e^x$ at $(0, 1)$.

(b) Find the equation of the circle best fitting $y = e^x$ at $(0, 1)$.

Q[11](*):

Consider the motion of a thumbtack stuck in the tread of a tire which is on a bicycle moving at constant speed. This motion is given by the parametrized curve

$$\mathbf{r}(t) = \big(t - \sin t\,,\ 1 - \cos t\big)$$

with $t > 0$.

(a) Sketch the curve in the xy-plane for $0 < t < 4\pi$.

(b) Find and simplify the formula for the curvature $\kappa(t)$.

(c) Find the radius of curvature of the osculating circle to $\mathbf{r}(t)$ at $t = \pi$.

(d) Find the equation of the osculating circle to $\mathbf{r}(t)$ at $t = \pi$.

▸▸ Stage 3

Q[12]: Find the curvature κ as a function of arclength s (measured from $(0, 0)$) for the curve

$$x(\theta) = \int_0^\theta \cos\left(\tfrac{1}{2}\pi t^2\right) dt \qquad y(\theta) = \int_0^\theta \sin\left(\tfrac{1}{2}\pi t^2\right) dt$$

Q[13](*): Let C be the curve in $\mathbb{R}^2$ given by the graph of the function $y = \frac{x^3}{3}$. Let $\kappa(x)$ be the curvature of C at the point $(x, x^3/3)$. Find all points where $\kappa(x)$ attains its maximal values, or else explain why such points do not exist. What are the limits of $\kappa(x)$ as $x \to \infty$ and $x \to -\infty$?

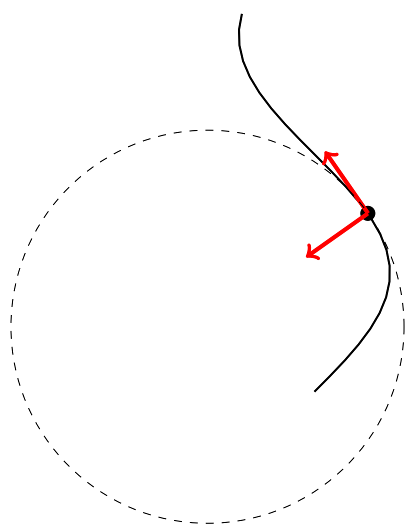
1.4▲ Curves in Three Dimensions

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: In the sketch below of a three-dimensional curve and its osculating circle at a point, label $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$. Will $\hat{\mathbf{B}}$ be pointing out of the paper towards the reader, or into the paper away from the reader?

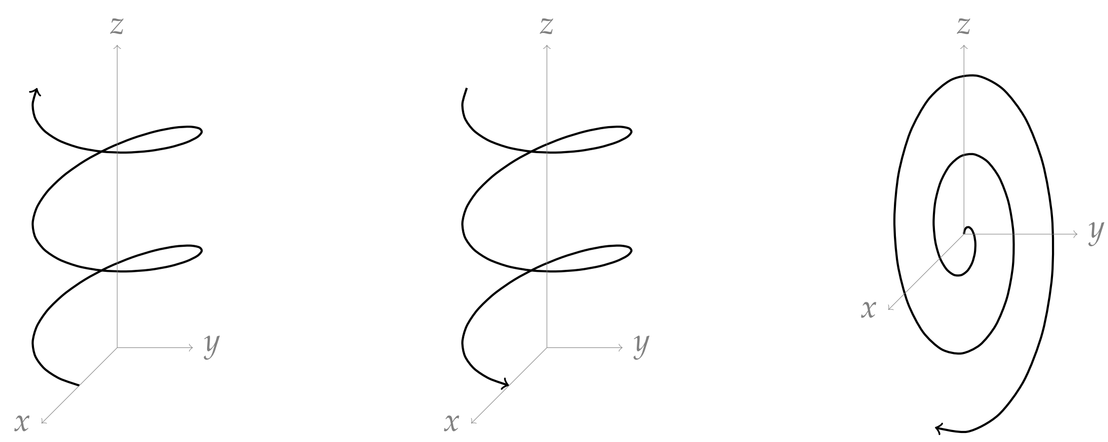


Q[2]: In the formula

$$\frac{\mathrm{d}s}{\mathrm{d}t}(t) = |\mathbf{v}(t)| = |\mathbf{r}'(t)|$$

does s stand for speed, or for arclength?

Q[3]: Which curve (or curves) below have positive torsion, which have negative torsion, and which have zero torsion? The arrows indicate the direction of increasing t.



$\mathbf{a}(t) = (\cos t, -2\sin t, t/2)$ $\mathbf{b}(t) = (\cos t, 2\sin t, -t/2)$ $\mathbf{c}(t) = (0, t/2\sin t, t\cos t)$

Q[4]: Consider a curve that is parametrized by arc length s.

(a) Show that if the curve has curvature $\kappa(s) = 0$ for all s, then the curve is a straight line.

(b) Show that if the curve has curvature $\kappa(s) > 0$ and torsion $\tau(s) = 0$ for all s, then the curve lies in a plane.

(c) Show that if the curve has curvature $\kappa(s) = \kappa_0$, a strictly positive constant, and torsion $\tau(s) = 0$ for all s, then the curve is a circle.

Q[5](*): The surface $z = x^2 + y^2$ is sliced by the plane $x = y$. The resulting curve is oriented from $(0, 0, 0)$ to $(1, 1, 2)$.

(a) Sketch the curve from $(0, 0, 0)$ to $(1, 1, 2)$.

(b) Sketch $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$ and $\hat{\mathbf{B}}$ at $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

(c) Find the torsion at $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

▸▸ Stage 2

Q[6](*): Let C be the space curve

$$\mathbf{r}(t) = \left(e^t - e^{-t}\right)\hat{\boldsymbol{\imath}} + \left(e^t + e^{-t}\right)\hat{\boldsymbol{\jmath}} + 2t\,\hat{\mathbf{k}}$$

(a) Find $\mathbf{r}'$, $\mathbf{r}''$ and the curvature of C.

(b) Find the length of the curve between $\mathbf{r}(0)$ and $\mathbf{r}(1)$.

Q[7]: Find the torsion of $\mathbf{r}(t) = (t, t^2, t^3)$ at the point $(2, 4, 8)$.

Q[8]: Find the unit tangent, unit normal and binormal vectors and the curvature and torsion of the curve

$$\mathbf{r}(t) = t\,\hat{\boldsymbol{\imath}} + \frac{t^2}{2}\,\hat{\boldsymbol{\jmath}} + \frac{t^3}{3}\,\hat{\mathbf{k}}$$

Q[9]: For some constant c, define $\mathbf{r}(t) = (t^3, t, e^{ct})$. For which value(s) of c is $\tau(5) = 0$? For each of those values of c, find an equation for the plane containing the osculating circle to the curve at $t = 5$.

Q[10](*):

(a) Consider the parametrized space curve

$$\mathbf{r}(t) = \left(t^2, t, t^3\right)$$

Find an equation for the plane passing through $(1, 1, 1)$ with normal vector tangent to $\mathbf{r}$ at that point.

(b) Find the curvature of the curve from (a) as a function of the parameter t.

Q[11](*): Let C be the osculating circle to the helix $\mathbf{r}(t) = \big(\cos t\,,\ \sin t\,,\ t\big)$ at the point where $t = \pi/6$. Find:

(a) the radius of curvature of C

(b) the center of C

(c) the unit normal to the plane of C

Q[12](*):

(a) Consider the parametrized space curve

$$\mathbf{r}(t) = (\cos(t), \sin(t), t^2)$$

Find a parametric form for the tangent line at the point corresponding to $t = \pi$.

(b) Find the tangential component $a_T(t)$ of acceleration, as a function of t, for the parametrized space curve $\mathbf{r}(t)$.

Q[13](*): Suppose, in terms of the time parameter t , a particle moves along the path $\mathbf{r}(t) = (\sin t - t\cos t)\,\hat{\boldsymbol{\imath}} + (\cos t + t\sin t)\,\hat{\boldsymbol{\jmath}} + t^2\,\hat{\mathbf{k}}$, $1 \leqslant t < \infty$.

(a) Find the speed of the particle at time t.

(b) Find the tangential component of acceleration at time t.

(c) Find the normal component of acceleration at time t.

(d) Find the curvature of the path at time t.

Q[14](*): Assume the paraboloid $z = x^2 + y^2$ and the plane $2x + z = 8$ intersect in a curve C. C is traversed counter-clockwise if viewed from the positive z-axis.

(a) Parametrize the curve C.

(b) Find the unit tangent vector $\hat{\mathbf{T}}$, the principal normal vector $\hat{\mathbf{N}}$, the binormal vector $\hat{\mathbf{B}}$ and the curvature κ all at the point $(2, 0, 4)$.

Q[15](*): Consider the curve C given by

$$\mathbf{r}(t) = \frac{1}{3}t^3\,\hat{\boldsymbol{\imath}} + \frac{1}{\sqrt{2}}t^2\,\hat{\boldsymbol{\jmath}} + t\,\hat{\mathbf{k}}, \qquad -\infty < t < \infty.$$

(a) Find the unit tangent $\hat{\mathbf{T}}(t)$ as a function of t.

(b) Find the curvature $\kappa(t)$ as a function of t.

(c) Determine the principal normal vector $\hat{\mathbf{N}}$ at the point $\left(\frac{8}{3}, 2\sqrt{2}, 2\right)$.

Q[16](∗): Suppose the curve C is the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $x + y + z = 1$.

(a) Find a parameterization of C.

(b) Determine the curvature of C.

(c) Find the points at which the curvature is maximum and determine the value of the curvature at these points.

Q[17](∗): Let

$$\mathbf{r}(t) = t^2\,\hat{\imath} + 2t\,\hat{\jmath} + \ln t\,\hat{\mathbf{k}}$$

Compute the unit tangent and unit normal vectors $\hat{\mathbf{T}}(t)$ and $\hat{\mathbf{N}}(t)$. Compute the curvature $\kappa(t)$. Simplify whenever possible!

Q[18](∗):

(a) Find the length of the curve $\mathbf{r}(t) = \left(1, \frac{t^2}{2}, \frac{t^3}{3}\right)$ for $0 \leqslant t \leqslant 1$.

(b) Find the principal unit normal vector $\hat{\mathbf{N}}$ to $\mathbf{r}(t) = \cos(t)\,\hat{\imath} + \sin(t)\,\hat{\jmath} + t\,\hat{\mathbf{k}}$ at $t = \pi/4$.

(c) Find the curvature of $\mathbf{r}(t) = \cos(t)\,\hat{\imath} + \sin(t)\,\hat{\jmath} + t\,\hat{\mathbf{k}}$ at $t = \pi/4$.

Q[19](∗): A particle moves along a curve with position vector given by

$$\mathbf{r}(t) = \left(t+2\,,\, 1-t\,,\, t^2/2\right)$$

for $-\infty < t < \infty$.

(a) Find the velocity as a function of t.

(b) Find the speed as a function of t.

(c) Find the acceleration as a function of t.

(d) Find the curvature as a function of t.

(e) Recall that the decomposition of the acceleration into tangential and normal components is given by the formula

$$\mathbf{r}''(t) = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\hat{\mathbf{T}}(t) + \kappa(t)\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2\hat{\mathbf{N}}(t)$$

Use this formula and your answers to the previous parts of this question to find $\hat{\mathbf{N}}(t)$, the principal unit normal vector, as a function of t.

(f) Find an equation for the osculating plane (the plane which best fits the curve) at the point corresponding to $t = 0$.

(g) Find the centre of the osculating circle at the point corresponding to $t = 0$.

Q[20](∗): Consider the curve C given by

$$\mathbf{r}(t) = \frac{t^3}{3}\,\hat{\boldsymbol{\imath}} + \frac{t^2}{\sqrt{2}}\,\hat{\boldsymbol{\jmath}} + t\,\hat{\mathbf{k}} \qquad -\infty < t < \infty$$

(a) Find the unit tangent $\hat{\mathbf{T}}(t)$ as a function of t.

(b) Find the curvature $\kappa(t)$ as a function of t.

(c) Evaluate $\kappa(t)$ at $t = 0$.

(d) Determine the principal normal vector $\hat{\mathbf{N}}(t)$ at $t = 0$.

(e) Compute the binormal vector $\hat{\mathbf{B}}(t)$ at $t = 0$.

Q[21](∗): A curve in $\mathbb{R}^3$ is given by $\mathbf{r}(t) = (t^2\,,\,t\,,\,t^3)$.

(a) Find the parametric equations of the tangent line to the curve at the point $(1, -1, -1)$.

(b) Find an equation for the osculating plane of the curve at the point $(1, 1, 1)$.

Q[22](∗): A curve in $\mathbb{R}^3$ is given by

$$\mathbf{r}(t) = (\sin t - t\cos t)\,\hat{\boldsymbol{\imath}} + (\cos t + t\sin t)\,\hat{\boldsymbol{\jmath}} + t^2\,\hat{\mathbf{k}}, \qquad 0 \leqslant t < \infty$$

(a) Find the length of the curve $\mathbf{r}(t)$ from $\mathbf{r}(0) = (0, 1, 0)$ to $\mathbf{r}(\pi) = (\pi, -1, \pi^2)$.

(b) Find the curvature of the curve at time $t > 0$.

Q[23](∗): At time $t = 0$, NASA launches a rocket which follows a trajectory so that its position at any time t is

$$x = \frac{4\sqrt{2}}{3}t^{3/2},\ y = \frac{4\sqrt{2}}{3}t^{3/2},\ z = t(2 - t)$$

(a) Assuming that the flight ends when $z = 0$, find out how far the rocket travels.

(b) Find the unit tangent and unit normal to the trajectory at its highest point.

(c) Also, compute the curvature of the trajectory at its highest point.

Q[24](∗): Consider a particle travelling in space along the path parametrized by

$$x = \cos^3 t,\ y = \sin^3 t,\ z = 2\sin^2 t$$

(a) Calculate the arc length of this path for $0 \leqslant t \leqslant \pi/2$.

(b) Find the vectors $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$, $\hat{\mathbf{B}}$ for the particle at $t = \pi/6$.

Q[25]: Suppose that the curve C is the intersection of the cylinder $x^2 + y^2 = 1$ with the surface $z = x^2 - y^2$.

(a) Find a parameterization of C.

(b) Determine the curvature of C at the point $\big(1/\sqrt{2}\,, 1/\sqrt{2}\,, 0\big)$.

(c) Find the osculating plane to C at the point $\big(1/\sqrt{2}, 1/\sqrt{2}, 0\big)$. In general, the osculating plane to a curve $\mathbf{r}(t)$ at the point $\mathbf{r}(t_0)$ is the plane which fits the curve best at $\mathbf{r}(t_0)$. It passes through $\mathbf{r}(t_0)$ and has normal vector $\hat{\mathbf{B}}(t_0)$.

(d) Find the radius and the centre of the osculating circle to C at the point $\big(1/\sqrt{2}, 1/\sqrt{2}, 0\big)$.

▸▸ Stage 3

Q[26](∗): Under the influence of a force field $\mathbf{F}$, a particle of mass 2 kg is moving with constant speed 3 m/s along the path given as the intersection of the plane $z = x$ and the parabolic cylinder $z = y^2$, in the direction of increasing y. Find $\mathbf{F}$ at the point $(1, 1, 1)$. (Length is measured in m along the three coordinate axes.)

Q[27](∗): Consider the curve C in 3 dimensions given by

$$\mathbf{r}(t) = 2t\hat{\boldsymbol{\imath}} + t^2\hat{\boldsymbol{\jmath}} + \sqrt{3}t^2\hat{\mathbf{k}}$$

for $t \in \mathbb{R}$.

(a) Compute the unit tangent vector $\mathbf{T}(t)$.

(b) Compute the unit normal vector $\mathbf{N}(t)$.

(c) Show that the binormal vector $\mathbf{B}$ to this curve does not depend on t and is one of the following vectors:

$$① \begin{bmatrix} 1/2 \\ -\sqrt{3}/2 \\ 0 \end{bmatrix} \qquad ② \begin{bmatrix} 0 \\ \sqrt{3}/2 \\ 1/2 \end{bmatrix} \qquad ③ \begin{bmatrix} 0 \\ -\sqrt{3}/2 \\ 1/2 \end{bmatrix} \qquad ④ \begin{bmatrix} 0 \\ -1/2 \\ \sqrt{3}/2 \end{bmatrix}$$

This implies that C is a plane curve.

(d) According to your choice of vector ①, ②, ③ or ④, give the equation of the plane containing C.

(e) Compute the curvature $\kappa(t)$ of the curve.

(f) Are there point(s) where the curvature is maximal? If yes, give the coordinates of the point(s). If no, justify your answer.

(g) Are there point(s) where the curvature is minimal? If yes, give the coordinates of the point(s). If no, justify your answer.

(h) Let

$$\mathbf{u} := 2\hat{\boldsymbol{\imath}}, \quad \mathbf{v} := \hat{\boldsymbol{\jmath}} + \sqrt{3}\,\hat{\mathbf{k}} \quad \mathbf{w} := -\sqrt{3}\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}$$

(i) Express $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\mathbf{k}}$ in terms of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

(ii) Using (i), write $\mathbf{r}(t)$ in the form

$$a(t)\mathbf{u} + b(t)\mathbf{v} + c(t)\mathbf{w}$$

where $a(t)$, $b(t)$ and $c(t)$ are functions you have to determine. You should find that one of these functions is zero.

(iii) Draw the curve given by $\big(a(t), b(t)\big)$ in the xy-plane.

(iv) Is the drawing consistent with parts (f) and (g)? Explain.

Q[28](*): Recall that if $\hat{\mathbf{T}}$ is the unit tangent vector to an oriented curve with arclength parameter s, then the curvature κ and the principle normal vector $\hat{\mathbf{N}}$ are defined by the equation

$$\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s} = \kappa\,\hat{\mathbf{N}}$$

Moreover, the torsion τ and the binormal vector $\hat{\mathbf{B}}$ are defined by the equations

$$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}, \qquad \frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}s} = -\tau\,\hat{\mathbf{N}}$$

Show that

$$\frac{\mathrm{d}\hat{\mathbf{N}}}{\mathrm{d}s} = -\kappa\,\hat{\mathbf{T}} + \tau\,\hat{\mathbf{B}}$$

Q[29](*): A skier descends the hill $z = \sqrt{4 - x^2 - y^2}$ along a trail with parameterization

$$x = \sin(2\theta), \qquad y = 1 - \cos(2\theta), \qquad z = 2\cos\theta, \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

Let P denote the point on the trail where $x = 1$.

(a) Find the vectors $\hat{\mathbf{T}}, \hat{\mathbf{N}}, \hat{\mathbf{B}}$ and the curvature κ of the ski trail at the point P.

(b) The skier's acceleration at P is $\mathbf{a} = (-2, 3, -2\sqrt{2})$. Find, at P,

(i) the rate of change of the skier's speed and

(ii) the skier's velocity (a vector).

Q[30](*): A particle moves so that its position vector is given by $\mathbf{r}(t) = \big(\cos t\,,\, \sin t\,,\, c\sin t\big)$, where $t > 0$ and c is a constant.

(a) Find the velocity $\mathbf{v}(t)$ and the acceleration $\mathbf{a}(t)$ of the particle.

(b) Find the speed $v(t) = |\mathbf{v}(t)|$ of the particle.

(c) Find the tangential component of the acceleration of the particle.

(d) Show that the trajectory of this particle lies in a plane.

Q[31](*): A race track between two hills is described by the parametric curve

$$\mathbf{r}(\theta) = \Big(4\cos\theta\,,\, 2\sin\theta\,,\, \frac{1}{4}\cos(2\theta)\Big), \qquad 0 \leqslant \theta \leqslant 2\pi$$

(a) Compute the curvature of the track at the point $\big(-4, 0, \frac{1}{4}\big)$.

(b) Compute the radius of the circle that best approximates the bend at the point $\big(-4, 0, \frac{1}{4}\big)$ (that is, the radius of the osculating circle at that point).

(c) A car drives down the track so that its position at time t is given by $\mathbf{r}(t^2)$. (Note the relationship between t and θ is $\theta = t^2$). Compute the following quantities.

(i) The speed at the point $\left(-4, 0, \frac{1}{4}\right)$.

(ii) The acceleration at the point $\left(-4, 0, \frac{1}{4}\right)$.

(iii) The magnitude of the normal component of the acceleration at the point $\left(-4, 0, \frac{1}{4}\right)$.

1.6▲ Integrating Along a Curve

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: Give an equation for arclength of a curve C as a line integral.

Q[2]:

(a) Show that the integral $\int_C f(x,y)\, ds$ along the curve $\mathcal{C}$ given in polar coordinates by $r = r(\theta)$, $\theta_1 \leqslant \theta \leqslant \theta_2$, is

$$\int_{\theta_1}^{\theta_2} f\big(r(\theta)\cos\theta, r(\theta)\sin\theta\big)\sqrt{r(\theta)^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}(\theta)\right)^2}\, \mathrm{d}\theta$$

(b) Compute the arc length of $r = 1 + \cos\theta,\ 0 \leqslant \theta \leqslant 2\pi$. You may use the formula

$$1 + \cos\theta = 2\cos^2\frac{\theta}{2}$$

to simplify the computation.

▸▸ Stage 2

Q[3]: Calculate $\int_C \left(\frac{xy}{z}\right) \mathrm{d}s$, where C is the curve $\left(\frac{2}{3}t^3\ ,\ \sqrt{3}t^2\ ,\ 3t\right)$ from $t = 1$ to $t = 2$.

Q[4]: A hoop of radius r traces out the curve $x^2 + y^2 = 1$, where x and y are measured in metres. At a point (x, y), its density is x^2 kg per metre. What is the mass of the hoop?

Q[5]: Compute $\int_C (xy + z)\mathrm{d}s$ where C is the straight line from $(1, 2, 3)$ to $(2, 4, 5)$.

Q[6]: Evaluate the path integral $\int_C f(x,y,z)\, \mathrm{d}s$ for

(a) $f(x,y,z) = x\cos z, \quad \mathcal{C} : \mathbf{r}(t) = t\hat{\imath} + t^2\hat{\jmath}, 0 \leqslant t \leqslant 1.$

(b) $f(x,y,z) = \frac{x+y}{y+z}, \qquad \mathcal{C} : \mathbf{r}(t) = \left(t, \frac{2}{3}t^{3/2}, t\right), 1 \leqslant t \leqslant 2.$

Q[7]: Evaluate $\int_C \sin x\, \mathrm{d}s$, where C is the curve $(\operatorname{arcsec}(t), \ln t)$, $1 \leqslant t \leqslant \sqrt{2}$.

Q[8](*): A particle of mass $m = 1$ has position $\mathbf{r}(0) = \hat{\boldsymbol{\jmath}}$ and velocity $\mathbf{v}_0 = \hat{\boldsymbol{\imath}} + \hat{\mathbf{k}}$ at time $t = 0$. The particle moves under a force

$$\mathbf{F}(t) = \hat{\boldsymbol{\jmath}} - \sin t\,\hat{\mathbf{k}}$$

where t denotes time.

(a) Find the position $\mathbf{r}(t)$ of the particle as a function of t.

(b) Find the position $\mathbf{r}(t_1)$ of the particle when it crosses the plane $x = \pi/2$ for the first time at t_1.

(c) Determine the work done by $\mathbf{F}$ in moving the particle from $\mathbf{r}(0)$ to $\mathbf{r}(t_1)$.

▸▸ Stage 3

Q[9](*): Evaluate the line integral $\int_C \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}s$ where $\mathbf{F}(x,y) = xy^2\,\hat{\boldsymbol{\imath}} + ye^x\,\hat{\boldsymbol{\jmath}}$, C is the boundary of the rectangle R: $0 \leqslant x \leqslant 3$, $-1 \leqslant y \leqslant 1$, and $\hat{\mathbf{n}}$ is the unit vector, normal to C, pointing to the outside of the rectangle.

Q[10](*): Let $\mathcal{C}$ be the curve given by

$$\mathbf{r}(t) = t\cos t\,\hat{\boldsymbol{\imath}} + t\sin t\,\hat{\boldsymbol{\jmath}} + t^2\,\hat{\mathbf{k}}, \qquad 0 \leqslant t \leqslant \pi$$

(a) Find the unit tangent $\hat{\mathbf{T}}$ to $\mathcal{C}$ at the point $(-\pi, 0, \pi^2)$.

(b) Calculate the line integral

$$\int_C \sqrt{x^2+y^2}\,\mathrm{d}s$$

(c) Find the equation of a smooth surface in 3-space containing the curve $\mathcal{C}$.

(d) Sketch the curve $\mathcal{C}$.

Q[11]: A wire traces out a path C described by the curve $(t + \frac{1}{2}t^2\ ,\ t - \frac{1}{2}t^2\ ,\ \frac{4}{3}\,t^{3/2})$, $0 \leqslant t \leqslant 4$. Its density at the point (x,y,z) is $\rho(x,y,z) = \left(\frac{x+y}{2}\right)$. Find its centre of mass.

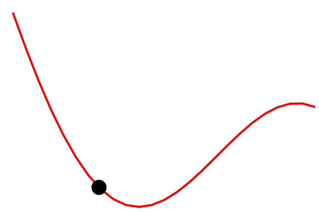
1.7▴ Sliding on a Curve

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS. *You may assume the acceleration due to gravity is $g = 9.8$ m/s^2. You may also assume that the systems described function as they do in the book: so tracks are frictionless, etc., unless otherwise mentioned.*

▸▸ Stage 1

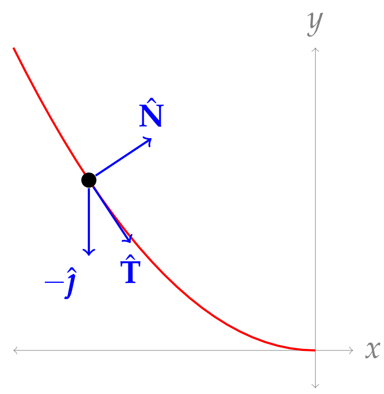
Q[1]: The figure below represents a bead sliding down a wire. Sketch vectors representing the normal force the wire exerts on the bead, and the force of gravity.



Assume the top of the page is "straight up."

Q[2]: In the definition $E = \frac{1}{2}m|\mathbf{v}|^2 + mgy$, $\mathbf{v}$ is the derivative of position with respect to what quantity?

Q[3]: A bead slides down a wire with the shape shown below, $x < 0$.



Let $W\hat{\mathbf{N}}$ be the normal force exerted by the wire when the bead is at position x. Note $W > 0$. Is $\frac{\mathrm{d}W}{\mathrm{d}x}$ positive or negative?

Q[4]: A skateboarder is rolling on a frictionless, very tall parabolic ramp with cross-section described by $y = x^2$. Given a boarder of mass m with system energy E, what is the highest elevation the skater reaches? How does this compare to a circular culvert?

▸▸ Stage 2

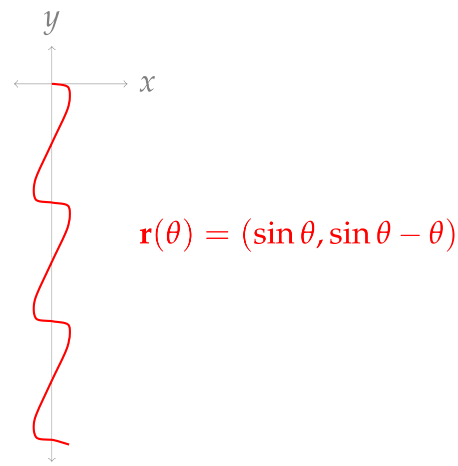
Q[5]: A skateboarder of mass 100 kg is freely rolling in a frictionless circular culvert of radius 5 m. If the skateboarder oscillates between vertical heights of 0 and 3 m, what is the energy E of the system?

Q[6]: A skateboarder is rolling on a frictionless circular culvert of radius 5 m. What should their speed be when they're at the bottom of the culvert ($y = 0$) for them to make it all the way around?

Q[7]: A ball of mass 1 kg rolls down a track with the shape $\mathbf{r}(\theta) = (3\cos\theta, 5\sin\theta, 4 + 4\cos\theta)$ for $0 \leqslant \theta \leqslant \frac{\pi}{2}$. Coordinates are measured in metres, and the z axis is vertical (so the force due to gravity is $-mg\hat{\mathbf{k}}$.)

When $\theta = \pi/4$, the particle has instantaneous velocity $|\mathbf{v}(t)| = 5$ m/s. What is the normal force exerted by the track at that time? Give your answer as a vector.

Q[8]: A bead of mass $\frac{1}{9.8}$ kg slides down a wire in the shape of the curve $\mathbf{r}(\theta) = (\sin\theta, \sin\theta - \theta)$, $\theta \geqslant 0$, with coordinates measured in metres. The bead will break off the wire when the wire exerts a force of 100 N on the bead.



If the bead breaks off the wire at $\theta = \frac{13\pi}{3}$, how fast is the bead moving at that point?

Q[9]: A skier is gliding down a hill. The hill can be described as $\mathbf{r}(t) = (\ln t, 1 - t)$, $1/e \leqslant t \leqslant e$, with coordinates measured in kilometres. How fast would the skier have to be moving in order to catch air?

▸▸ Stage 3

Q[10]: A wire follows the arclength-parametrized path $\mathbf{r}(s) = (x(s), y(s))$. A bead, equipped with a jet pack, slides down the wire. The jet pack can exert a variable force in a direction tangent to the wire, $U\hat{\mathbf{T}}$. Assuming the bead slides with constant speed $\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right| = c\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right| = c$, find a simplified equation for U, the signed magnitude of the force exerted by the jet pack.

Let the acceleration due to gravity be g, and let the mass of the bead with its jet pack be m. Give U as a function of s.

Remark: most beads this author has seen did not have jet packs. However, in modelling a frictionful[3] system, friction acts as a force that is directly opposing the direction of motion — much like our jet pack.

Q[11]: A snowmachine is cautiously descending a hill in low gear. Its engine provides a force $M\hat{\mathbf{T}}$ parallel to the direction of motion. The engine provides whatever force is necessary to keep the snowmachine moving at a constant speed, $|\mathbf{v}|$. Its treads do not slip.

(a) Give a formula for M in terms of the mass m of the snowmachine, the acceleration due to gravity g, and the tangent vector $\hat{\mathbf{T}}$ to the hill.

3 Frictionated? Frictiony? Befrictioned?

(b) Let $\hat{\mathbf{T}}$ point in the downhill direction. Do you expect M to be positive or negative as the snowmachine moves downhill?

(c) Find M for the hill of shape $y = 1 + \cos x$ (measured in metres) at the point $x = \frac{3\pi}{4}$ for a snowmachine of mass 200 kg.

Q[12]: A skateboarder rolls along a culvert with elliptical cross-section described by $\mathbf{r}(\theta) = (4\cos\theta, 3(1+\sin\theta))$, $0 \leqslant \theta \leqslant 2\pi$, with coordinates measured in metres.

(a) Give the height y_S (in terms of m, g, and E) where the skater's speed is zero.

(b) Write an equation relating E, m, g, and y_A, where y_A is the y-value where the skater would become airborne, i.e. where $W = 0$. (You do not have to solve for y_A explicitly.)

(c) Suppose the skater has speed 11 m/s at the bottom of the culvert. Which of the following describes their journey: they make it all the way around; they roll back and forth in the bottom half; or they make it onto the ceiling, then fall off?

Q[13]: A frictionless roller-coaster track has the form of one turn of the circular helix with parametrization $(a\cos\theta, a\sin\theta, b\theta)$. A car leaves the point where $\theta = 2\pi$ with zero velocity and moves under gravity to the point where $\theta = 0$. By Newton's law of motion, the position $\mathbf{r}(t)$ of the car at time t obeys

$$m\mathbf{r}''(t) = \mathbf{N}\big(\mathbf{r}(t)\big) - mg\hat{\mathbf{k}}$$

Here m is the mass of the car, g is a constant, $-mg\hat{\mathbf{k}}$ is the force due to gravity and $\mathbf{N}\big(\mathbf{r}(t)\big)$ is the force that the roller-coaster track applies to the car to keep the car on the track. Since the track is frictionless, $\mathbf{N}\big(\mathbf{r}(t)\big)$ is always perpendicular to $\mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)$.

(a) Prove that $E(t) = \frac{1}{2}m|\mathbf{v}(t)|^2 + mg\mathbf{r}(t)\cdot\hat{\mathbf{k}}$ is a constant, independent of t. (This is called "conservation of energy".)

(b) Prove that the speed $|\mathbf{v}|$ at the point θ obeys $|\mathbf{v}|^2 = 2gb(2\pi - \theta)$.

(c) Find the time it takes to reach $\theta = 0$.

1.8▲ Polar Coordinates

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: Consider the points

$$(x_1, y_1) = (3, 0) \qquad (x_2, y_2) = (1, 1) \qquad (x_3, y_3) = (0, 1)$$
$$(x_4, y_4) = (-1, 1) \qquad (x_5, y_5) = (-2, 0)$$

For each $1 \leqslant i \leqslant 5$,

- sketch, in the xy-plane, the point (x_i, y_i) and
- find the polar coordinates r_i and θ_i, with $0 \leqslant \theta_i < 2\pi$, for the point (x_i, y_i).

Q[2]:

(a) Find *all* pairs (r, θ) such that

$$(-2, 0) = \big(r\cos\theta\,,\ r\sin\theta\big)$$

(b) Find *all* pairs (r, θ) such that

$$(1, 1) = \big(r\cos\theta\,,\ r\sin\theta\big)$$

(c) Find *all* pairs (r, θ) such that

$$(-1, -1) = \big(r\cos\theta\,,\ r\sin\theta\big)$$

Q[3]: Consider the points

$$(x_1, y_1) = (3, 0) \qquad (x_2, y_2) = (1, 1) \qquad (x_3, y_3) = (0, 1)$$
$$(x_4, y_4) = (-1, 1) \qquad (x_5, y_5) = (-2, 0)$$

Also define, for each angle θ, the vectors

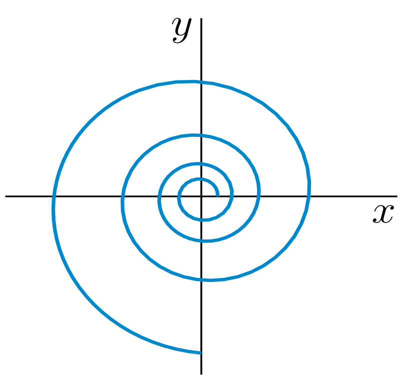
$$\hat{\mathbf{e}}_r(\theta) = \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}} \qquad \hat{\mathbf{e}}_\theta(\theta) = -\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}$$

(a) Determine, for each angle θ, the lengths of the vectors $\hat{\mathbf{e}}_r(\theta)$ and $\hat{\mathbf{e}}_\theta(\theta)$ and the angle between the vectors $\hat{\mathbf{e}}_r(\theta)$ and $\hat{\mathbf{e}}_\theta(\theta)$. Compute $\hat{\mathbf{e}}_r(\theta) \times \hat{\mathbf{e}}_\theta(\theta)$ (viewing $\hat{\mathbf{e}}_r(\theta)$ and $\hat{\mathbf{e}}_\theta(\theta)$ as vectors in three dimensions with zero $\hat{\mathbf{k}}$ components).

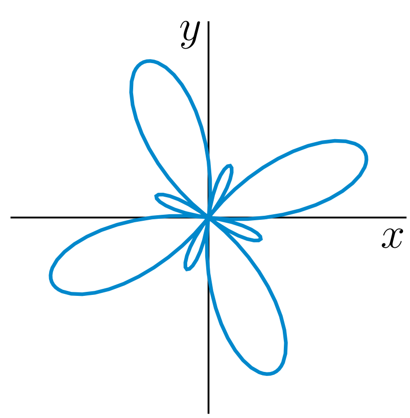
(b) For each $1 \leqslant i \leqslant 5$, sketch, in the xy-plane, the point (x_i, y_i) and the vectors $\hat{\mathbf{e}}_r(\theta_i)$ and $\hat{\mathbf{e}}_\theta(\theta_i)$. In your sketch of the vectors, place the tails of the vectors $\hat{\mathbf{e}}_r(\theta_i)$ and $\hat{\mathbf{e}}_\theta(\theta_i)$ at (x_i, y_i).

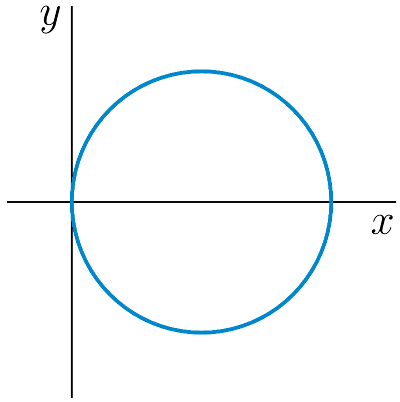
Q[4](∗): Match the following equations with the corresponding pictures. Cartesian coordinates are (x, y) and polar coordinates are (r, θ).

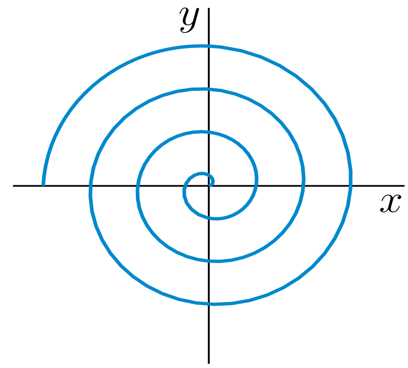
(A)

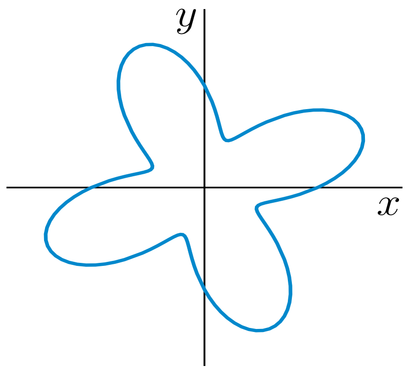


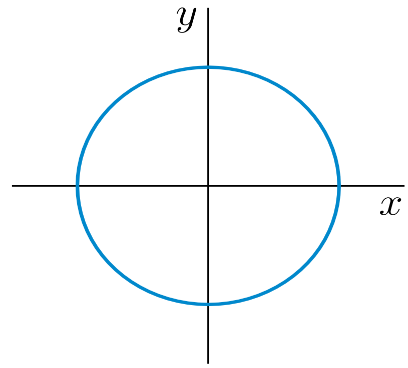
(B)



(C) 

(D) 

(E) 

(F) 

(a) $r = 2 + \sin(4\theta)$	(b) $r = 1 + 2\sin(4\theta)$	(c) $r = 1$
(d) $r = 2\cos(\theta)$, $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	(e) $r = e^{\theta/10} + e^{-\theta/10}$	(f) $r = \theta$

▸▸ Stage 2

Q[5]: Recall that a point with polar coordinates r and θ has $x = r\cos\theta$ and $y = r\sin\theta$. Let $r = f(\theta)$ be the equation of a plane curve in polar coordinates. Find the curvature of this curve at a general point θ.

Q[6]: Find the curvature of the cardioid $r = a(1 - \cos\theta)$.

Chapter 2

VECTOR FIELDS

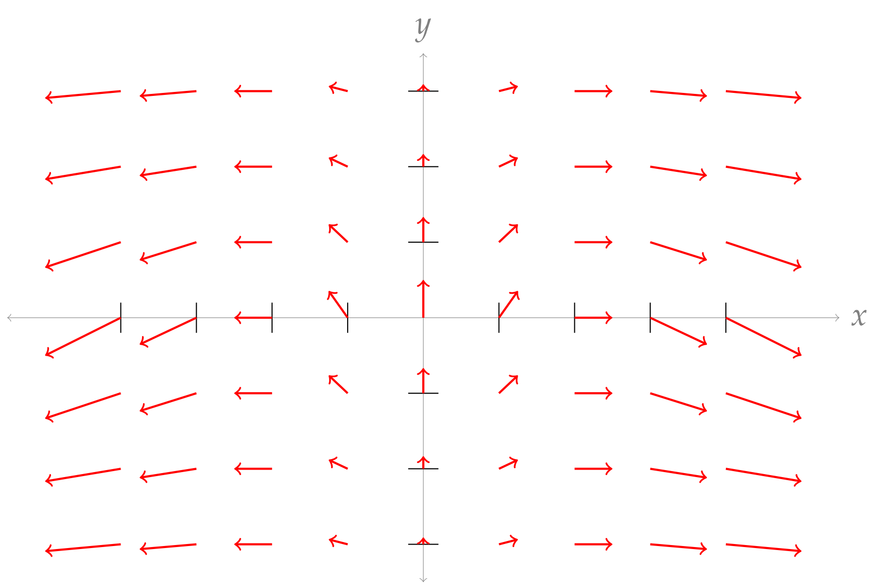
2.1 ▲ Definitions and First Examples

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

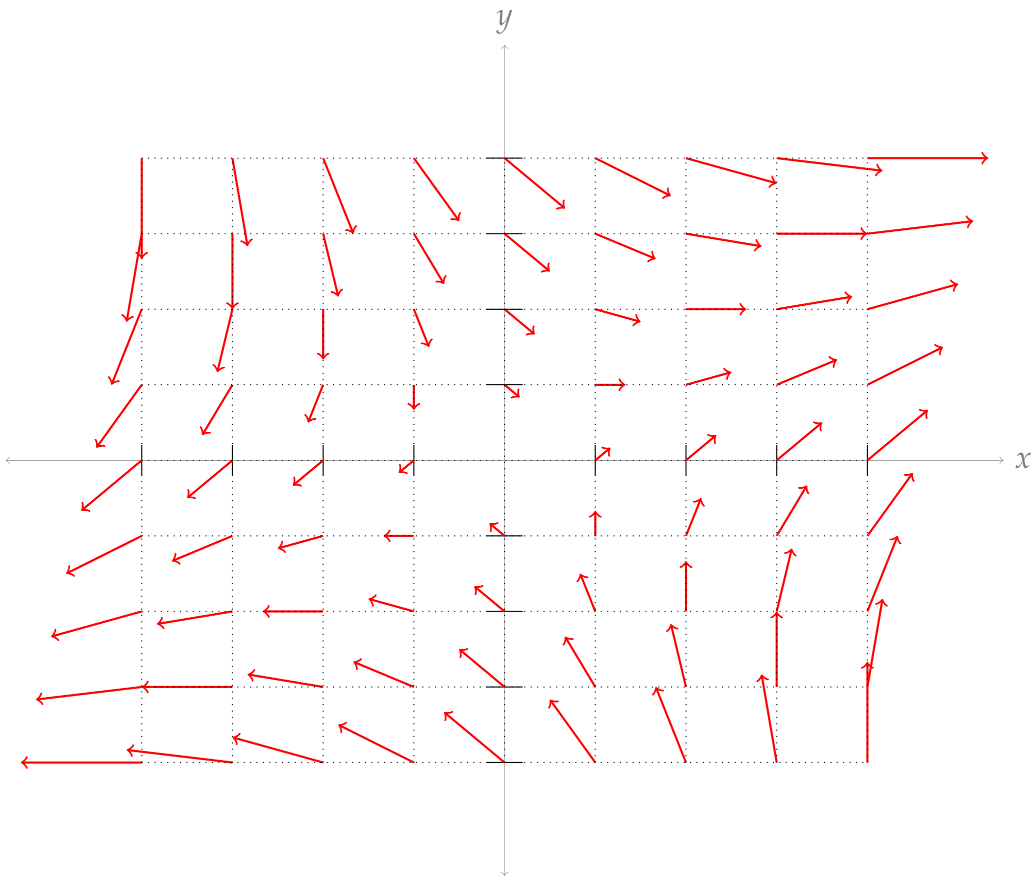
Q[1]: Below is a sketch of the vector field $\mathbf{v}(x, y)$.



Find the regions where the x-coordinates and y-coordinates are positive, negative, and zero: $\mathbf{v}(x,y)\cdot\hat{\imath}\begin{cases} >0 & \text{when } \boxed{} \\ =0 & \text{when } \boxed{} \\ <0 & \text{when } \boxed{} \end{cases}$ $\mathbf{v}(x,y)\cdot\hat{\jmath}\begin{cases} >0 & \text{when } \boxed{} \\ =0 & \text{when } \boxed{} \\ <0 & \text{when } \boxed{} \end{cases}$

You may assume that $\mathbf{v}(x,y)$ behaves as expected at the points you don't see. That is, the samples are representative of a smooth, continuous vector-valued function. You may also assume the tick marks on the axes correspond to unit distances.

Q[2]: Below is a sketch of the vector field $\mathbf{v}(x,y)$.



Find the regions where the x-coordinates and y-coordinates are positive, negative, and zero:

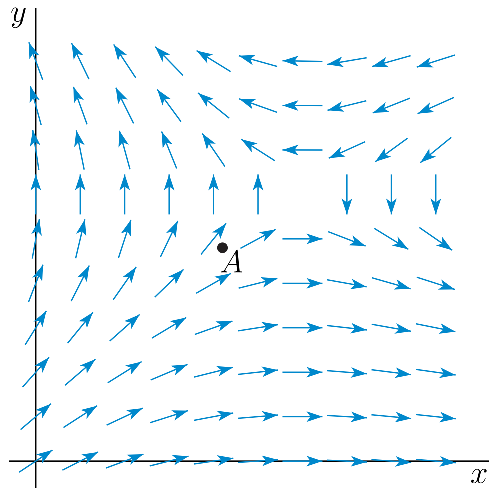
$\mathbf{v}(x,y)\cdot\hat{\imath}\begin{cases} >0 & \text{when } \boxed{} \\ =0 & \text{when } \boxed{} \\ <0 & \text{when } \boxed{} \end{cases}$ $\mathbf{v}(x,y)\cdot\hat{\jmath}\begin{cases} >0 & \text{when } \boxed{} \\ =0 & \text{when } \boxed{} \\ <0 & \text{when } \boxed{} \end{cases}$

You may assume that the samples shown are representative of the general behaviour of $\mathbf{v}(x,y)$. You may also assume the tick marks on the axes correspond to unit distances.

Q[3]: A platform with many small conveyor belts is aligned on a coordinate plane. Every conveyor belt moves an object on top of it in the direction of the origin, and a conveyor belt at position (x,y) causes an object on top of it to move with speed y. Assume the objects do not interfere with one another.

Give a vector-valued formula for the velocity of an object at position (x, y).

Q[4]: Let $\mathbf{F} = P\,\hat{\imath} + Q\,\hat{\jmath}$ be the two-dimensional vector field sketched below.



Determine the signs of P, Q, $\frac{\partial Q}{\partial x}$ and $\frac{\partial Q}{\partial y}$ at the point A.

Q[5]: Imagine that the vector field $\mathbf{v}(x, y) = x\,\hat{\imath} + y\,\hat{\jmath}$ is the velocity field of a moving fluid.

(a) At time 0 you drop a twig into the fluid at the point $(1, 1)$. What is the approximate position of the twig at time $t = 0.01$?

(b) At time 0 you drop a twig into the fluid at the point $(0, 0)$. What is the position of the twig at time $t = 0.01$?

(c) At time 0 you drop a twig into the fluid at the point $(0, 0)$. What is the position of the twig at time $t = 10$?

Q[6]: Imagine that the vector field $\mathbf{v}(x, y) = 2x\,\hat{\imath} - \hat{\jmath}$ is the velocity field of a moving fluid. At time 0 you drop a twig into the fluid at the point $(0, 0)$. What is the position of the twig at time $t = 10$?

▸▸ Stage 2

Q[7]: A platform with many small conveyor belts is aligned on a coordinate plane. Every conveyor belt moves an object on top of it in the direction of the origin, and a conveyor belt at position (x, y) causes an object on top of it to move with speed y. Assume the objects do not interfere with one another.

Give a vector-valued formula for the velocity of an object at position (x, y).

Q[8]: Friendly bees fly towards your face from all directions. The speed of each bee is inversely proportional to its distance from your face. Find a vector field for the velocity of the swarm.

Q[9]: Sketch the vector field $\mathbf{v}(x, y) = (x^2, y)$.

Q[10]: Sketch the *direction* field of $\mathbf{v}(x, y) = \left(\sqrt{x^2 + y^2}\,,\ \sqrt{(x-1)^2 + (y-1)^2}\right)$.

Q[11]: Sketch the *direction* field of $\mathbf{v}(x,y) = (x^2 + xy, y^2 - xy)$.

Q[12]: Sketch the vector field $\mathbf{v}(x,y) = \left[\frac{1/3}{\sqrt{x^2+y^2}}(x,y) + \frac{1/3}{\sqrt{(x-1)^2+y^2}}(x-1,y)\right]$.

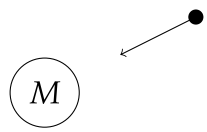
Q[13]: Sketch each of the following vector fields, by drawing a figure like Figure 2.1.1 in the CLP-4 text.

(a) $\mathbf{v}(x,y) = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}}$.

(b) $\mathbf{v}(x,y) = 2x\,\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}$.

(c) $\mathbf{v}(x,y) = \frac{y\hat{\boldsymbol{\imath}} - x\hat{\boldsymbol{\jmath}}}{\sqrt{x^2+y^2}}$.

Q[14]: A body of mass M exerts a force of magnitude $\frac{GM}{D^2}$ on a particle of unit mass distance D away from itself, where G is a physical constant. The force acts in the direction from the particle to the body.

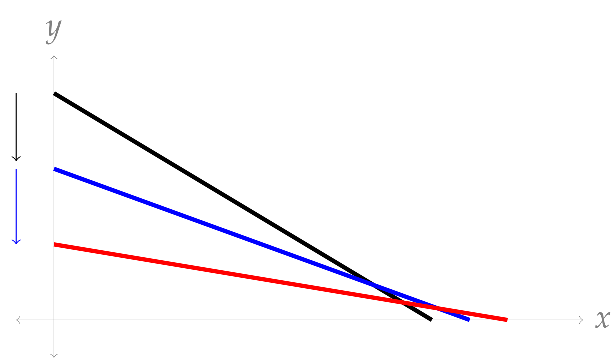


Suppose a mass of 5 kg sits at position $(0,0)$, a mass of 3 kg sits at position $(2,3)$, and a mass of 7 kg sits at position $(4,0)$ on a coordinate plane. Give the vector field $\mathbf{f}(x,y)$ of the net gravitational force exerted on a unit mass at position (x,y).

▸▸ Stage 3

Q[15]:

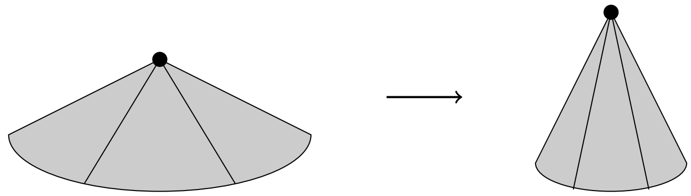
a. A pole leans against a vertical wall. The pole has length 2, and it touches the wall at height $H = 1$. The pole slides down, still touching the wall, with its height decreasing at a rate of $\frac{dH}{dt} = 0.5$.



Find a vector function $\mathbf{v} : [0,2] \to \mathbb{R}^2$ for the velocity, when $H = 1$, of a point on the pole that is p units from the lower end, using the coordinate system from the sketch above.

b. The frame of an umbrella is constructed by attaching straight, rigid poles to a common centre. The poles are all the same length, so they form radii of a circle.

The frame is lifted from the centre of the circle. The edges of the frame drag on the ground, keeping the frame in the shape of a right circular cone that is becoming taller and thinner.



Suppose the length of each pole is 2 metres, and the centre of the frame is being lifted at a rate of 50 cm/s. Give a vector field for the velocity $\mathbf{V}(x, y, z)$ of a point (x, y, z) on the frame when its centre is 1 metre above the ground.

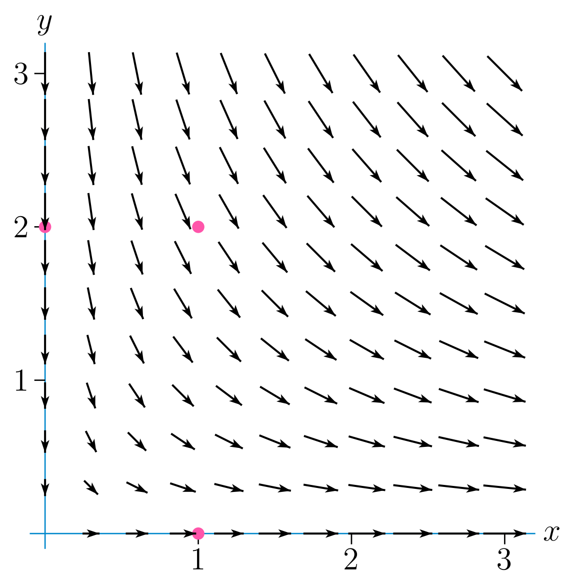
Let the ground have height $z = 0$, and let the centre of the frame sit directly above the origin.

2.2▲ Field Lines

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

Q[1]: Suppose that the vector field $\mathbf{v}(x, y)$ sketched below represents the velocity of moving water at the point (x, y) in the first quadrant of the xy-plane.



Sketch the path followed by a rubber ducky dropped in at the point

(a) $(0, 2)$

(b) $(1, 0)$

(c) $(1, 2)$

Q[2]: Find a vector field $\mathbf{v}(x, y)$ for which

$$x(t) = e^{-t}\cos t$$
$$y(t) = e^{-t}\sin t$$

is a field line.

▸▸ Stage 2

Q[3](*): Consider the function $f(x, y) = xy$.

(a) Explicitly determine the field lines (flow lines) of $\mathbf{F}(x, y) = \nabla f$.

(b) Sketch the field lines of $\mathbf{F}$ and the level curves of f in the same diagram.

Q[4](*): Find the field line of the vector field $\mathbf{F} = 2y\,\hat{\boldsymbol{\imath}} + \frac{x}{y^2}\,\hat{\boldsymbol{\jmath}} + e^y\hat{\mathbf{k}}$ that passes through $(1, 1, e)$.

Q[5](*): Find and sketch the field lines of the vector field $\mathbf{F} = x\,\hat{\boldsymbol{\imath}} + 3y\,\hat{\boldsymbol{\jmath}}$.

2.3▴ Conservative Vector Fields

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: We've seen two calculations of the energy E of a system. Equation 1.7.1 told us $E = \frac{1}{2}m|\mathbf{v}|^2 + mgy$, while Example 2.3.3 says $\frac{1}{2}m|\mathbf{v}(t)|^2 - \varphi\big(x(t), y(t), z(t)\big) = E$.

Consider a force given by $\mathbf{F} = \nabla\varphi$ for some differentiable function $\varphi : \mathbb{R}^3 \to \mathbb{R}$. A particle of mass m is being acted on by $\mathbf{F}$ and no other forces, and its position at time t is given by $(x(t), y(t), 0)$.

True or false: $mgy(t) = -\varphi(x(t), y(t), 0)$.

Q[2]: For each of the following fields, decide which of the following holds:

A. The screening test for conservative vector fields tells us $\mathbf{F}$ is conservative.

B. The screening test for conservative vector fields tells us $\mathbf{F}$ is **not** conservative.

C. The screening test for conservative vector fields does not tell us whether $\mathbf{F}$ is conservative or not.

(The screening test is Theorem 2.3.9 in the text.)

a. $\mathbf{F} = x\hat{\boldsymbol{\imath}} + z\hat{\boldsymbol{\jmath}} + y\hat{\mathbf{k}}$

b. $\mathbf{F} = y^2z\hat{\boldsymbol{\imath}} + x^2z\hat{\boldsymbol{\jmath}} + x^2y\hat{\mathbf{k}}$

c. $\mathbf{F} = (ye^{xy}+1)\hat{\boldsymbol{\imath}} + (xe^{xy}+z)\hat{\boldsymbol{\jmath}} + \left(\frac{1}{z}+y\right)\hat{\mathbf{k}}$

d. $\mathbf{F} = y\cos(xy)\hat{\boldsymbol{\imath}} + x\sin(xy)\hat{\boldsymbol{\jmath}}$

Q[3]: Suppose $\mathbf{F}$ is conservative and let a, b, and c be constants. Find a potential for $\mathbf{F}+(a,b,c)$, OR give a conservative field $\mathbf{F}$ and constants a, b, and c for which $\mathbf{F}+(a,b,c)$ is not conservative.

Q[4]: Prove, or find a counterexample to, each of the following statements.

a. If $\mathbf{F}$ is a conservative field and $\mathbf{G}$ is a non-conservative field, then $\mathbf{F}+\mathbf{G}$ is non-conservative.

b. If $\mathbf{F}$ and $\mathbf{G}$ are both non-conservative fields, then $\mathbf{F}+\mathbf{G}$ is non-conservative.

c. If $\mathbf{F}$ and $\mathbf{G}$ are both conservative fields, then $\mathbf{F}+\mathbf{G}$ is conservative.

▸▸ Stage 2

Q[5](∗): Let D be the domain consisting of all (x,y) such that $x>1$, and let $\mathbf{F}$ be the vector field

$$\mathbf{F} = -\frac{y}{x^2+y^2}\,\hat{\boldsymbol{\imath}} + \frac{x}{x^2+y^2}\,\hat{\boldsymbol{\jmath}}$$

Is $\mathbf{F}$ conservative on D? Give reasons for your answer.

Q[6]: Find a potential φ for $\mathbf{F}(x,y) = (x+y)\hat{\boldsymbol{\imath}} + (x-y)\hat{\boldsymbol{\jmath}}$, or prove none exists.

Q[7]: Find a potential φ for $\mathbf{F}(x,y) = \left(\frac{1}{x}-\frac{1}{y}\right)\hat{\boldsymbol{\imath}} + \left(\frac{x}{y^2}\right)\hat{\boldsymbol{\jmath}}$, or prove none exists.

Q[8]: Find a potential φ for $\mathbf{F}(x,y,z) = \left(x^2yz+xz\right)\hat{\boldsymbol{\imath}} + \left(\frac{1}{3}x^3z+y\right)\hat{\boldsymbol{\jmath}} + \left(\frac{1}{3}x^3y+\frac{1}{2}x^2+y\right)\hat{\mathbf{k}}$, or prove none exists.

Q[9]: Find a potential φ for

$$\mathbf{F}(x,y) = \left(\frac{x}{x^2+y^2+z^2}\right)\hat{\boldsymbol{\imath}} + \left(\frac{y}{x^2+y^2+z^2}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{z}{x^2+y^2+z^2}\right)\hat{\mathbf{k}},$$

or prove none exists.

Q[10]: Determine whether or not each of the following vector fields are conservative. Find the potential if it is.

(a) $\mathbf{F}(x,y,z) = x\hat{\boldsymbol{\imath}} - 2y\hat{\boldsymbol{\jmath}} + 3z\hat{\mathbf{k}}$

(b) $\mathbf{F}(x,y) = \frac{x\hat{\boldsymbol{\imath}}-y\hat{\boldsymbol{\jmath}}}{x^2+y^2}$

Q[11]: Let $\mathbf{F} = e^{(z^2)}\,\hat{\boldsymbol{\imath}} + 2Byz^3\,\hat{\boldsymbol{\jmath}} + \left(Axze^{(z^2)} + 3By^2z^2\right)\hat{\mathbf{k}}$.

(a) For what values of the constants A and B is the vector field $\mathbf{F}$ conservative on $\mathbb{R}^3$?

(b) If A and B have values found in (a), find a potential function for $\mathbf{F}$.

▸▸ Stage 3

Q[12]: Find the velocity field for a two dimensional incompressible fluid when there is a point source of strength m at the origin. That is, fluid is emitted from the origin at area rate $2\pi m$ cm^2/sec. Show that this velocity field is conservative and find its potential.

Q[13]: A particle of mass 10 kg moves in the force field $\mathbf{F} = \boldsymbol{\nabla}\varphi$, where $\varphi(x, y, z) = -(x^2 + y^2 + z^2)$. When its potential energy is 0, the particle is at the origin, and it moves with a velocity 2 m/s.

Following Example 2.3.3, give a region the particle can never escape.

Q[14]: A particle with constant mass $m = 1/2$ moves under a force field $\mathbf{F} = \hat{\boldsymbol{\jmath}} + 3\sqrt[3]{z}\,\hat{\mathbf{k}}$. At position $(0, 0, 0)$, its speed is 1. What is its speed at $(1, 1, 1)$?

(You may assume without proof that the particle does indeed reach the point $(1, 1, 1)$.)

Q[15]: For some differentiable, real-valued functions $f, g, h : \mathbb{R} \to \mathbb{R}$, we define

$$\mathbf{F} = 2f(x)f'(x)\hat{\boldsymbol{\imath}} + g'(y)h(z)\hat{\boldsymbol{\jmath}} + g(y)h'(z).$$

Verify that $\mathbf{F}$ is conservative.

Q[16]: Describe the region in $\mathbb{R}^3$ where the field

$$\mathbf{F} = \left\langle xy, xz, y^2 + z \right\rangle$$

has curl $\mathbf{0}$.

2.4▴ Line Integals

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: Evaluate $\int_C x^2y^2\,\mathrm{d}x + x^3y\,\mathrm{d}y$ counterclockwise around the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$.

Q[2]: For each of the following fields, decide which of the following holds:

A. The characterization of conservative vector fields, Theorem 2.4.7 (with Theorem 2.3.9), tells us $\mathbf{F}$ is conservative.

B. The characterization of conservative vector fields, Theorem 2.4.7 (with Theorem 2.3.9), tells us $\mathbf{F}$ is **not** conservative.

C. The characterization of conservative vector fields, Theorem 2.4.7 (with Theorem 2.3.9), does not tell us whether $\mathbf{F}$ is conservative or not.

a. $\mathbf{F} = x\hat{\boldsymbol{\imath}} + z\hat{\boldsymbol{\jmath}} + y\hat{\mathbf{k}}$

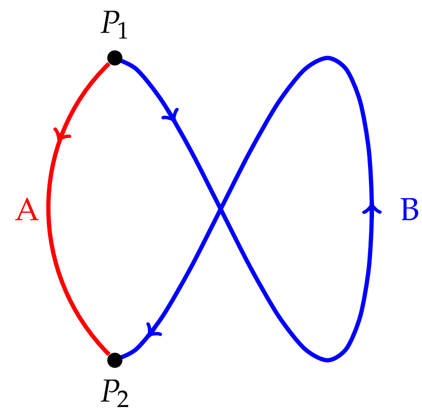
b. $\mathbf{F} = y^2z\hat{\boldsymbol{\imath}} + x^2z\hat{\boldsymbol{\jmath}} + x^2y\hat{\mathbf{k}}$

c. $\mathbf{F} = (ye^{xy} + 1)\hat{\boldsymbol{\imath}} + (xe^{xy} + z)\hat{\boldsymbol{\jmath}} + \left(\frac{1}{z} + y\right)\hat{\mathbf{k}}$

d. $\mathbf{F} = y\cos(xy)\hat{\boldsymbol{\imath}} + x\sin(xy)\hat{\boldsymbol{\jmath}}$

Q[3]: Let $\varphi(x, y, z) = e^{x^2+y^2} + \cos(z^2)$, and define $\mathbf{F} = \boldsymbol{\nabla}\varphi$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ over the closed curve C that is an ellipse traversed clockwise, centred at $(1, 2, 3)$, passing through the points $(\sqrt{5} - 1, -2, \sqrt{5} - 3)$, $((\sqrt{5} - 2)/2, -1/2, (\sqrt{5} - 6)/2)$, and $(-2, \sqrt{3} - 2, \sqrt{3} - 3)$.

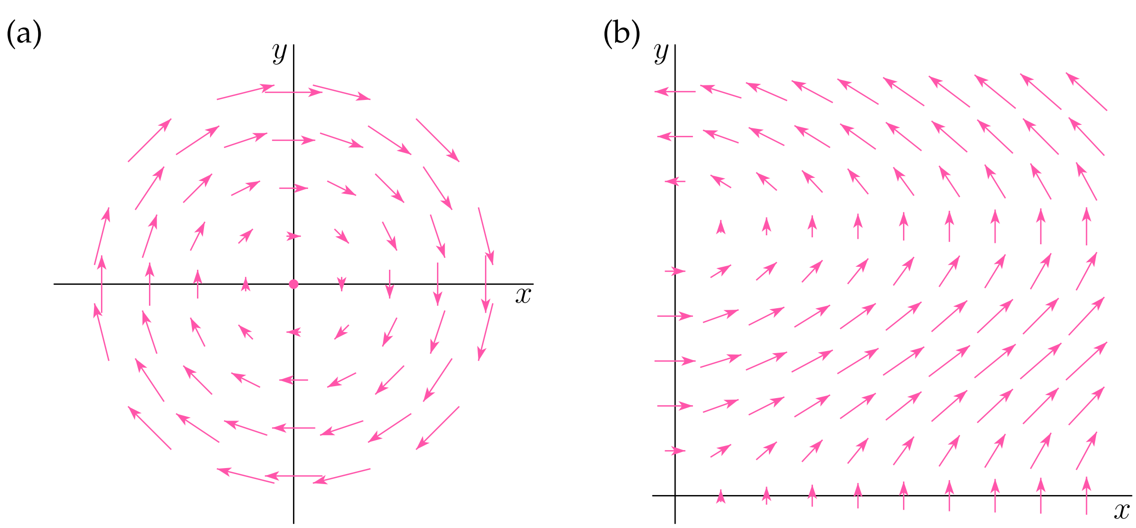
Q[4]: Let P_1 and P_2 be points in $\mathbb{R}^2$. Let A and B be paths from P_1 to P_2, as shown below.



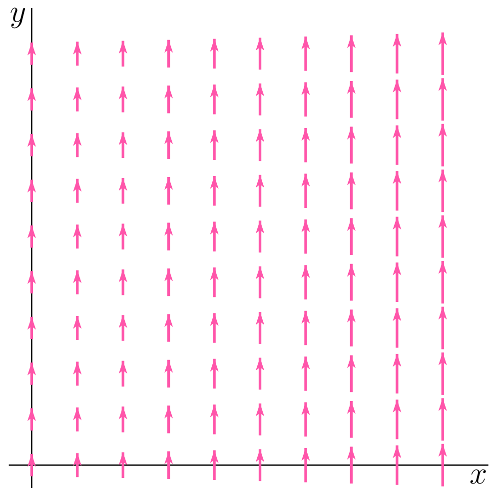
Suppose $\mathbf{F}$ is a conservative vector field in $\mathbb{R}^2$ with $\int_A \mathbf{F} \cdot d\mathbf{r} = 5$. What is $\int_B \mathbf{F} \cdot d\mathbf{r}$?

Q[5](*): Let $\mathbf{F}(x, y, z) = e^x \sin y\,\hat{\boldsymbol{\imath}} + \left[ae^x \cos y + bz\right]\hat{\boldsymbol{\jmath}} + cx\,\hat{\mathbf{k}}$. For which values of the constants a, b, c is $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed paths C?

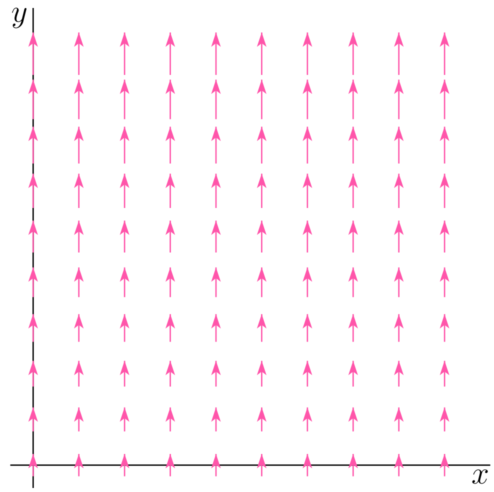
Q[6]: Consider the four vector fields sketched below. Exactly one of those vector fields is conservative. Determine which three vector fields are not conservative and explain why.



(c)



(d)



Q[7](∗): Consider the vector field

$$\mathbf{F}(x,y,z) = \frac{x-2y}{x^2+y^2}\,\hat{\imath} + \frac{2x+y}{x^2+y^2}\,\hat{\jmath} + z\,\hat{\mathbf{k}}$$

(a) Determine the domain of **F**.

(b) Compute $\nabla \times \mathbf{F}$. Simplify the result.

(c) Evaluate the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r}$$

where C is the circle of radius 2 in the plane $z = 3$, centered at $(0,0,3)$ and traversed counter-clockwise if viewed from the positive z-axis, i.e. viewed "from above".

(d) Is **F** conservative?

Q[8]: Find the work, $\int_C \mathbf{F}\cdot d\mathbf{r}$, done by the force field $\mathbf{F} = (x+y)\hat{\imath} + (x-z)\hat{\jmath} + (z-y)\hat{\mathbf{k}}$ in moving an object from $(1,0,-1)$ to $(0,-2,3)$. Does the work done depend on the path used to get from $(1,0,-1)$ to $(0,-2,3)$?

▸▸ Stage 2

Q[9]: Consider the vector field

$$\mathbf{V}(x,y) = (e^x \cos y + x^2, x^2 y + 3)$$

Evaluate the line integral $\int_C \mathbf{V}\cdot d\mathbf{r}$ along the oriented curve C obtained by moving from $(0,0)$ to $(1,0)$ to $(1,\pi)$ and finally to $(0,\pi)$ along straight line segments.

Q[10]: Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ for

(a) $\mathbf{F}(x,y) = xy\,\hat{\imath} - x^2\,\hat{\jmath}$ along $y = x^2$ from $(0,0)$ to $(1,1)$.

(b) $\mathbf{F}(x,y,z) = (x-z)\,\hat{\imath} + (y-z)\,\hat{\jmath} - (x+y)\,\hat{\mathbf{k}}$ along the polygonal path from $(0,0,0)$ to $(1,0,0)$ to $(1,1,0)$ to $(1,1,1)$.

Q[11](∗): Let $\mathcal{C}$ be the part of the curve of intersection of $xyz = 8$ and $x = 2y$ which lies between the points $(2, 1, 4)$ and $(4, 2, 1)$. Calculate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where

$$\mathbf{F} = x^2\,\hat{\boldsymbol{\imath}} + (x - 2y)\,\hat{\boldsymbol{\jmath}} + x^2y\,\hat{\mathbf{k}}$$

Q[12](∗): Let $\mathbf{F} = e^x \sin y\,\hat{\boldsymbol{\imath}} + [ae^x \cos y + bz]\,\hat{\boldsymbol{\jmath}} + cx\,\hat{\mathbf{k}}$. For which values of the constants a, b, c is $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed paths C?

Q[13]: Let $\mathbf{F} = 6x^2yz^2\,\hat{\boldsymbol{\imath}} + (2x^3z^2 + 2y - xz)\,\hat{\boldsymbol{\jmath}} + 4x^3yz\,\hat{\mathbf{k}}$ and let $\mathbf{G} = yz\,\hat{\boldsymbol{\imath}} + xy\,\hat{\mathbf{k}}$.

(a) For what value of the constant λ is the vector field $\mathbf{H} = \mathbf{F} + \lambda\mathbf{G}$ conservative on 3-space?

(b) Find a scalar potential $\phi(x, y, z)$ for the conservative field $\mathbf{H}$ referred to in part (a).

(c) Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ if C is the curve of intersection of the two surfaces $z = x$ and $y = e^{xz}$ from the point $(0, 1, 0)$ to the point $(1, e, 1)$.

Q[14](∗): Find the work done by the force field $\mathbf{F}(x,y,z) = (x - y^2\,,\, y - z^2\,,\, z - x^2)$ on a particle that moves along the line segment from $(0, 0, 1)$ to $(2, 1, 0)$.

Q[15](∗): Let $\mathbf{F} = \frac{x}{x^2+y^2}\,\hat{\boldsymbol{\imath}} + \frac{y}{x^2+y^2}\,\hat{\boldsymbol{\jmath}} + x^3\,\hat{\mathbf{k}}$. Let P be the path which starts at $(1, 0, 0)$, ends at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2}\ln 2\right)$ and follows

$$x^2 + y^2 = 1 \qquad xe^z = 1$$

Find the work done in moving a particle along P in the field $\mathbf{F}$.

Q[16](∗): Let $\mathbf{F} = \big(yz\cos x\,,\, z\sin x + 2yz\,,\, y\sin x + y^2 - \sin z\big)$ and let C be the line segment $\mathbf{r}(t) = (t, t, t)$, for $0 \leqslant t \leqslant \pi/2$. Evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$.

Q[17](∗): Let C be the upper half of the unit circle centred on $(1, 0)$ (i.e. that part of the circle which lies above the x-axis), oriented clockwise. Compute the line integral $\int_C xy\,dy$.

Q[18](∗): Show that the following line integral is independent of path and evaluate the integral.

$$\int_C (ye^x + \sin y)\,dx + (e^x + \sin y + x\cos y)\,dy$$

where C is any path from $(1, 0)$ to $(0, \pi/2)$.

Q[19](∗): Evaluate the integral

$$\int_C xy\,dx + yz\,dy + zx\,dz$$

around the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, oriented clockwise as seen from the point $(1, 1, 1)$.

Q[20](*): Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}$ is the conservative vector field

$$\mathbf{F}(x,y,z) = \big(y + ze^x, x + e^y \sin z, z + e^x + e^y \cos z\big)$$

and C is the curve given by the parametrization

$$\mathbf{r}(t) = (t, e^t, \sin t), \qquad t \text{ from } 0 \text{ to } \pi.$$

Q[21](*):

(a) For which values of the constants α, β and γ is the vector field

$$\mathbf{F}(x,y,z) = \alpha e^y\,\hat{\boldsymbol{\imath}} + (xe^y + \beta\cos z)\,\hat{\boldsymbol{\jmath}} - \gamma y \sin z\,\hat{\mathbf{k}}$$

conservative?

(b) For those values of α, β and γ found in part (a), calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve parametrized by $x = t^2$, $y = e^t$, $z = \pi t$, $0 \leqslant t \leqslant 1$.

Q[22](*): Consider the vector field $\mathbf{F}(x,y,z) = (\cos x, 2 + \sin y, e^z)$.

(a) Compute the curl of $\mathbf{F}$.

(b) Is there a function f such that $\mathbf{F} = \boldsymbol{\nabla} f$? Justify your answer.

(c) Compute the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve C parametrized by $\mathbf{r}(t) = (t, \cos t, \sin t)$ with $0 \leqslant t \leqslant 3\pi$.

Q[23](*):

(a) Consider the vector field

$$\mathbf{F}(x,y,z) = (z + e^y, xe^y - e^z \sin y, 1 + x + e^z \cos y)$$

Find the curl of $\mathbf{F}$. Is $\mathbf{F}$ conservative?

(b) Find the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ of the field $\mathbf{F}$ from (a) where C is the curve with parametrization

$$\mathbf{r}(t) = (t^2, \sin t, \cos^2 t)$$

where t ranges from 0 to π.

Q[24](*): A physicist studies a vector field $\mathbf{F}$. From experiments, it is known that $\mathbf{F}$ is of the form

$$\mathbf{F} = (x-a)ye^x\,\hat{\boldsymbol{\imath}} + (xe^x + z^3)\,\hat{\boldsymbol{\jmath}} + byz^2\,\hat{\mathbf{k}}$$

where a and b are some real numbers. From theoretical considerations, it is known that $\mathbf{F}$ is conservative.

(a) Determine a and b.

(b) Find a potential $f(x,y,z)$ such that $\boldsymbol{\nabla} f = \mathbf{F}$.

(c) Evaluate the line intgeral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve defined by

$$\mathbf{r}(t) = \big(t\,,\, \cos 2t\,,\, \cos t\big), \qquad 0 \leqslant t \leqslant \pi$$

(d) Evaluate the line integral

$$I = \int_C (x+1)ye^x \, \mathrm{d}x + (xe^x + z^3) \, \mathrm{d}y + 4yz^2 \, \mathrm{d}z,$$

where C is the same curve as in part (c). [Note: the "4" in the last term is not a misprint!].

Questions 25 and 26 ask you to evaluate line integrals of vector fields that are not conservative, but that can be expressed as a sum of a conservative vector field and another vector field that can be written concisely.

Q[25](∗): Let

$$\mathbf{F} = \left(y^2e^{3z} + Axy^3\right) \hat{\boldsymbol{\imath}} + \left(2xye^{3z} + 3x^2y^2\right) \hat{\boldsymbol{\jmath}} + Bxy^2e^{3z} \, \hat{\mathbf{k}}$$

(a) Find all values of A and B for which the vector field $\mathbf{F}$ is conservative.

(b) If A and B have values found in (a), find a potential function for $\mathbf{F}$.

(c) Let C be the curve with parametrization $\mathbf{r}(t) = e^{2t} \, \hat{\boldsymbol{\imath}} + e^{-t} \, \hat{\boldsymbol{\jmath}} + \ln(1+t) \, \hat{\mathbf{k}}$ from $(1,1,0)$ to $\left(e^2, \frac{1}{e}, \ln 2\right)$. Evaluate

$$\int_C \left(y^2e^{3z} + xy^3\right) \mathrm{d}x + \left(2xye^{3z} + 3x^2y^2\right) \mathrm{d}y + 3xy^2e^{3z} \, \mathrm{d}z.$$

Q[26](∗):

(a) For which value(s) of the constants a, b is the vector field

$$\mathbf{F} = \left(2x\sin(\pi y) - e^z\right)\hat{\boldsymbol{\imath}} + \left(ax^2\cos(\pi y) - 3e^z\right)\hat{\boldsymbol{\jmath}} - \left(x + by\right)e^z\hat{\mathbf{k}}$$

conservative?

(b) Let $\mathbf{F}$ be a conservative field from part (a). Find all functions φ for which $\mathbf{F} = \boldsymbol{\nabla}\varphi$.

(c) Let $\mathbf{F}$ be a conservative field from part (a). Evaluate $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where C is the intersection of $y = x$ and $z = \ln(1+x)$ from $(0,0,0)$ to $(1,1,\ln 2)$.

(d) Evaluate $\int_C \mathbf{G} \cdot \mathrm{d}\mathbf{r}$ where

$$\mathbf{G} = \left(2x\sin(\pi y) - e^z\right) \hat{\boldsymbol{\imath}} + \left(\pi x^2\cos(\pi y) - 3e^z\right) \hat{\boldsymbol{\jmath}} - xe^z \, \hat{\mathbf{k}}$$

and C is the intersection of $y = x$ and $z = \ln(1+x)$ from $(0,0,0)$ to $(1,1,\ln 2)$.

Q[27](∗): Consider the vector field

$$\mathbf{F}(x,y,z) = -2y\cos x \sin x \, \hat{\boldsymbol{\imath}} + \left(\cos^2 x + (1+yz)e^{yz}\right) \hat{\boldsymbol{\jmath}} + y^2e^{yz} \, \hat{\mathbf{k}}$$

(a) Find a real valued function $f(x,y,z)$ such that $\mathbf{F} = \boldsymbol{\nabla} f$.

(b) Evaluate the line integral

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

where C is the arc of the curve $\mathbf{r}(t) = \left(t, e^t, t^2 - \pi^2\right)$, $0 \leqslant t \leqslant \pi$, traversed from $(0,1,-\pi^2)$ to $(\pi, e^\pi, 0)$.

Q[28](*): Consider the vector field $\mathbf{F}(x, y, z) = 2x\,\hat{\imath} + 2y\,\hat{\jmath} + 2z\,\hat{\mathbf{k}}$.

(a) Compute $\nabla \times \mathbf{F}$.

(b) If C is any path from $(0, 0, 0)$ to (a_1, a_2, a_3) and $\mathbf{a} = a_1\,\hat{\imath} + a_2\,\hat{\jmath} + a_3\,\hat{\mathbf{k}}$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = \mathbf{a} \cdot \mathbf{a}$.

Q[29](*): Let C be the parameterized curve given by

$$\mathbf{r}(t) = \big(\cos t, \sin t, t\big), \qquad 0 \leqslant t \leqslant \frac{\pi}{2}$$

and let

$$\mathbf{F} = \big(e^{yz}\,,\ xze^{yz} + ze^{y}\,,\ xye^{yz} + e^{y}\big)$$

(a) Compute and simplify $\nabla \times \mathbf{F}$.

(b) Compute the work integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Q[30](*):

(a) Show that the planar vector field

$$\mathbf{F}(x, y) = \big(2xy\cos(x^2)\,,\ \sin(x^2) - \sin(y)\big)$$

is conservative.

(b) Find a potential function for $\mathbf{F}$.

(c) For the vector field $\mathbf{F}$ from above compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the part of the graph $x = \sin(y)$ from $y = \pi/2$ to $y = \pi$.

Q[31](*): Consider the following force field, in which m, n, p, q are constants:

$$\mathbf{F} = (mxyz + z^2 - ny^2)\,\hat{\imath} + (x^2z - 4xy)\,\hat{\jmath} + (x^2y + pxz + qz^3)\,\hat{\mathbf{k}}$$

(a) Find all values of m, n, p, q such that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for all piecewise smooth closed curves $\mathcal{C}$ in $\mathbb{R}^3$.

(b) For every possible choice of m, n, p, q in (a), find the work done by $\mathbf{F}$ in moving a particle from the bottom to the top of the sphere $x^2 + y^2 + z^2 = 2z$. (The direction of $\hat{\mathbf{k}}$ defines "up".)

▸▸ Stage 3

Q[32]: Let C be the curve from $(0, 0, 0)$ to $(1, 1, 1)$ along the intersection of the surfaces $y = x^2$ and $z = x^3$.

(a) Find $\int_C \rho\, ds$ if s is arc length along C and $\rho = 8x + 36z$.

(b) Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ if $\mathbf{F} = \sin y\,\hat{\imath} + (x\cos y + z)\,\hat{\jmath} + (y + z)\,\hat{\mathbf{k}}$.

Q[33](*): The curve C is the helix that winds around the cylinder $x^2 + y^2 = 1$ (counterclockwise, as viewed from the positive z-axis, looking down on the xy-plane). It starts at the point $(1, 0, 0)$, winds around the cylinder once, and ends at the point $(1, 0, 1)$. Compute the line integral of the vector field

$$\mathbf{F}(x, y, z) = (-y, x, z^2)$$

along C.

Q[34](*): Evaluate the line integrals below. (Use any method you like.)

(a) $\int_C (x^2 + y)\,\mathrm{d}x + x\,\mathrm{d}y$, where C is the arc of the parabola $y = 9 - x^2$ from $(-3, 0)$ to $(3, 0)$.

(b) $\int_C \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}s$, where $\mathbf{F}(x, y) = 2x^2\hat{\imath} + ye^x\hat{\jmath}$, C is the boundary of the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$. Here $\hat{\mathbf{n}}$ is the unit normal vector pointing outward from the square, and s is arc length.

Q[35](*): A particle of mass $m = 1$ has position $\mathbf{r}_0 = \hat{\jmath}$ and velocity $\mathbf{v}_0 = \hat{\imath} + \hat{\mathbf{k}}$ at time $t = 0$. The particle moves under a force $\mathbf{F}(t) = \hat{\jmath} - \sin t\,\hat{\mathbf{k}}$, where t denotes time.

(a) Find the position $\mathbf{r}(t)$ of the particle as a function of t.

(b) Find the position $\mathbf{r}_1$ of the particle when it crosses the plane $x = \pi/2$ for the first time after time $t = 0$.

(c) Determine the work done by $\mathbf{F}$ in moving the particle from $\mathbf{r}_0$ to $\mathbf{r}_1$.

Questions 36 and 37 ask you to find a path that leads to a particular value of a line integral. Many such paths are possible — you only need to find one.

Q[36](*):

(a) Consider the vector field $\mathbf{F}(x, y) = (3y, x - 1)$ in $\mathbb{R}^2$. Compute the line integral

$$\int_L \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

where L is the line segment from $(2, 2)$ to $(1, 1)$.

(b) Find an oriented path C from $(2, 2)$ to $(1, 1)$ such that

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 4$$

where $\mathbf{F}$ is the vector field from (a).

Q[37](*): Let $\mathbf{F} = (2y + 2)\,\hat{\imath}$ be a vector field on $\mathbb{R}^2$. Find an oriented curve C from $(0, 0)$ to $(2, 0)$ such that $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 8$.

Q[38](*): Let

$$\mathbf{F}(x, y) = \big(1, yg(y)\big)$$

and suppose that $g(y)$ is a function defined everywhere with everywhere continuous partials. Show that for any curve C whose endpoints P and Q lie on the x-axis,

$$\text{distance between } P \text{ and } Q = \left| \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} \right|$$

Q[39](∗): Let S be the surface $z = 2 + x^2 - 3y^2$ and let $\mathbf{F}(x, y, z) = (xz + axy^2)\hat{\imath} + yz\hat{\jmath} + z^2\hat{\mathbf{k}}$. Consider the points $P_1 = (1, 1, 0)$ and $P_2 = (0, 0, 2)$ on the surface S.

Find a value of the constant a so that $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any two curves C_1 and C_2 on the surface S from P_1 to P_2.

Q[40](∗): Consider the vector field $\mathbf{F}$ defined as

$$\mathbf{F}(x, y, z) = \Big((1 + ax^2)ye^{3x^2} - bxz\cos(x^2z)\,,\ xe^{3x^2}\,,\ x^2\cos(x^2z)\Big)$$

where a and b are real valued constants.

(a) Compute $\nabla \times \mathbf{F}$.

(b) Determine for which values a and b the vector field $\mathbf{F}$ is conservative.

(c) For the values of a and b obtained in part (b), find a potential function f such that $\nabla f = \mathbf{F}$.

(d) Evaluate the line integral

$$\int_C \Big(ye^{3x^2} + 2xz\cos(x^2z)\Big)dx + xe^{3x^2}dy + x^2\cos(x^2z)dz$$

where C is the arc of the curve (t, t, t^3) starting at the point $(0, 0, 0)$ and ending at the point $(1, 1, 1)$.

Q[41](∗): Let C be the curve from $(0, 0, 0)$ to $(1, 1, 1)$ along the intersection of the surfaces $y = x^2$ and $z = x^3$.

(a) Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ if $\mathbf{F} = (xz - y)\,\hat{\imath} + (z + x)\,\hat{\jmath} + y\,\hat{\mathbf{k}}$.

(b) Find $\int_C \rho\, ds$ if s is arc length along C and $\rho(x, y, z) = 8x + 36z$.

(c) Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ if $\mathbf{F} = \sin y\,\hat{\imath} + (x\cos y + z)\,\hat{\jmath} + (y + z)\,\hat{\mathbf{k}}$.

Q[42](∗): The vector field $\mathbf{F}(x, y, z) = Ax^3y^2z\,\hat{\imath} + \big(z^3 + Bx^4yz\big)\,\hat{\jmath} + \big(3yz^2 - x^4y^2\big)\,\hat{\mathbf{k}}$ is conservative on $\mathbb{R}^3$.

(a) Find the values of the constants A and B.

(b) Find a potential φ such that $\mathbf{F} = \nabla\varphi$ on $\mathbb{R}^3$.

(c) If $\mathcal{C}$ is the curve $y = -x,\ z = x^2$ from $(0, 0, 0)$ to $(1, -1, 1)$, evaluate $I = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

(d) Evaluate $J = \int_{\mathcal{C}} (z - 4x^3y^2z)dx + (z^3 - x^4yz)dy + (3yz^2 - x^4y^2)dz$, where $\mathcal{C}$ is the curve of part (c).

(e) Let $\mathcal{T}$ be the closed triangular path with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, oriented counterclockwise as seen from the point $(1, 1, 1)$. Evaluate $\int_{\mathcal{T}} (z\hat{\imath} + \mathbf{F}) \cdot d\mathbf{r}$.

Q[43](∗): A particle of mass

$$m = 2$$

is acted on by a force

$$\mathbf{F} = \left(4t\,,\, 6t^2\,,\, -4t\right)$$

At $t = 0$, the particle has velocity zero and is located at the point $(1, 2, 3)$.

(a) Find the velocity vector $\mathbf{v}(t)$ for $t \geqslant 0$.

(b) Find the position vector $\mathbf{r}(t)$ for $t \geqslant 0$.

(c) Find $\kappa(t)$ the curvature of the path traversed by the particle for $t \geqslant 0$.

(d) Find the work done by the force on the particle from $t = 0$ to $t = T$.

Q[44](*): The position of an airplane at time t is given by $x = y = \frac{4\sqrt{2}}{3}t^{3/2}$, $z = t(2 - t)$ from take-off at $t = 0$ to landing at $t = 2$.

(a) What is the total distance the plane travels on this flight?

(b) Find the radius of curvature κ at the apex of the flight, which occurs at $t = 1$.

(c) Two external forces are applied to the plane during the flight: the force of gravity $\mathbf{G} = -Mg\,\hat{\mathbf{k}}$, where M is the mass of the plane and g is a constant; and a friction force $\mathbf{F} = -|\mathbf{v}|^2\mathbf{v}$, where $\mathbf{v}$ is the velocity of the plane. Find the work done by each of these forces during the flight.

(d) One half-hour later, a bird follows the exact same flight — path as the plane, travelling at a constant speed $v = 3$. One can show that at the apex of the path, i.e. when the bird is at $\left(\frac{4\sqrt{2}}{3}, \frac{4\sqrt{2}}{3}, 1\right)$, the principal unit normal $\hat{\mathbf{N}}$ to the path points in the $-\hat{\mathbf{k}}$ direction. Find the bird's (vector) acceleration at that moment.

Chapter 3

SURFACE INTEGRALS

3.1 ▲ Parametrized Surfaces

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: Parametrize the surface given by $z = e^{x+1} + xy$ in terms of x and y.

Q[2](*): Let S be the surface given by

$$\mathbf{r}(u,v) = \big(u+v\,,\, u^2+v^2\,,\, u-v\big), \qquad -2 \leqslant u \leqslant 2,\ -2 \leqslant v \leqslant 2$$

This is a surface you are familiar with. What surface is it (it may be just a portion of one of the following)?

sphere helicoid ellipsoid saddle parabolic bowl cylinder cone plane

▸▸ Stage 2

Q[3](*): Suppose S is the part of the hyperboloid $x^2 + y^2 - 2z^2 = 1$ that lies inside the cylinder $x^2 + y^2 = 9$ and above the plane $z = 1$ (i.e. for which $z \geqslant 1$).

Which of the following are parameterizations of S?

(a) The vector function

$$\mathbf{r}(u,v) = u\,\hat{\boldsymbol{\imath}} + v\,\hat{\boldsymbol{\jmath}} + \frac{\sqrt{u^2+v^2-1}}{\sqrt{2}}\,\hat{\mathbf{k}}$$

with domain $D = \big\{ (u,v) \,\big|\, 2 \leqslant u^2+v^2 \leqslant 9 \big\}$.

(b) The vector function

$$\mathbf{r}(u,v) = u\sin v\,\hat{\boldsymbol{\imath}} - u\cos v\,\hat{\boldsymbol{\jmath}} + \sqrt{\frac{u^2}{2} - \frac{1}{2}}\,\hat{\mathbf{k}}$$

with domain $D = \{\, (u,v) \mid \sqrt{3} \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\pi \,\}$.

(c) The vector function

$$\mathbf{r}(u,v) = \sqrt{1+2v^2}\cos u\,\hat{\boldsymbol{\imath}} + \sqrt{1+2v^2}\sin u\,\hat{\boldsymbol{\jmath}} + v\,\hat{\mathbf{k}}$$

with domain $D = \{\, (u,v) \mid 0 \leqslant u \leqslant 2\pi,\ 1 \leqslant v \leqslant 2 \,\}$.

(d) The vector function

$$\mathbf{r}(u,v) = \sqrt{1+u}\sin v\,\hat{\boldsymbol{\imath}} + \sqrt{1+u}\cos v\,\hat{\boldsymbol{\jmath}} + \sqrt{\frac{u}{2}}\,\hat{\mathbf{k}}$$

with domain $D = \{\, (u,v) \mid 2 \leqslant u \leqslant 8,\ 0 \leqslant v \leqslant 2\pi \,\}$.

(e) The vector function

$$\mathbf{r}(u,v) = \sqrt{u}\cos v\,\hat{\boldsymbol{\imath}} - \sqrt{u}\sin v\,\hat{\boldsymbol{\jmath}} + \frac{\sqrt{u+1}}{\sqrt{2}}\,\hat{\mathbf{k}}$$

with domain $D = \{\, (u,v) \mid 3 \leqslant u \leqslant 9,\ 0 \leqslant v \leqslant 2\pi \,\}$.

Q[4](∗): Suppose the surface S is the part of the sphere $x^2+y^2+z^2 = 2$ that lies inside the cylinder $x^2+y^2 = 1$ and for which $z \geqslant 0$. Which of the following are parameterizations of S?

(a) $\mathbf{r}(\phi,\theta) = 2\sin\phi\cos\theta\,\hat{\boldsymbol{\imath}} + 2\cos\phi\,\hat{\boldsymbol{\jmath}} + 2\sin\phi\sin\theta\,\hat{\mathbf{k}}$

$0 \leqslant \phi \leqslant \frac{\pi}{4}, 0 \leqslant \theta \leqslant 2\pi$

(b) $\mathbf{r}(x,y) = x\,\hat{\boldsymbol{\imath}} - y\,\hat{\boldsymbol{\jmath}} + \sqrt{2-x^2-y^2}\,\hat{\mathbf{k}}$

$x^2+y^2 \leqslant 1$

(c) $\mathbf{r}(u,\theta) = u\sin\theta\,\hat{\boldsymbol{\imath}} + u\cos\theta\,\hat{\boldsymbol{\jmath}} + \sqrt{2-u^2}\,\hat{\mathbf{k}}$

$0 \leqslant u \leqslant 2, 0 \leqslant \theta \leqslant 2\pi$

(d) $\mathbf{r}(\phi,\theta) = \sqrt{2}\sin\phi\cos\theta\,\hat{\boldsymbol{\imath}} + \sqrt{2}\sin\phi\sin\theta\,\hat{\boldsymbol{\jmath}} + \sqrt{2}\cos\phi\,\hat{\mathbf{k}}$

$0 \leqslant \phi \leqslant \frac{\pi}{4}, 0 \leqslant \theta \leqslant 2\pi$

(e) $\mathbf{r}(\phi,z) = -\sqrt{2-z^2}\sin\phi\,\hat{\boldsymbol{\imath}} + \sqrt{2-z^2}\cos\phi\,\hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}}$

$0 \leqslant \phi \leqslant 2\pi, 1 \leqslant z \leqslant \sqrt{2}$

Q[5](∗): Let S be the part of the paraboloid $z+x^2+y^2 = 4$ lying between the planes $z = 0$ and $z = 1$. For each of the following, indicate whether or not it correctly parameterizes the surface S.

(a) $\mathbf{r}(u,v) = u\,\hat{\boldsymbol{\imath}} + v\,\hat{\boldsymbol{\jmath}} + (4-u^2-v^2)\,\hat{\mathbf{k}}, \qquad 0 \leqslant u^2+v^2 \leqslant 1$

(b) $\mathbf{r}(u,v) = (\sqrt{4-u}\,\cos v)\,\hat{\boldsymbol{\imath}} + (\sqrt{4-u}\,\sin v)\,\hat{\boldsymbol{\jmath}} + u\,\hat{\mathbf{k}}, \qquad 0 \leqslant u \leqslant 1, 0 \leqslant v \leqslant 2\pi$

(c) $\mathbf{r}(u,v) = (u\cos v)\,\hat{\boldsymbol{\imath}} + (u\sin v)\,\hat{\boldsymbol{\jmath}} + (4-u^2)\,\hat{\mathbf{k}}, \qquad \sqrt{3} \leqslant u \leqslant 2, 0 \leqslant v \leqslant 2\pi$

▸▸ Stage 3

Q[6](∗): Consider the following surfaces

- S_1 is the hemisphere given by the equation $x^2 + y^2 + z^2 = 4$ with $z \geqslant 0$.
- S_2 is the cylinder given by the equation $x^2 + y^2 = 1$.
- S_3 is the cone given by the equation $z^2 = x^2 + y^2$ with $z \geqslant 0$.

Consider the following parameterizations:

A. $\mathbf{r}(\theta, \phi) = \big(\sqrt{4}\cos\theta\sin\phi\,,\ \sqrt{4}\sin\theta\sin\phi\,,\ \sqrt{4}\cos\phi\big), \qquad 0 \leqslant \theta \leqslant 2\pi, \quad 0 \leqslant \phi \leqslant \pi/6$

B. $\mathbf{r}(\theta, \phi) = \big(\sqrt{4}\cos\theta\sin\phi\,,\ \sqrt{4}\sin\theta\sin\phi\,,\ \sqrt{4}\cos\phi\big), \qquad 0 \leqslant \theta \leqslant 2\pi, \quad 0 \leqslant \phi \leqslant \pi/4$

C. $\mathbf{r}(\theta, \phi) = \big(\sqrt{4}\cos\theta\sin\phi\,,\ \sqrt{4}\sin\theta\sin\phi\,,\ \sqrt{4}\cos\phi\big), \qquad 0 \leqslant \theta \leqslant 2\pi, \quad 0 \leqslant \phi \leqslant \pi/3$

D. $\mathbf{r}(\theta, z) = \big(\sqrt{4 - z^2}\cos\theta\,,\ \sqrt{4 - z^2}\sin\theta\,,\ z\big) \qquad 0 \leqslant \theta \leqslant 2\pi, \quad 1 \leqslant z \leqslant 2$

E. $\mathbf{r}(\theta, z) = \big(\sqrt{4 - z^2}\cos\theta\,,\ \sqrt{4 - z^2}\sin\theta\,,\ z\big) \qquad 0 \leqslant \theta \leqslant 2\pi, \quad \sqrt{2} \leqslant z \leqslant 2$

F. $\mathbf{r}(\theta, z) = \big(\sqrt{4 - z^2}\cos\theta\,,\ \sqrt{4 - z^2}\sin\theta\,,\ z\big) \qquad 0 \leqslant \theta \leqslant 2\pi, \quad \sqrt{3} \leqslant z \leqslant 2$

G. $\mathbf{r}(\theta, z) = \big(z\cos\theta\,,\ z\sin\theta\,,\ z\big) \qquad 0 \leqslant \theta \leqslant 2\pi, \quad 0 \leqslant z \leqslant 1$

H. $\mathbf{r}(\theta, z) = \big(z\cos\theta\,,\ z\sin\theta\,,\ z\big) \qquad 0 \leqslant \theta \leqslant 2\pi, \quad 0 \leqslant z \leqslant \sqrt{2}$

I. $\mathbf{r}(\theta, z) = \big(z\cos\theta\,,\ z\sin\theta\,,\ z\big) \qquad 0 \leqslant \theta \leqslant 2\pi, \quad 0 \leqslant z \leqslant \sqrt{3}$

J. $\mathbf{r}(x, y) = \big(x\,,\ y\,,\ \sqrt{x^2 + y2}\big) \qquad x^2 + y^2 \leqslant 1$

K. $\mathbf{r}(x, y) = \big(x\,,\ y\,,\ \sqrt{x^2 + y2}\big) \qquad x^2 + y^2 \leqslant \sqrt{2}$

L. $\mathbf{r}(x, y) = \big(x\,,\ y\,,\ \sqrt{x^2 + y2}\big) \qquad x^2 + y^2 \leqslant 2$

For each of the following, choose from above all of the valid parameterization of each of the given surfaces. Note that there may be one or more valid parameterization for each surface, and not necessarily all of the above parameterizations will be used.

(a) The part of S_1 contained inside S_2:

(b) The part of S_1 contained inside S_3:

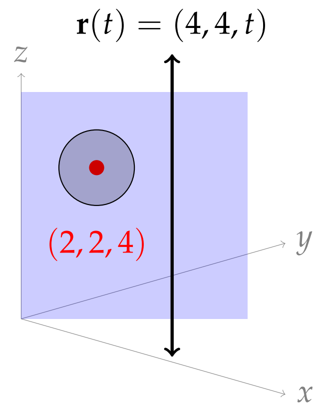
(c) The part of S_3 contained inside S_2:

(d) The part of S_3 contained inside S_1:

Q[7]: Parametrize a solid of rotation about a line not parallel to an axis. Maybe first show that the plane you're rotating is normal to that axis.

(a) Give a parametric equation for the circle of radius 1, centred at $(2, 2, 4)$, lying in the plane $x = y$.

(b) Give a parametrized equation for the surface formed by rotating the circle from part (a) about the line $\mathbf{r}(t) = 4\hat{\imath} + 4\hat{\jmath} + t\hat{\mathbf{k}}$.



3.2▲ Tangent Planes

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: Is it reasonable to say that the surfaces $x^2 + y^2 + (z-1)^2 = 1$ and $x^2 + y^2 + (z+1)^2 = 1$ are tangent to each other at $(0, 0, 0)$?

Q[2]: Let the point $\mathbf{r}_0 = (x_0, y_0, z_0)$ lie on the surface $G(x, y, z) = 0$. Assume that $\nabla G(x_0, y_0, z_0) \neq \mathbf{0}$. Suppose that the parametrized curve $\mathbf{r}(t) = \big(x(t), y(t), z(t)\big)$ is contained in the surface and that $\mathbf{r}(t_0) = \mathbf{r}_0$. Show that the tangent line to the curve at $\mathbf{r}_0$ lies in the tangent plane to $G = 0$ at $\mathbf{r}_0$.

Q[3]: Find the parametric equations of the normal line to the surface $z = f(x, y)$ at the point $\big(x_0\,,\ y_0\,,\ z_0 = f(x_0, y_0)\big)$. By definition, the normal line in question is the line through (x_0, y_0, z_0) whose direction vector is perpendicular to the surface at (x_0, y_0, z_0).

Q[4]: Let $F(x_0, y_0, z_0) = G(x_0, y_0, z_0) = 0$ and let the vectors $\nabla F(x_0, y_0, z_0)$ and $\nabla G(x_0, y_0, z_0)$ be nonzero and not be parallel to each other. Find the equation of the normal plane to the curve of intersection of the surfaces $F(x, y, z) = 0$ and $G(x, y, z) = 0$ at (x_0, y_0, z_0). By definition, that normal plane is the plane through (x_0, y_0, z_0) whose normal vector is the tangent vector to the curve of intersection at (x_0, y_0, z_0).

Q[5]: Let $f(x_0, y_0) = g(x_0, y_0)$ and let $\big(f_x(x_0, y_0), f_y(x_0, y_0)\big) \neq \big(g_x(x_0, y_0), g_y(x_0, y_0)\big)$. Find the equation of the tangent line to the curve of intersection of the surfaces $z = f(x, y)$ and $z = g(x, y)$ at $(x_0\,,\ y_0\,,\ z_0 = f(x_0, y_0))$.

▸▸ Stage 2

Q[6](*): Let $f(x, y) = \dfrac{x^2 y}{x^4 + 2y^2}$. Find the tangent plane to the surface $z = f(x, y)$ at the point $\left(-1\,,\,1\,,\,\frac{1}{3}\right)$.

Q[7](∗): Find the tangent plane to

$$\frac{27}{\sqrt{x^2+y^2+z^2+3}} = 9$$

at the point $(2,1,1)$.

Q[8](∗): Consider the surface $z = f(x,y)$ defined implicitly by the equation $xyz^2 + y^2z^3 = 3 + x^2$. Use a 3–dimensional gradient vector to find the equation of the tangent plane to this surface at the point $(-1,1,2)$. Write your answer in the form $z = ax + by + c$, where a, b and c are constants.

Q[9](∗): A surface is given by

$$z = x^2 - 2xy + y^2.$$

(a) Find the equation of the tangent plane to the surface at $x = a$, $y = 2a$.

(b) For what value of a is the tangent plane parallel to the plane $x - y + z = 1$?

Q[10](∗): A surface S is given by the parametric equations

$$\begin{aligned} x &= 2u^2 \\ y &= v^2 \\ z &= u^2 + v^3 \end{aligned}$$

Find an equation for the tangent plane to S at the point $(8,1,5)$.

Q[11](∗): Let S be the surface given by

$$\mathbf{r}(u,v) = \big(u+v\,,\,u^2+v^2\,,\,u-v\big), \qquad -2 \leqslant u \leqslant 2,\ -2 \leqslant v \leqslant 2$$

Find the tangent plane to the surface at the point $(2,2,0)$.

Q[12](∗): Find the tangent plane and normal line to the surface $z = f(x,y) = \frac{2y}{x^2+y^2}$ at $(x,y) = (-1,2)$.

Q[13](∗): Find all the points on the surface $x^2 + 9y^2 + 4z^2 = 17$ where the tangent plane is parallel to the plane $x - 8z = 0$.

Q[14](∗): Let S be the surface $z = x^2 + 2y^2 + 2y - 1$. Find all points $P(x_0, y_0, z_0)$ on S with $x_0 \neq 0$ such that the normal line at P contains the origin $(0,0,0)$.

Q[15](∗): Find all points on the hyperboloid $z^2 = 4x^2 + y^2 - 1$ where the tangent plane is parallel to the plane $2x - y + z = 0$.

▸▸ Stage 3

Q[16](∗):

(a) Find a vector perpendicular at the point $(1, 1, 3)$ to the surface with equation $x^2 + z^2 = 10$.

(b) Find a vector tangent at the same point to the curve of intersection of the surface in part (a) with surface $y^2 + z^2 = 10$.

(c) Find parametric equations for the line tangent to that curve at that point.

Q[17](∗): Let P be the point where the curve

$$\mathbf{r}(t) = t^3\,\hat{\boldsymbol{\imath}} + t\,\hat{\boldsymbol{\jmath}} + t^2\,\hat{\mathbf{k}}, \qquad (0 \leqslant t < \infty)$$

intersects the surface

$$z^3 + xyz - 2 = 0$$

Find the (acute) angle between the curve and the surface at P.

Q[18]: Find all horizontal planes that are tangent to the surface with equation

$$z = xye^{-(x^2+y^2)/2}$$

What are the largest and smallest values of z on this surface?

3.3▲ Surface Integrals

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: Let $0 < \theta < \frac{\pi}{2}$, and $a, b > 0$. Denote by S the part of the surface $z = y\,\tan\theta$ with $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$.

(a) Find the surface area of S without using any calculus.

(b) Find the surface area of S by using (3.3.2) in the CLP-4 text.

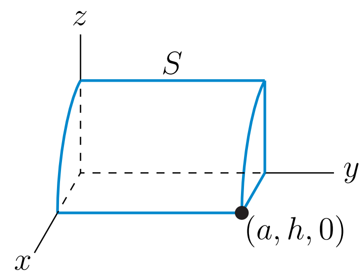
Q[2]: Let $a, b, c > 0$. Denote by S the triangle with vertices $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$.

(a) Find the surface area of S in three different ways, each using (3.3.2) in the CLP-4 text.

(b) Denote by T_{xy} the projection of S onto the xy-plane. (It is the triangle with vertices $(0, 0, 0)$ $(a, 0, 0)$ and $(0, b, 0)$.) Similarly use T_{xz} to denote the projection of S onto the xz-plane and T_{yz} to denote the projection of S onto the yz-plane. Show that

$$\text{Area}(S) = \sqrt{\text{Area}(T_{xy})^2 + \text{Area}(T_{xz})^2 + \text{Area}(T_{yz})^2}$$

Q[3]: Let $a, h > 0$. Denote by S the part of the cylinder $x^2 + z^2 = a^2$ with $x \geqslant 0$, $0 \leqslant y \leqslant h$ and $z \geqslant 0$.



(a) Find the surface area of S without using any calculus.

(b) Parametrize S by

$$\mathbf{r}(\theta, y) = a\cos\theta\,\hat{\imath} + y\,\hat{\jmath} + a\sin\theta\,\hat{\mathbf{k}} \quad 0 \leqslant \theta \leqslant \frac{\pi}{2},\ 0 \leqslant y \leqslant h$$

Find the surface area of S by using (3.3.1) in the CLP-4 text.

▸▸ Stage 2

Q[4]: Let S be the part of the surface $z = xy$ lying inside the cylinder $x^2 + y^2 = 3$. Find the moment of inertia of S about the z-axis, that is,

$$I = \iint_S (x^2 + y^2)\,\mathrm{d}S$$

Q[5](*): Find the surface area of the part of the paraboloid $z = a^2 - x^2 - y^2$ which lies above the xy–plane.

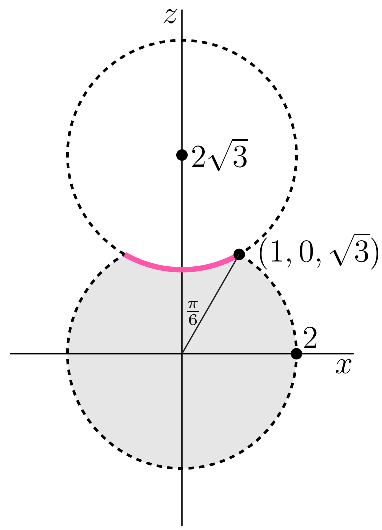
Q[6](*): Find the area of the portion of the cone $z^2 = x^2 + y^2$ lying between the planes $z = 2$ and $z = 3$.

Q[7](*): Determine the surface area of the surface given by $z = \frac{2}{3}\big(x^{3/2} + y^{3/2}\big)$, over the square $0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1$.

Q[8](*):

(a) To find the surface area of the surface $z = f(x, y)$ above the region D, we integrate $\iint_D F(x, y)\,\mathrm{d}A$. What is $F(x, y)$?

(b) Consider a "Death Star", a ball of radius 2 centred at the origin with another ball of radius 2 centred at $(0, 0, 2\sqrt{3})$ cut out of it. The diagram below shows the slice where $y = 0$.



(i) The Rebels want to paint part of the surface of Death Star hot pink; specifically, the concave part (indicated with a thick line in the diagram). To help them determine how much paint is needed, carefully fill in the missing parts of this integral:

$$\text{surface area} = \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} \_\_\_\_\_ \, \mathrm{d}r \, \mathrm{d}\theta$$

(ii) What is the total surface area of the Death Star?

Q[9](∗): Find the area of the cone $z^2 = x^2 + y^2$ between $z = 1$ and $z = 16$.

Q[10](∗): Find the surface area of that part of the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ which lies within the cylinder $\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$.

Q[11]: The cylinder $x^2 + y^2 = 2x$ cuts out a portion S of the upper half of the cone $x^2 + y^2 = z^2$. Compute

$$\iint_S (x^4 - y^4 + y^2 z^2 - z^2 x^2 + 1) \, \mathrm{d}S$$

Q[12]: Find the surface area of the torus obtained by rotating the circle $(x - R)^2 + z^2 = r^2$ (the circle is contained in the xz-plane) about the z-axis.

Q[13]: A spherical shell of radius a is centred at the origin. Find the centroid (i.e. the centre of mass with constant density) of the part of the sphere that lies in the first octant.

Q[14]: Find the area of that part of the cylinder $x^2 + y^2 = 2ay$ lying outside $z^2 = x^2 + y^2$.

Q[15](∗): Let a and b be positive constants, and let $\mathcal{S}$ be the part of the conical surface

$$a^2 z^2 = b^2 (x^2 + y^2)$$

where $0 \leqslant z \leqslant b$. Consider the surface integral

$$I = \iint_{\mathcal{S}} (x^2 + y^2) \, \mathrm{d}S.$$

(a) Express I as a double integral over a disk in the xy-plane.

(b) Use the parametrization $x = t\cos\theta$, $y = t\sin\theta$, etc., to express I as a double integral over a suitable region in the $t\theta$-plane.

(c) Evaluate I using the method of your choice.

Q[16]: Evaluate, for each of the following, the flux $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ where $\hat{\mathbf{n}}$ is the outward normal to the surface S.

(a) $\mathbf{F} = (x^2+y^2+z^2)^n(x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{\mathbf{k}})$ and the surface S is the sphere $x^2+y^2+z^2 = a^2$.

(b) $\mathbf{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{\mathbf{k}}$ and S is the surface of the rectangular box $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$, $0 \leqslant z \leqslant c$.

(c) $\mathbf{F} = y\,\hat{\imath} + z\,\hat{\mathbf{k}}$ and S is the surface of the solid cone $0 \leqslant z \leqslant 1 - \sqrt{x^2+y^2}$.

Q[17](∗): Let $\mathcal{S}$ be the part of the surface $x^2+y^2+2z = 2$ that lies above the square $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$.

(a) Find $\displaystyle\iint_{\mathcal{S}} \frac{x^2+y^2}{\sqrt{1+x^2+y^2}}\,\mathrm{d}S.$

(b) Find the flux of $\mathbf{F} = x\hat{\imath} + y\hat{\jmath} + z\hat{\mathbf{k}}$ upward through $\mathcal{S}$.

Q[18](∗): Let $\mathcal{S}$ be the part of the surface $z = xy$ that lies above the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$ in the xy-plane.

(a) Find $\displaystyle\iint_{\mathcal{S}} \frac{x^2y}{\sqrt{1+x^2+y^2}}\,\mathrm{d}S.$

(b) Find the flux of $\mathbf{F} = x\hat{\imath} + y\hat{\jmath} + \hat{\mathbf{k}}$ upward through $\mathcal{S}$.

Q[19](∗): Find the area of the part of the surface $z = y^{3/2}$ that lies above $0 \leqslant x, y \leqslant 1$.

Q[20](∗): Let $\mathcal{S}$ be spherical cap which consists of the part of the sphere $x^2+y^2+(z-2)^2 = 4$ which lies under the plane $z = 1$. Let $f(x,y,z) = (2-z)(x^2+y^2)$. Calculate

$$\iint_{\mathcal{S}} f(x,y,z)\,\mathrm{d}S$$

Q[21](∗):

(a) Find a parametrization of the surface S of the cone whose vertex is at the point $(0,0,3)$, and whose base is the circle $x^2+y^2 = 4$ in the xy-plane. Only the cone surface belongs to S, not the base. Be careful to include the domain for the parameters.

(b) Find the z-coordinate of the centre of mass of the surface S from (a).

Q[22](∗): Let S be the surface of a cone of height a and base radius a. The surface S does not include the base of the cone or the interiour of the cone. Find the centre of mass of S.

Locate the cone in a coordinate system so that its base is in the xy-plane, and its vertex on the z-axis. So the vertex will be the point $(0, 0, a)$. The base is a circle of radius a in the xy-plane with centre at the origin. The cone surface is characterized by the fact that for every point of S, the distance from the z-axis and the distance from the xy-plane add up to a.

Q[23](∗): Let S be the portion of the elliptical cylinder $x^2 + \frac{1}{4}y^2 = 1$ lying between the planes $z = 0$ and $z = 1$ and let $\hat{\mathbf{n}}$ denote the outward normal to S. Let $\mathbf{F} = x\,\hat{\imath} + xyz\,\hat{\jmath} + zy^4\,\hat{\mathbf{k}}$. Calculate the flux integral $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S$ directly, using an appropriate parameterization of S.

Q[24](∗):

Evaluate the flux integral

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S$$

where $\mathbf{F}(x, y, z) = (x + 1)\,\hat{\imath} + (y + 1)\,\hat{\jmath} + 2z\,\hat{\mathbf{k}}$, and S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the triangle $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1 - x$. S is oriented so that its unit normal has a negative z-component.

Q[25](∗): Evaluate the surface integral

$$\iint_S xy^2\,\mathrm{d}S$$

where S is the part of the sphere $x^2 + y^2 + z^2 = 2$ for which $x \geqslant \sqrt{y^2 + z^2}$.

Q[26](∗): Let S be the surface given by the equation

$$x^2 + z^2 = \sin^2 y$$

lying between the planes $y = 0$ and $y = \pi$. Evaluate the integral

$$\iint_S \sqrt{1 + \cos^2 y}\,\mathrm{d}S$$

Q[27](∗): Let S be the part of the paraboloid $z = 1 - x^2 - y^2$ lying above the xy-plane. At (x, y, z) S has density

$$\rho(x, y, z) = \frac{z}{\sqrt{5 - 4z}}$$

Find the centre of mass of S.

Q[28](∗): Let S be the part of the plane

$$x + y + z = 2$$

that lies in the first octant oriented so that $\hat{\mathbf{n}}$ has a positive $\hat{\mathbf{k}}$ component. Let

$$\mathbf{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{\mathbf{k}}$$

Evaluate the flux integral

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$$

Q[29](∗): Find the net flux $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$ of the vector field $\mathbf{F}(x,y,z) = (x,y,z)$ upwards (with respect to the z-axis) through the surface S parametrized $\mathbf{r} = \big(uv^2\,,\, u^2v\,,\, uv\big)$ for $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant 3$.

Q[30](∗): Let S be the surface obtained by revolving the curve $z = e^y$, $0 \leqslant y \leqslant 1$, around the y-axis, with the orientation of S having $\hat{\mathbf{n}}$ pointing toward the y-axis.

(a) Draw a picture of S and find a parameterization of S.

(b) Compute the integral $\iint_S e^y\, \mathrm{d}S$.

(c) Compute the flux integral $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$ where $\mathbf{F} = (x,0,z)$.

Q[31](∗): Compute the net outward flux of the vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}}}{\sqrt{x^2+y^2+z^2}}$$

across the boundary of the region *between* the spheres of radius 1 and radius 2 centred at the origin.

Q[32](∗): Evaluate the surface integral $\iint_S z^2\, \mathrm{d}S$ where S is the part of the cone $x^2 + y^2 = 4z^2$ where $0 \leqslant x \leqslant y$ and $0 \leqslant z \leqslant 1$.

Q[33](∗): Compute the flux integral $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$, where

$$\mathbf{F} = \Big(-\frac{1}{2}x^3 - xy^2\,,\, -\frac{1}{2}y^3\,,\, z^2 \Big)$$

and S is the part of the paraboloid $z = 5 - x^2 - y^2$ lying inside the cylinder $x^2 + y^2 \leqslant 4$, with orientation pointing downwards.

Q[34](∗): Let the thin shell S consist of the part of the surface $z^2 = 2xy$ with $x \geqslant 1$, $y \geqslant 1$ and $z \leqslant 2$. Find the mass of S if it has surface density given by $\rho(x,y,z) = 3z$ kg per unit area.

Q[35](∗): Let S be the portion of the paraboloid $x = y^2 + z^2$ that satisfies $x \leqslant 2y$. Its unit normal vector $\hat{\mathbf{n}}$ is so chosen that $\hat{\mathbf{n}} \cdot \hat{\boldsymbol{\imath}} > 0$. Find the flux of $\mathbf{F} = 2\,\hat{\boldsymbol{\imath}} + z\,\hat{\boldsymbol{\jmath}} + y\,\hat{\mathbf{k}}$ out of S.

Q[36](∗): Let S denote the portion of the paraboloid $z = 1 - \frac{1}{4}x^2 - y^2$ for which $z \geqslant 0$. Orient S so that its unit normal has a positive $\hat{k}$ component. Let

$$\mathbf{F}(x,y,z) = (3y^2 + z)\,\hat{\boldsymbol{\imath}} + (x - x^2)\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}$$

Evaluate the surface integral $\iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$.

Q[37]: Let S be the boundary of the apple core bounded by the sphere $x^2 + y^2 + z^2 = 16$ and the hyperboloid $x^2 + y^2 - z^2 = 8$. Find the flux integral $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$ where $\mathbf{F} = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}}$ and $\hat{\mathbf{n}}$ is the outward normal to the surface S.

▸▸ Stage 3

Q[38](∗):

(a) Consider the surface S given by the equation

$$x^2 + z^2 = \cos^2 y$$

Find an equation for the tangent plane to S at the point $\left(\frac{1}{2}, \frac{\pi}{4}, \frac{1}{2}\right)$.

(b) Compute the integral

$$\iint_S \sin y \, \mathrm{d}S$$

where S is the part of the surface from (a) lying between the planes $y = 0$ and $y = \frac{1}{2}\pi$.

Q[39](∗): Let f be a function on $\mathbb{R}^3$ such that all its first order partial derivatives are continuous. Let S be the surface $\{ (x, y, z) \mid f(x, y, z) = c \}$ for some $c \in \mathbb{R}$. Assume that $\nabla f \neq \mathbf{0}$ on S. Let $\mathbf{F}$ be the gradient field $\mathbf{F} = \nabla f$.

(a) Let C be a piecewise smooth curve contained in S (not necessarily closed). Must it be true that $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$? Explain why.

(b) Prove that for any vector field $\mathbf{G}$,

$$\iint_S (\mathbf{F} \times \mathbf{G}) \cdot \hat{\mathbf{n}} \, \mathrm{d}S = 0.$$

Q[40](∗):

(a) Give parametric descriptions of the form $\mathbf{r}(u, v) = \big(x(u, v)\,,\, y(u, v)\,,\, z(u, v)\big)$ for the following surfaces. Be sure to state the domains of your parametrizations.

(i) The part of the plane $2x + 4y + 3z = 16$ in the first octant

$$\{ (x, y, z) \mid x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0 \}$$

(ii) The cap of the sphere $x^2 + y^2 + z^2 = 16$ for $4/\sqrt{2} \leqslant z \leqslant 4$.

(iii) The hyperboloid $z^2 = 1 + x^2 + y^2$ for $1 \leqslant z \leqslant 10$.

(b) Use your parametrization from part (a) to compute the surface area of the cap of the sphere $x^2 + y^2 + z^2 = 16$ for $4/\sqrt{2} \leqslant z \leqslant 4$.

Q[41](∗): Let S be the part of the sphere $x^2 + y^2 + z^2 = 2$ where $y \geqslant 1$, oriented away from the origin.

(a) Compute

$$\iint_S y^3 \, \mathrm{d}S$$

(b) Compute

$$\iint_S \left(xy\,\hat{\boldsymbol{\imath}} + xz\,\hat{\boldsymbol{\jmath}} + zy\,\hat{\mathbf{k}}\right) \cdot \hat{\mathbf{n}}\,\mathrm{d}S$$

Q[42](*): Let $\mathcal{S}$ be the part of the surface $(x+y+1)^2 + z^2 = 4$ which lies in the first octant. Find the flux of **F** downwards through $\mathcal{S}$ where

$$\mathbf{F} = xy\,\hat{\boldsymbol{\imath}} + (z - xy)\,\hat{\boldsymbol{\jmath}}$$

Chapter 4

INTEGRAL THEOREMS

4.1▲ Gradient, Divergence and Curl

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1](*): Let $\mathbf{F} = P\,\hat{\imath} + Q\,\hat{\jmath}$ be the two dimensional vector field shown below.

(a) Assuming that the vector field in the picture is a force field, the work done by the vector field on a particle moving from point A to B along the given path is:

(A) Positive

(B) Negative

(C) Zero

(D) Not enough information to determine.

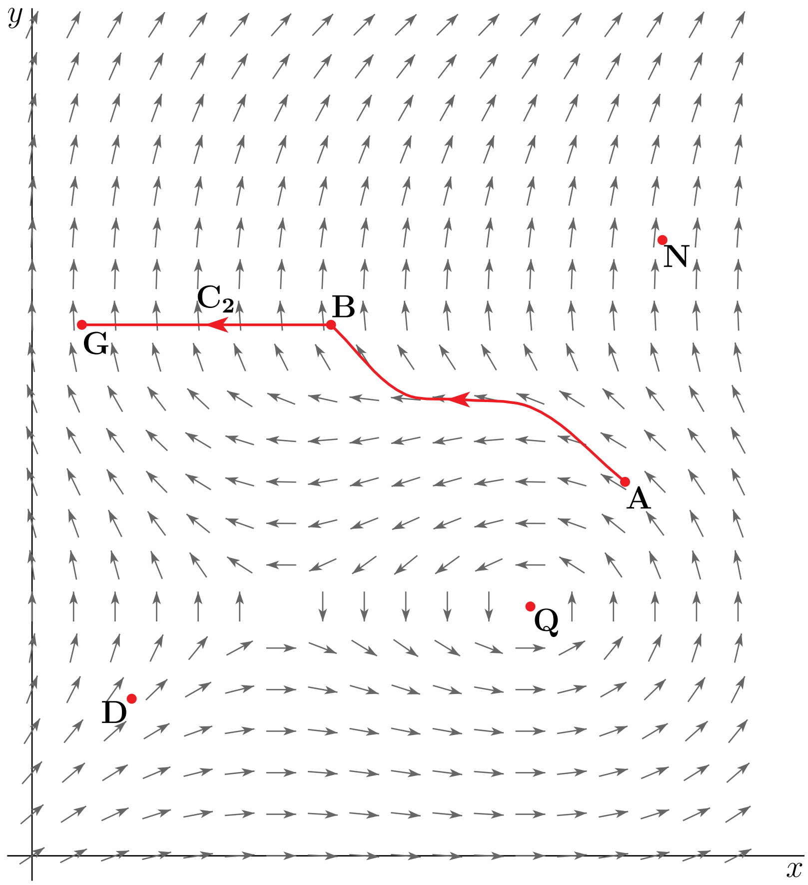
(b) Which statement is the most true about the line integral $\int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$:

(A) $\int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} > 0$

(B) $\int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$

(C) $\int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} < 0$

(D) Not enough information to determine.



(c) $\nabla \cdot \mathbf{F}$ at the point N (in the picture) is:

(A) Positive

(B) Negative

(C) Zero

(D) Not enough information to determine.

(d) $Q_x - P_y$ at the point Q is:

(A) Positive

(B) Negative

(C) Zero

(D) Not enough information to determine.

(e) Assuming that $\mathbf{F} = P\,\hat{\boldsymbol{\imath}} + Q\,\hat{\boldsymbol{\jmath}}$, which of the following statements is correct about $\frac{\partial P}{\partial x}$ at the point D?

(A) $\frac{\partial P}{\partial x} = 0$ at D.

(B) $\frac{\partial P}{\partial x} > 0$ at D.

(C) $\frac{\partial P}{\partial x} < 0$ at D.

(D) The sign of $\frac{\partial P}{\partial x}$ at D can not be determined by the given information.

Q[2]: Does $\nabla \times \mathbf{F}$ have to be perpendicular to $\mathbf{F}$?

Q[3]: Verify the vector identities

(a) $\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f$

(b) $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$

(c) $\nabla^2(fg) = f\,\nabla^2 g + 2\nabla f \cdot \nabla g + g\,\nabla^2 f$

▸▸ Stage 2

Q[4]: Evaluate $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ for each of the following vector fields.

(a) $\mathbf{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{\mathbf{k}}$

(b) $\mathbf{F} = xy^2\hat{\imath} - yz^2\hat{\jmath} + zx^2\hat{\mathbf{k}}$

(c) $\mathbf{F} = \frac{x\hat{\imath}+y\hat{\jmath}}{\sqrt{x^2+y^2}}$ (the polar basis vector $\hat{\mathbf{r}}$ in 2d)

(d) $\mathbf{F} = \frac{-y\hat{\imath}+x\hat{\jmath}}{\sqrt{x^2+y^2}}$ (the polar basis vector $\hat{\boldsymbol{\theta}}$ in 2d)

Q[5](∗):

(a) Compute and simplify $\nabla \cdot \left(\frac{\mathbf{r}}{r}\right)$ for $\mathbf{r} = (x, y, z)$ and $r = |(x, y, z)|$. Express your answer in terms of r.

(b) Compute $\nabla \times \left(yz\,\hat{\imath} + 2xz\,\hat{\jmath} + e^{xy}\,\hat{\mathbf{k}}\right)$.

Q[6](∗): In the following, we use the notation $\mathbf{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{\mathbf{k}}$, $r = |\mathbf{r}|$, and k is some number $k = 0, 1, -1, 2, -2, \ldots$.

(a) Find the value k for which

$$\nabla(r^k) = -3\frac{\mathbf{r}}{r^5}$$

(b) Find the value k for which

$$\nabla \cdot (r^k\mathbf{r}) = 5r^2$$

(c) Find the value k for which

$$\nabla^2(r^k) = \frac{2}{r^4}$$

Q[7](∗): Let $\mathbf{r}$ be the vector field $\mathbf{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{\mathbf{k}}$ and let r be the function $r = |\mathbf{r}|$. Let $\mathbf{a}$ be the *constant* vector $\mathbf{a} = a_1\,\hat{\imath} + a_2\,\hat{\jmath} + a_3\,\hat{\mathbf{k}}$. Compute and simplify the following quantities. Answers must be expressed in terms of $\mathbf{a}$, $\mathbf{r}$, and r. There should be no x's, y's, or z's in your answers.

(a) $\nabla \cdot \mathbf{r}$

(b) $\nabla(r^2)$

(c) $\nabla \times (\mathbf{r} \times \mathbf{a})$

(d) $\nabla \cdot \big(\nabla(r)\big)$

Q[8](*): Let

$$\mathbf{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{\mathbf{k}}, \qquad r = |\mathbf{r}|$$

(a) Compute a where $\nabla\left(\frac{1}{r}\right) = -r^a\,\mathbf{r}$.

(b) Compute a where $\nabla \cdot \big(r\,\mathbf{r}\big) = ar$.

(c) Compute a where $\nabla \cdot \big(\nabla(r^3)\big) = ar$.

Q[9]: Find, if possible, a vector field $\mathbf{A}$ that has $\hat{\mathbf{k}}$ component $A_3 = 0$ and that is a vector potential for

(a) $\mathbf{F} = (1 + yz)\hat{\imath} + (2y + zx)\hat{\jmath} + (3z^2 + xy)\hat{\mathbf{k}}$

(b) $\mathbf{G} = yz\hat{\imath} + zx\hat{\jmath} + xy\hat{\mathbf{k}}$

▸▸ Stage 3

Q[10](*): Let

$$\mathbf{F} = \frac{-z}{x^2 + z^2}\,\hat{\imath} + y\,\hat{\jmath} + \frac{x}{x^2 + z^2}\,\hat{\mathbf{k}}$$

(a) Determine the domain of $\mathbf{F}$.

(b) Determine the curl of $\mathbf{F}$. Simplify if possible.

(c) Determine the divergence of $\mathbf{F}$. Simplify if possible.

(d) Is $\mathbf{F}$ conservative? Give a reason for your answer.

Q[11](*): A physicist studies a vector field $\mathbf{F}$ in her lab. She knows from theoretical considerations that $\mathbf{F}$ must be of the form $\mathbf{F} = \nabla \times \mathbf{G}$, for some smooth vector field $\mathbf{G}$. Experiments also show that $\mathbf{F}$ must be of the form

$$\mathbf{F}(x, y, z) = (xz + xy)\hat{\imath} + \alpha(yz - xy)\hat{\jmath} + \beta(yz + xz)\hat{\mathbf{k}}$$

where α and β are constant.

(a) Determine α and β.

(b) Further experiments show that $\mathbf{G} = xyz\hat{\imath} - xyz\hat{\jmath} + g(x, y, z)\hat{\mathbf{k}}$. Find the unknown function $g(x, y, z)$.

Q[12]: A rigid body rotates at an angular velocity of Ω rad/sec about an axis that passes through the origin and has direction $\hat{\mathbf{a}}$. When you are standing at the head of $\hat{\mathbf{a}}$ looking towards the origin, the rotation is counterclockwise. Set $\boldsymbol{\Omega} = \Omega\hat{\mathbf{a}}$.

(a) Show that the velocity of the point $\mathbf{r} = (x, y, z)$ on the body is $\boldsymbol{\Omega} \times \mathbf{r}$.

(b) Evaluate $\nabla \times (\boldsymbol{\Omega} \times \mathbf{r})$ and $\nabla \cdot (\boldsymbol{\Omega} \times \mathbf{r})$, treating $\boldsymbol{\Omega}$ as a constant.

(c) Find the speed of the students in a classroom located at latitude 49° N due to the rotation of the Earth. Ignore the motion of the Earth about the Sun, the Sun in the Galaxy and so on. The radius of the Earth is 6378 km.

Q[13]: Suppose that the vector field $\mathbf{F}$ obeys $\nabla \cdot \mathbf{F} = 0$ in all of $\mathbb{R}^3$. Let

$$\mathbf{r}(t) = tx\,\hat{\boldsymbol{\imath}} + ty\,\hat{\boldsymbol{\jmath}} + tz\,\hat{\mathbf{k}}, \qquad 0 \leqslant t \leqslant 1$$

be a parametrization of the line segment from the origin to (x, y, z). Define

$$\mathbf{G}(x, y, z) = \int_0^1 t\,\mathbf{F}\big(\mathbf{r}(t)\big) \times \frac{d\mathbf{r}}{dt}(t)\,dt$$

Show that $\nabla \times \mathbf{G} = \mathbf{F}$ throughout $\mathbb{R}^3$.

4.2▲ The Divergence Theorem

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: Let V be the cube

$$V = \big\{\, (x, y, z) \bigm| 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1,\ 0 \leqslant z \leqslant 1 \,\big\}$$

and R be the square

$$R = \big\{\, (x, y) \bigm| 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 \,\big\}$$

and let $f(x, y, z)$ have continuous first partial derivatives.

(a) Use the fundamental theorem of calculus to show that

$$\iiint_V \frac{\partial f}{\partial z}(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_R f(x, y, 1)\,\mathrm{d}x\,\mathrm{d}y - \iint_R f(x, y, 0)\,\mathrm{d}x\,\mathrm{d}y$$

(b) Use the divergence theorem to show that

$$\iiint_V \frac{\partial f}{\partial z}(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_R f(x, y, 1)\,\mathrm{d}x\,\mathrm{d}y - \iint_R f(x, y, 0)\,\mathrm{d}x\,\mathrm{d}y$$

Q[2]:

(a) By applying the divergence theorem to $\mathbf{F} = \phi\,\mathbf{a}$, where $\mathbf{a}$ is an arbitrary constant vector, show that

$$\iiint\limits_V \boldsymbol{\nabla}\phi\,\mathrm{d}V = \iint\limits_{\partial V} \phi\,\hat{\mathbf{n}}\,\mathrm{d}S$$

(b) Show that the centroid $(\bar{x}, \bar{y}, \bar{z})$ of a solid V with volume $|V|$ is given by

$$(\bar{x}, \bar{y}, \bar{z}) = \frac{1}{2|V|}\iint\limits_{\partial V} (x^2 + y^2 + z^2)\,\hat{\mathbf{n}}\,\mathrm{d}S$$

▸▸ Stage 2

Q[3]: Let S be the unit sphere centered at the origin and oriented by the outward pointing normal. If

$$\mathbf{F}(x, y, z) = \big(x, y, z^2\big)$$

evaluate the flux of $\mathbf{F}$ through S

(a) directly and

(b) by applying the divergence theorem.

Q[4]: Evaluate, by two methods, the integral $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$, where $\mathbf{F} = z\,\hat{\mathbf{k}}$, S is the surface $x^2 + y^2 + z^2 = a^2$ and $\hat{\mathbf{n}}$ is the outward pointing unit normal to S.

(a) First, by direct computation of the surface integral.

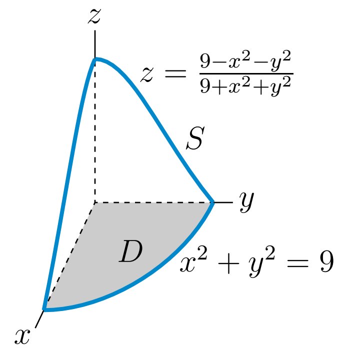
(b) Second, by using the divergence theorem.

Q[5]: Let

- $\mathbf{F} = zy^3\,\hat{\boldsymbol{\imath}} + yx\,\hat{\boldsymbol{\jmath}} + (2z + y^2)\hat{\mathbf{k}}$ and
- V be the solid in 3-space defined by

$$0 \leqslant z \leqslant \frac{9 - x^2 - y^2}{9 + x^2 + y^2}$$

  and
- D be the bottom surface of V. Because $\frac{9-x^2-y^2}{9+x^2+y^2}$ is positive for $x^2 + y^2 < 9$ and negative for $x^2 + y^2 > 9$, the bottom surface is $z = 0$, $x^2 + y^2 \leqslant 9$.
- Let S be the curved portion of the boundary of V. It is $z = \frac{9-x^2-y^2}{9+x^2+y^2}$, $x^2 + y^2 \leqslant 9$.



Denote by $|V|$ the volume of V and compute, in terms of $|V|$,

(a) $\displaystyle\iint_D \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$ with $\hat{\mathbf{n}}$ pointing downward

(b) $\displaystyle\iiint_V \boldsymbol{\nabla} \cdot F\, \mathrm{d}V$

(c) $\displaystyle\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$ with $\hat{\mathbf{n}}$ pointing outward

Use the divergence theorem to answer at least one of parts (a), (b) and (c).

Q[6]: Evaluate the integral $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$, where $\mathbf{F} = (x, y, 1)$ and S is the surface $z = 1 - x^2 - y^2$, for $x^2 + y^2 \leqslant 1$, by two methods.

(a) First, by direct computation of the surface integral.

(b) Second, by using the divergence theorem.

Q[7](∗):

(a) Find the divergence of the vector field $\mathbf{F} = (z + \sin y, zy, \sin x \cos y)$.

(b) Find the flux of the vector field $\mathbf{F}$ of (a) through the sphere of radius 3 centred at the origin in $\mathbb{R}^3$.

Q[8]: The sides of a grain silo are described by the portion of the cylinder $x^2 + y^2 = 1$ with $0 \leqslant z \leqslant 1$. The top of the silo is given by the portion of the sphere $x^2 + y^2 + z^2 = 2$ lying within the cylinder and above the xy-plane. Find the flux of the vector field

$$\mathbf{V}(x, y, z) = (x^2yz\,,\ yz + e^xz\,,\ x^2 + y)$$

out of the silo.

Q[9]: Let B be the ball of volume V centered at the point (x_0, y_0, z_0), and let S be the sphere that is the boundary of B. Find the flux of $\mathbf{F} = x^2\hat{\boldsymbol{\imath}} + xy\hat{\boldsymbol{\jmath}} + (3z - yz)\hat{\mathbf{k}}$ outward (from B) through S.

Q[10](∗): Let

$$\mathbf{F}(x, y, z) = \left(1 + z^{1+z^{1+z}}\,,\ 1 + z^{1+z^{1+z}}\,,\ 1\right)$$

Let S be the portion of the surface

$$x^2 + y^2 = 1 - z^4$$

which is above the xy-plane. What is the flux of $\mathbf{F}$ downward through S?

Q[11](*): Use the divergence theorem to find the flux of $x\hat{\imath} + y\hat{\jmath} + 2z\hat{\mathbf{k}}$ through the part of the ellipsoid

$$x^2 + y^2 + 2z^2 = 2$$

with $z \geqslant 0$. [Note: the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ has volume $\frac{4}{3}\pi abc$.]

Q[12](*): Let $\mathbf{F}(x, y, z) = \mathbf{r}/r^3$ where $\mathbf{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{\mathbf{k}}$ and $r = |\mathbf{r}|$.

(a) Find $\boldsymbol{\nabla} \cdot \mathbf{F}$.

(b) Find the flux of $\mathbf{F}$ outwards through the spherical surface $x^2 + y^2 + z^2 = a^2$.

(c) Do the results of (a) and (b) contradict the divergence theorem? Explain your answer.

(d) Let E be the solid region bounded by the surfaces $z^2 - x^2 - y^2 + 1 = 0$, $z = 1$ and $z = -1$. Let σ be the bounding surface of E. Determine the flux of $\mathbf{F}$ outwards through σ.

(e) Let R be the solid region bounded by the surfaces $z^2 - x^2 - y^2 + 4y - 3 = 0$, $z = 1$ and $z = -1$. Let Σ be the bounding surface of R. Determine the flux of $\mathbf{F}$ outwards through Σ.

Q[13](*): Consider the ellipsoid S given by

$$x^2 + \frac{y^2}{4} + \frac{z^2}{4} = 1$$

with the unit normal pointing outward.

(a) Parameterize S.

(b) Compute the flux $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$ of the vector field

$$\mathbf{F}(x, y, z) = (x, y, z)$$

(c) Verify your answer in (b) using the divergence theorem.

Q[14](*): Evaluate the flux integral $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$, where

$$\mathbf{F}(x, y, z) = \big(x^3 + \cos(y^2)\,,\, y^3 + ze^x\,,\, z^2 + \arctan(xy)\big)$$

and S is the surface of the solid region bounded by the cylinder $x^2 + y^2 = 2$ and the planes $z = 0$ and $z = 2x + 3$. The surface is positively oriented (its unit normal points outward).

Q[15](*): Find the flux of the vector field $(x + y, x + z, y + z)$ through the cylindrical surface whose equation is $x^2 + z^2 = 4$, and which extends from $y = 0$ to $y = 3$. (Only the curved part of the cylinder is included, not the two disks bounding it on the left and right.) The orientation of the surface is outward, i.e., pointing away from the y-axis.

Q[16](∗): The surface S is the part above the xy-plane of the surface obtained by revolving the graph of $z = 1 - x^4$ around the z-axis. The surface S is oriented such that the normal vector has positive z-component. The circle with radius 1 and centre at the origin in the xy-plane is the boundary of S.

Find the flux of the divergenceless vector field $\mathbf{F}(x, y, z) = (yz, x + z, x^2 + y^2)$ through S.

Q[17](∗):

Let S be the part of the paraboloid $z = 2 - x^2 - y^2$ contained in the cone $z = \sqrt{x^2 + y^2}$ and oriented in the upward direction. Let

$$\mathbf{F} = (\tan\sqrt{z} + \sin(y^3))\,\hat{\boldsymbol{\imath}} + e^{-x^2}\,\hat{\boldsymbol{\jmath}} + z\hat{\mathbf{k}}$$

Evaluate the flux integral $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S$.

Q[18](∗): Evaluate the surface integral

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S$$

where $\mathbf{F}(x, y, z) = \big(\cos z + xy^2\,,\, xe^{-z}\,,\, \sin y + x^2 z\big)$ and S is the boundary of the solid region enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$, with outward pointing normal.

Q[19](∗): Let S be the part of the sphere $x^2 + y^2 + z^2 = 4$ between the planes $z = 1$ and $z = 0$ oriented away from the origin. Let

$$\mathbf{F} = (e^y + xz)\,\hat{\boldsymbol{\imath}} + (zy + \tan(x))\,\hat{\boldsymbol{\jmath}} + (z^2 - 1)\,\hat{\mathbf{k}}$$

Compute the flux integral

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S.$$

Q[20](∗): Let B be the solid region lying between the planes $x = -1$, $x = 1$, $y = 0$, $y = 2$ and bounded below by the plane $z = 0$ and above by the plane $z + y = 3$. Let S be the surface of B. Find the flux of the vector field

$$\mathbf{F}(x, y, z) = \big(x^2 z + \cos\pi y\big)\,\hat{\boldsymbol{\imath}} + \big(yz + \sin\pi z\big)\,\hat{\boldsymbol{\jmath}} + (x - y^2)\,\hat{\mathbf{k}}$$

Q[21](∗): Let S be the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geqslant 0$, oriented with $\hat{\mathbf{n}}$ pointing away from the origin. Evaluate the flux integral

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S$$

where

$$\mathbf{F} = \big(x + \cos(z^2)\big)\,\hat{\boldsymbol{\imath}} + \big(y + \ln(x^2 + z^5)\big)\,\hat{\boldsymbol{\jmath}} + \sqrt{x^2 + y^2}\,\hat{\mathbf{k}}$$

Q[22](∗): Let E be the solid region between the plane $z = 4$ and the paraboloid $z = x^2 + y^2$. Let

$$\mathbf{F} = \Big(-\frac{1}{3}x^3 + e^{z^2}\Big)\hat{\imath} + \Big(-\frac{1}{3}y^3 + x\tan z\Big)\hat{\jmath} + 4z\hat{\mathbf{k}}$$

(a) Compute the flux of $\mathbf{F}$ outward through the boundary of E.

(b) Let S be the part of the paraboloid $z = x^2 + y^2$ lying below the $z = 4$ plane oriented so that $\hat{\mathbf{n}}$ has a positive $\hat{\mathbf{k}}$ component. Compute the flux of $\mathbf{F}$ through S.

Q[23](∗): Consider the vector field

$$\mathbf{F}(x, y, z) = \frac{x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{\mathbf{k}}}{\left[x^2 + y^2 + z^2\right]^{3/2}}$$

(a) Compute $\nabla \cdot \mathbf{F}$.

(b) Let S_1 be the sphere given by

$$x^2 + (y-2)^2 + z^2 = 9$$

oriented outwards. Compute $\iint_{S_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$.

(c) Let S_2 be the sphere given by

$$x^2 + (y-2)^2 + z^2 = 1$$

oriented outwards. Compute $\iint_{S_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$.

(d) Are your answers to (b) and (c) the same or different? Give a mathematical explanation of your answer.

Q[24](∗): Let $\mathbf{F}$ be the vector field defined by

$$\mathbf{F}(x, y, z) = \left(y^3 z + 2x\right)\hat{\imath} + \left(3y - e^{\sin z}\right)\hat{\jmath} + \left(e^{x^2+y^2} + z\right)\hat{\mathbf{k}}$$

Calculate the flux integral $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$ where S is the boundary surface of the solid region

$$E\ :\ 0 \leqslant x \leqslant 2, \quad 0 \leqslant y \leqslant 2, \quad 0 \leqslant z \leqslant 2 + y$$

with outer normal.

Q[25](∗): Consider the vector field

$$\mathbf{F}(x, y, z) = \left(z \arctan(y^2)\,,\ z^3 \ln(x^2+1)\,,\ 3z\right)$$

Let the surface S be the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the plane $z = 1$ and be oriented downwards.

(a) Find the divergence of $\mathbf{F}$.

(b) Compute the flux integral $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$.

Q[26](*): Let S be the sphere $x^2+y^2+z^2=3$ oriented inward. Compute the flux integral

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$$

where

$$\mathbf{F}=\big(xy^2+y^4z^6\,,\,yz^2+x^4z\,,\,zx^2+xy^4\big)$$

Q[27](*): Consider the vector field $\mathbf{F}(x,y,z)=-2xy\,\hat{\imath}+\big(y^2+\sin(xz)\big)\,\hat{\jmath}+(x^2+y^2)\,\hat{\mathbf{k}}$.

(a) Calculate $\nabla\cdot\mathbf{F}$.

(b) Find the flux of $\mathbf{F}$ through the surface S defined by

$$x^2+y^2+(z-12)^2=13^2,\ z\geqslant 0$$

using the outward normal to S.

Q[28](*): Let S be the portion of the hyperboloid $x^2+y^2-z^2=1$ between $z=-1$ and $z=1$. Find the flux of $\mathbf{F}=(x+e^{yz})\,\hat{\imath}+\big(2yz+\sin(xz)\big)\,\hat{\jmath}+(xy-z-z^2)\,\hat{\mathbf{k}}$ out of S (away from the origin).

Q[29](*): Let $\mathbf{F}$ be the vector field $\mathbf{F}(x,y,z)=(x^2-y-1)\,\hat{\imath}+(e^{\cos y}+z^3)\,\hat{\jmath}+(2xz+z^5)\,\hat{\mathbf{k}}$. Evaluate $\iint_S \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ where S is the part of the ellipsoid $x^2+y^2+2z^2=1$ with $z\geqslant 0$.

Q[30](*): Let S be the portion of the sphere $x^2+y^2+(z-1)^2=4$ that lies above the xy-plane. Find the flux of $\mathbf{F}=(x^2+e^{y^2})\,\hat{\imath}+(e^{x^2}+y^2)\,\hat{\jmath}+(4+5x)\,\hat{\mathbf{k}}$ outward across S.

Q[31](*): Find the flux of $\mathbf{F}=xy^2\hat{\imath}+x^2y\hat{\jmath}+\hat{\mathbf{k}}$ outward through the hemispherical surface

$$x^2+y^2+z^2=4,\qquad z\geqslant 0$$

Q[32](*): Let D be the cylinder $x^2+y^2\leqslant 1$, $0\leqslant z\leqslant 5$. Calculate the flux of the vector field

$$\mathbf{F}=(x+xye^z)\,\hat{\imath}+\tfrac{1}{2}y^2ze^z\,\hat{\jmath}+(3z-yze^z)\,\hat{\mathbf{k}}$$

outward through the curved part of the surface of D.

Q[33]: Find the flux of $\mathbf{F}=(y+xz)\hat{\imath}+(y+yz)\hat{\jmath}-(2x+z^2)\hat{\mathbf{k}}$ upward through the first octant part of the sphere $x^2+y^2+z^2=a^2$.

Q[34]: Let $\mathbf{F}=(x-yz)\hat{\imath}+(y+xz)\hat{\jmath}+(z+2xy)\hat{\mathbf{k}}$ and let

- S_1 be the portion of the cylinder $x^2+y^2=2$ that lies inside the sphere $x^2+y^2+z^2=4$
- S_2 be the portion of the sphere $x^2+y^2+z^2=4$ that lies outside the cylinder $x^2+y^2=2$
- V be the solid bounded by S_1 and S_2

Compute

(a) $\iint_{S_1} \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$ with $\hat{\mathbf{n}}$ pointing inward

(b) $\iiint_V \boldsymbol{\nabla} \cdot F \, \mathrm{d}V$

(c) $\iint_{S_2} \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$ with $\hat{\mathbf{n}}$ pointing outward

Use the divergence theorem to answer at least one of parts (a), (b) and (c).

▸▸ Stage 3

Q[35]: Let $\mathbf{E}(\mathbf{r})$ be the electric field due to a charge configuration that has density $\rho(\mathbf{r})$. Gauss' law states that, if V is any solid in $\mathbb{R}^3$ with surface ∂V, then the electric flux

$$\iint\limits_{\partial V} \mathbf{E} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = 4\pi Q \qquad \text{where} \qquad Q = \iiint\limits_{V} \rho \, \mathrm{d}V$$

is the total charge in V. Here, as usual, $\hat{\mathbf{n}}$ is the outward pointing unit normal to ∂V. Show that

$$\boldsymbol{\nabla} \cdot \mathbf{E}(\mathbf{r}) = 4\pi\rho(\mathbf{r})$$

for all $\mathbf{r}$ in $\mathbb{R}^3$. This is one of Maxwell's equations. Assume that $\boldsymbol{\nabla} \cdot \mathbf{E}(\mathbf{r})$ and $\rho(\mathbf{r})$ are well–defined and continuous everywhere.

Q[36]: Let V be a solid in $\mathbb{R}^3$ with surface ∂V. Show that

$$\iint\limits_{\partial V} \mathbf{r} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = 3 \, \mathrm{Volume}(V)$$

where $\mathbf{r} = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}}$ and, as usual, $\hat{\mathbf{n}}$ is the outer normal to ∂V. See if you can explain this result geometrically.

Q[37](∗): Let S be the sphere of radius 3, centered at the origin and with outward orientation. Given the vector field $\mathbf{F}(x, y, z) = (0, 0, x + z)$:

(a) Calculate (using the definition) the flux of $\mathbf{F}$ through S

$$\iint\limits_{S} \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$$

That is, compute the flux by evaluating the surface integral directly.

(b) Calculate the same flux using the divergence theorem.

Q[38](∗): Consider the cube of side length 1 that lies entirely in the first octant ($x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$) with one corner at the origin and another corner at point $(1, 1, 1)$. As such, one face lies in the plane $x = 0$, one lies in the plane $y = 0$, and another lies in the plane $z = 0$. The other three faces lie in the planes $x = 1$, $y = 1$, and $z = 1$. Denote S as the open surface that consists of the union of the 5 faces of the cube that do not lie in the plane $z = 0$. The surface S is oriented in such a way that the unit normal vectors point outwards (that is, the orientation of S is such that the unit normal vectors on the top face point towards positive z-directions). Determine the value of

$$I = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$$

where $\mathbf{F}$ is the vector field given by

$$\mathbf{F} = \left(y \cos(y^2) + z - 1 \,,\, \frac{z}{x+1} + 1 \,,\, xye^{z^2} \right)$$

Q[39](*):

(a) Find an upward pointing unit normal vector to the surface $z = xy$ at the point $(1, 1, 1)$.

(b) Now consider the part of the surface $z = xy$, which lies within the cylinder $x^2 + y^2 = 9$ and call it S. Compute the upward flux of $\mathbf{F} = (y, x, 3)$ through S.

(c) Find the flux of $\mathbf{F} = (y, x, 3)$ through the cylindrical surface $x^2 + y^2 = 9$ in between $z = xy$ and $z = 10$. The orientation is outward, away from the z-axis.

Q[40](*):

(a) Find the divergence of the vector field $\mathbf{F} = (x + \sin y, z + y, z^2)$.

(b) Find the flux of $\mathbf{F}$ through the upper hemisphere $x^2 + y^2 + z^2 = 25$, $z \geqslant 0$, oriented in the positive z-direction.

(c) Specify an oriented closed surface S, such that the flux $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$ is equal to -9.

Q[41](*): Evaluate the surface integrals. (Use any method you like.)

(a) $\iint_S z^2 \, \mathrm{d}S$, if S is the part of the cone $x^2 + y^2 = 4z^2$ where $0 \leqslant x \leqslant y$ and $0 \leqslant z \leqslant 1$.

(b) $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$, if $\mathbf{F} = z\hat{\mathbf{k}}$ and S is the rectangle with vertices $(0, 2, 0)$, $(0, 0, 4)$, $(5, 2, 0)$, $(5, 0, 4)$, oriented so that the normal vector points upward.

(c) $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$, where $\mathbf{F} = (y - z^2)\hat{\imath} + (z - x^2)\hat{\jmath} + z^2\hat{\mathbf{k}}$ and S is the boundary surface of the box $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 2$, $0 \leqslant z \leqslant 3$, with the normal vector pointing outward.

Q[42](*): Let σ_1 be the open surface given by $z = 1 - x^2 - y^2$, $z \geqslant 0$. Let σ_2 be the open surface given by $z = x^2 + y^2 - 1$, $z \leqslant 0$. Let σ_3 be the planar surface given by $z = 0$, $x^2 + y^2 \leqslant 1$. Let $\mathbf{F} = [a(y^2 + z^2) + bxz] \, \hat{\imath} + [c(x^2 + z^2) + dyz] \, \hat{\jmath} + x^2 \, \hat{\mathbf{k}}$ where a, b, c, and d are constants.

(a) Find the flux of $\mathbf{F}$ upwards across σ_1.

(b) Find all values of the constants a, b, c, and d so that the flux of $\mathbf{F}$ outwards across the closed surface $\sigma_1 \cup \sigma_3$ is zero.

(c) Find all values of the constants a, b, c, and d so that the flux of $\mathbf{F}$ outwards across the closed surface $\sigma_1 \cup \sigma_2$ is zero.

Q[43](*): Let S be the ellipsoid $x^2 + 2y^2 + 3z^2 = 16$ and $\hat{\mathbf{n}}$ its outward unit normal.

(a) Find $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ if $\mathbf{F}(x,y,z) = \dfrac{(x,y,z)-(2,1,1)}{\big[(x-2)^2+(y-1)^2+(z-1)^2\big]^{3/2}}$.

(b) Find $\iint_S \mathbf{G}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ if $\mathbf{G}(x,y,z) = \dfrac{(x,y,z)-(3,2,2)}{\big[(x-3)^2+(y-2)^2+(z-2)^2\big]^{3/2}}$.

Q[44](*): Let $\Omega\subset\mathbb{R}^3$ be a smoothly bounded domain, with boundary $\partial\Omega$ and outer unit normal $\hat{\mathbf{n}}$. Prove that for any vector field $\mathbf{F}$ which is continuously differentiable in $\Omega\cup\partial\Omega$,

$$\iiint_{\Omega} \boldsymbol{\nabla}\times\mathbf{F}\,\mathrm{d}V = -\iint_{\partial\Omega} \mathbf{F}\times\hat{\mathbf{n}}\,\mathrm{d}S$$

Q[45](*): Recall that if S is a smooth closed surface with outer normal field $\hat{\mathbf{n}}$, then for any smooth function $p(x,y,z)$ on $\mathbb{R}^3$, we have

$$\iint_S p\hat{\mathbf{n}}\ \mathrm{d}s = \iiint_E \nabla p\ \mathrm{d}V$$

where E is the solid bounded by S. Show that as a consequence, the total force exerted on the surface of a solid body contained in a gas of constant pressure is zero. (Recall that the pressure acts in the direction normal to the surface.)

Q[46](*): Let $\mathbf{F}$ be a smooth 3-dimensional vector field such that the flux of $\mathbf{F}$ out of the sphere $x^2+y^2+z^2=a^2$ is equal to $\pi(a^3+2a^4)$ for every $a>0$. Calculate $\boldsymbol{\nabla}\cdot\mathbf{F}(0,0,0)$.

Q[47](*): Let $\mathbf{F}=(x^2+y^2+z^2)\,\hat{\boldsymbol{\imath}}+(e^{x^2}+y^2)\,\hat{\boldsymbol{\jmath}}+(3+x+z)\,\hat{\mathbf{k}}$ and let S be the part of the surface $x^2+y^2+z^2=2az+3a^2$ having $z\geqslant 0$, oriented with normal pointing away from the origin. Here $a>0$ is a constant. Compute the flux of $\mathbf{F}$ through S.

Q[48](*): Let $u=u(x,y,z)$ be a solution of Laplace's Equation,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0,$$

in $\mathbb{R}^3$. Let $\mathcal{R}$ be a smooth solid in $\mathbb{R}^3$.

(a) Prove that the total flux of $\mathbf{F}=\nabla u$ out through the boundary of $\mathcal{R}$ is zero.

(b) Prove that the total flux of $\mathbf{G}=u\nabla u$ out through the boundary of $\mathcal{R}$ equals

$$\iiint_{\mathcal{R}} \Big[\Big(\frac{\partial u}{\partial x}\Big)^2+\Big(\frac{\partial u}{\partial y}\Big)^2+\Big(\frac{\partial u}{\partial z}\Big)^2\Big]\,\mathrm{d}V$$

Q[49](*): Let $\mathcal{R}$ be the part of the solid cylinder $x^2+(y-1)^2\leqslant 1$ satisfying $0\leqslant z\leqslant y^2$; let $\mathcal{S}$ be the boundary of $\mathcal{R}$. Given $\mathbf{F}=x^2\,\hat{\boldsymbol{\imath}}+2y\,\hat{\boldsymbol{\jmath}}-2z\,\hat{\mathbf{k}}$,

(a) Find the total flux of $\mathbf{F}$ outward through $\mathcal{S}$.

(b) Find the total flux of $\mathbf{F}$ outward through the (vertical) cylindrical sides of $\mathcal{S}$.
Hint: $\displaystyle\int_0^\pi \sin^n\theta\,\mathrm{d}\theta = \frac{n-1}{n}\int_0^\pi \sin^{n-2}\theta\,\mathrm{d}\theta$ for $n=2,3,4,\ldots$.

Q[50](*): A smooth surface $\mathcal{S}$ lies above the plane $z = 0$ and has as its boundary the circle $x^2 + y^2 = 4y$ in the plane $z = 0$. This circle also bounds a disk D in that plane. The volume of the 3-dimensional region R bounded by $\mathcal{S}$ and D is 10 cubic units. Find the flux of

$$\mathbf{F}(x, y, z) = (x + x^2y)\hat{\imath} + (y - xy^2)\hat{\jmath} + (z + 2x + 3y)\hat{\mathbf{k}}$$

through $\mathcal{S}$ in the direction outward from R.

4.3▲ Green's Theorem

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: Let R be the square

$$R = \left\{ (x, y) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 \right\}$$

and let $f(x, y)$ have continuous first partial derivatives.

(a) Use the fundamental theorem of calculus to show that

$$\iint_R \frac{\partial f}{\partial y}(x, y)\, \mathrm{d}x\, \mathrm{d}y = \int_0^1 f(x, 1)\, \mathrm{d}x - \int_0^1 f(x, 0)\, \mathrm{d}x$$

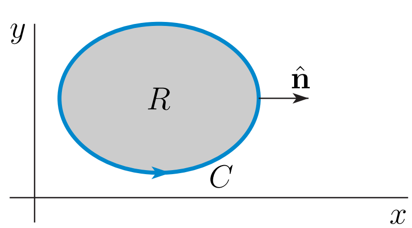
(b) Use Green's theorem to show that

$$\iint_R \frac{\partial f}{\partial y}(x, y)\, \mathrm{d}x\, \mathrm{d}y = \int_0^1 f(x, 1)\, \mathrm{d}x - \int_0^1 f(x, 0)\, \mathrm{d}x$$

Q[2]: Let R be a finite region in the xy-plane, whose boundary, C, consists of a single, piecewise smooth, simple closed curve that is oriented couterclockwise. "Simple" means that the curve does not intersect itself. Use Green's theorem to show that

$$\iint_R \nabla \cdot \mathbf{F}\, \mathrm{d}x\, \mathrm{d}y = \oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}s$$

where $\mathbf{F} = F_1\,\hat{\imath} + F_2\,\hat{\jmath}$, $\hat{\mathbf{n}}$ is the outward unit normal to C and s is the arclength along C.



So, by Green's theorem,

$$\oint_C \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}s = \oint_C \left[F_1\frac{\mathrm{d}y}{\mathrm{d}s} - F_2\frac{\mathrm{d}x}{\mathrm{d}s}\right]\,\mathrm{d}s = \oint_C [-F_2\,\mathrm{d}x + F_1\,\mathrm{d}y] = \iint_R \left[\frac{\partial}{\partial x}F_1 - \frac{\partial}{\partial y}(-F_2)\right]\,\mathrm{d}x\,\mathrm{d}y$$
$$= \iint_R \nabla\cdot\mathbf{F}\,\mathrm{d}x\,\mathrm{d}y$$

Q[3]: Integrate $\dfrac{1}{2\pi}\displaystyle\oint_C \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2}$ counterclockwise around

(a) the circle $x^2+y^2=a^2$

(b) the boundary of the square with vertices $(-1,-1)$, $(-1,1)$, $(1,1)$ and $(1,-1)$

(c) the boundary of the region $1 \leqslant x^2+y^2 \leqslant 2,\ y \geqslant 0$

Q[4]: Show that

$$\frac{\partial}{\partial x}\Big(\frac{x}{x^2+y^2}\Big) = \frac{\partial}{\partial y}\Big(\frac{-y}{x^2+y^2}\Big)$$

for all $(x,y)\neq(0,0)$. Discuss the connection between this result and the results of Q[3].

▸▸ Stage 2

Q[5]: Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = x^2y^2\,\hat{\imath} + 2xy\,\hat{\jmath}$ and C is the boundary of the square in the xy-plane having one vertex at the origin and diagonally opposite vertex at the point $(3,3)$, oriented counterclockwise.

Q[6]: Evaluate $\displaystyle\oint_C (x\sin y^2 - y^2)\,\mathrm{d}x + (x^2y\cos y^2 + 3x)\,\mathrm{d}y$ where C is the counterclockwise boundary of the trapezoid with vertices $(0,-2)$, $(1,-1)$, $(1,1)$ and $(0,2)$.

Q[7](*): Evaluate $I = \displaystyle\oint_{\mathcal{C}} \Big(\frac{1}{3}x^2y^3 - x^4y\Big)\,\mathrm{d}x + \big(xy^4 + x^3y^2\big)\,\mathrm{d}y$ counterclockwise around the boundary of the half-disk $0 \leqslant y \leqslant \sqrt{4-x^2}$.

Q[8](*): Let $\mathcal{C}$ be the counterclockwise boundary of the rectangle with vertices $(1,0)$, $(3,0)$, $(3,1)$ and $(1,1)$. Evaluate

$$\oint_{\mathcal{C}} \big(3y^2 + 2xe^{y^2}\big)\,\mathrm{d}x + \big(2yx^2e^{y^2}\big)\,\mathrm{d}y$$

Q[9](*): Consider the closed region enclosed by the curves $y = x^2+4x+4$ and $y = 4-x^2$. Let C be its boundary and suppose that C is oriented counter-clockwise.

(a) Draw the *oriented* curve C carefully in the xy-plane.

(b) Determine the value of

$$\oint_C xy\,\mathrm{d}x + (e^y + x^2)\mathrm{d}y$$

Q[10](*): Let

$$\mathbf{F}(x,y) = \big(y^2 - e^{-y^2} + \sin x\,,\, 2xye^{-y^2} + x\big)$$

Let C be the boundary of the triangle with vertices $(0,0)$, $(1,0)$ and $(1,2)$, oriented counter-clockwise. Compute

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}$$

Q[11](*): Suppose the curve C is the boundary of the region enclosed between the curves $y = x^2 - 4x + 3$ and $y = 3 - x^2 + 2x$. Determine the value of the line integral

$$\int_C \big(2xe^y + \sqrt{2+x^2}\big)\,\mathrm{d}x + x^2(2+e^y)\,\mathrm{d}y$$

where C is traversed counter-clockwise.

Q[12](*): Let

$$\mathbf{F}(x,y) = \big(\tfrac{3}{2}y^2 + e^{-y} + \sin x\big)\,\hat{\boldsymbol{\imath}} + \big(\tfrac{1}{2}x^2 + x - xe^{-y}\big)\,\hat{\boldsymbol{\jmath}}$$

Find $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$, where C is the boundary of the triangle $(0,0)$, $(1,-2)$, $(1,2)$, oriented anticlockwise.

Q[13](*):

(a) Use Green's theorem to evaluate the line integral

$$\int_C \frac{-y}{x^2+y^2}\mathrm{d}x + \frac{x}{x^2+y^2}\mathrm{d}y$$

where C is the arc of the parabola $y = \frac{1}{4}x^2 + 1$ from $(-2,2)$ to $(2,2)$.

(b) Use Green's theorem to evaluate the line integral

$$\int_C \frac{-y}{x^2+y^2}\mathrm{d}x + \frac{x}{x^2+y^2}\mathrm{d}y$$

where C is the arc of the parabola $y = x^2 - 2$ from $(-2,2)$ to $(2,2)$.

(c) Is the vector field

$$\mathbf{F} = \frac{-y}{x^2+y^2}\hat{\boldsymbol{\imath}} + \frac{x}{x^2+y^2}\hat{\boldsymbol{\jmath}}$$

conservative? Provide a reason for your answer based on your answers to the previous parts of this question.

Q[14](*): Suppose the curve C is the boundary of the region enclosed between the curves $y = x^2 - 4x + 3$ and $y = 3 - x^2 + 2x$. Determine the value of the line integral

$$\int_C \big(2xe^y + \sqrt{2} + x^2\big)\mathrm{d}x + x^2\big(2 + e^y\big)\mathrm{d}y$$

where C is traversed counter-clockwise.

Q[15](*): Let $\mathbf{F}(x,y) = P\,\hat{\imath} + Q\,\hat{\jmath}$ be a smooth plane vector field defined for $(x,y) \neq (0,0)$, and suppose $Q_x = P_y$ for $(x,y) \neq (0,0)$. In the following $I_j = \int_{C_j} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ for integer j, and all C_j are positively oriented circles. Suppose $I_1 = \pi$ where C_1 is the circle $x^2 + y^2 = 1$.

(a) Find I_2 for $C_2 : (x-2)^2 + y^2 = 1$. Explain briefly.

(b) Find I_3 for $C_3 : (x-2)^2 + y^2 = 9$. Explain briefly.

(c) Find I_4 for $C_4 : (x-2)^2 + (y-2)^2 = 9$. Explain briefly.

Q[16](*): Consider the vector field $\mathbf{F} = P\,\hat{\imath} + Q\,\hat{\jmath}$, where

$$P = \frac{x+y}{x^2+y^2}, \qquad Q = \frac{y-x}{x^2+y^2}$$

(a) Compute and simplify $Q_x - P_y$.

(b) Compute the integral $\int_{C_R} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ directly using a parameterization, where C_R is the circle of radius R, centered at the origin, and oriented in the counterclockwise direction.

(c) Is $\mathbf{F}$ conservative? Carefully explain how your answer fits with the results you got in the first two parts.

(d) Use Green's theorem to compute $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where C is the triangle with vertices $(1,1)$, $(1,0)$, $(0,1)$ oriented in the counterclockwise direction.

(e) Use Green's theorem to compute $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where C is the triangle with vertices $(-1,-1)$, $(1,0)$, $(0,1)$ oriented in the counterclockwise direction.

Q[17](*):

(a) Evaluate

$$\int_C \sqrt{1+x^3}\,\mathrm{d}x + \big(2xy^2 + y^2\big)\,\mathrm{d}y$$

where C is the unit circle $x^2 + y^2 = 1$, oriented counterclockwise.

(b) Evaluate

$$\int_C \sqrt{1+x^3}\,\mathrm{d}x + \big(2xy^2 + y^2\big)\,\mathrm{d}y$$

where C is now the part of the unit circle $x^2 + y^2 = 1$, with $x \geqslant 0$, still oriented counterclockwise.

▸▸ Stage 3

Q[18](∗): Evaluate the line integral

$$\int_C (x^2 + ye^x)\,\mathrm{d}x + (x\cos y + e^x)\,\mathrm{d}y$$

where C is the arc of the curve $x = \cos y$ for $-\pi/2 \leqslant y \leqslant \pi/2$, traversed in the direction of increasing y.

Q[19](∗): Use Green's theorem to establish that if C is a simple closed curve in the plane, then the area A enclosed by C is given by

$$A = \frac{1}{2}\oint_C x\,\mathrm{d}y - y\,\mathrm{d}x$$

Use this to calculate the area inside the curve $x^{2/3} + y^{2/3} = 1$.

Q[20](∗): Let $\mathbf{F}(x,y) = (x + 3y)\,\hat{\boldsymbol{\imath}} + (x + y)\,\hat{\boldsymbol{\jmath}}$ and $\mathbf{G}(x,y) = (x + y)\,\hat{\boldsymbol{\imath}} + (2x - 3y)\,\hat{\boldsymbol{\jmath}}$ be vector fields. Find a number A such that for each circle C in the plane

$$\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = A\oint_C \mathbf{G}\cdot\mathrm{d}\mathbf{r}$$

Q[21](∗): Let $\mathbf{F}(x,y) = \frac{y^3}{(x^2+y^2)^2}\hat{\boldsymbol{\imath}} - \frac{xy^2}{(x^2+y^2)^2}\hat{\boldsymbol{\jmath}}$, $(x,y) \neq (0,0)$.

(a) Compute $\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ where C is the unit circle in the xy-plane, positively oriented.

(b) Use (a) and Green's theorem to find $\oint_{C_0} \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ where C_0 is the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$, positively oriented.

Q[22](∗): Let $\mathcal{C}_1$ be the circle $(x-2)^2 + y^2 = 1$ and let $\mathcal{C}_2$ be the circle $(x-2)^2 + y^2 = 9$. Let $\mathbf{F} = -\frac{y}{x^2+y^2}\,\hat{\boldsymbol{\imath}} + \frac{x}{x^2+y^2}\,\hat{\boldsymbol{\jmath}}$. Find the integrals $\oint_{\mathcal{C}_1} \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ and $\oint_{\mathcal{C}_2} \mathbf{F}\cdot\mathrm{d}\mathbf{r}$.

Q[23](∗): Let R be the region in the first quadrant of the xy-plane bounded by the coordinate axes and the curve $y = 1 - x^2$. Let $\mathcal{C}$ be the boundary of R, oriented counterclockwise.

(a) Evaluate $\int_{\mathcal{C}} x\,\mathrm{d}s$.

(b) Evaluate $\int_{\mathcal{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r}$, where $\mathbf{F}(x,y) = \big(\sin(x^2) - xy\big)\,\hat{\boldsymbol{\imath}} + \big(x^2 + \cos(y^2)\big)\,\hat{\boldsymbol{\jmath}}$.

Q[24](∗): Let C be the curve defined by the intersection of the surfaces $z = x + y$ and $z = x^2 + y^2$.

(a) Show that C is a simple closed curve.

(b) Evaluate $\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ where

(i) $\mathbf{F} = x^2\,\hat{\boldsymbol{\imath}} + y^2\,\hat{\boldsymbol{\jmath}} + 3e^z\,\hat{\mathbf{k}}$.

(ii) $\mathbf{F} = y^2\,\hat{\boldsymbol{\imath}} + x^2\,\hat{\boldsymbol{\jmath}} + 3e^z\,\hat{\mathbf{k}}$.

Q[25]: Find a smooth, simple, closed, counterclockwise oriented curve, C, in the xy-plane for the which the value of the line integral $\oint_C (y^3 - y)\,dx - 2x^3\,dy$ is a maximum among all smooth, simple, closed, counterclockwise oriented curves.

4.4▲ Stokes' Theorem

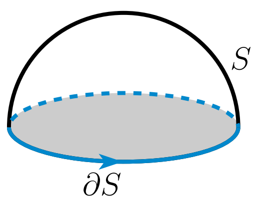
Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

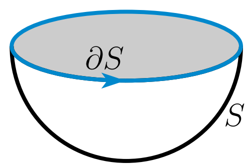
▸▸ Stage 1

Q[1]: Each of the figures below contains a sketch of a surface S and its boundary ∂S. Stokes' theorem says that $\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$ if $\hat{\mathbf{n}}$ is a correctly oriented unit normal vector to S. Add to each sketch a typical such normal vector.

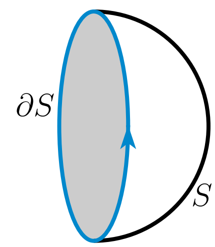
(a)



(b)

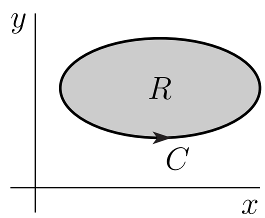


(c)



Q[2]: Let

- R be a finite region in the xy-plane,
- the boundary, C, of R consist of a single piecewise smooth, simple closed curve
  - that is oriented (i.e. an arrow is put on C) consistently with R in the sense that if you walk along C in the direction of the arrow, then R is on your left



- $F_1(x,y)$ and $F_2(x,y)$ have continuous first partial derivatives at every point of R.

Use Stokes' theorem to show that

$$\oint_C \left[F_1(x,y)\,dx + F_2(x,y)\,dy\right] = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$$

i.e. to show Green's theorem.

Q[3]: Verify the identity $\oint_C \phi \nabla \psi \cdot d\mathbf{r} = -\oint_C \psi \nabla \phi \cdot d\mathbf{r}$ for any continuously differentiable scalar fields ϕ and ψ and curve C that is the boundary of a piecewise smooth surface.

▸▸ Stage 2

Q[4]: Let C be the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the surface $z = y^2$ oriented in the counterclockwise direction as seen from $(0, 0, 100)$. Let $\mathbf{F} = (x^2 - y\,,\, y^2 + x\,,\, 1)$. Calculate $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$

(a) by direct evaluation

(b) by using Stokes' Theorem.

Q[5]: Evaluate $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where $\mathbf{F} = ye^x\,\hat{\boldsymbol{\imath}} + (x + e^x)\,\hat{\boldsymbol{\jmath}} + z^2\,\hat{\mathbf{k}}$ and C is the curve

$$\mathbf{r}(t) = (1 + \cos t)\,\hat{\boldsymbol{\imath}} + (1 + \sin t)\,\hat{\boldsymbol{\jmath}} + (1 - \sin t - \cos t)\,\hat{\mathbf{k}} \qquad 0 \leqslant t \leqslant 2\pi$$

Q[6](*): Find the value of $\iint_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S$ where $\mathbf{F} = \big(z - y\,,\, x\,,\, -x\big)$ and S is the hemisphere

$$\big\{\, (x, y, z) \in \mathbb{R}^3 \,\big|\, x^2 + y^2 + z^2 = 4,\ z \geqslant 0 \,\big\}$$

oriented so the surface normals point away from the centre of the hemisphere.

Q[7](*):

Let $\mathcal{S}$ be the part of the surface $z = 16 - \left(x^2 + y^2\right)^2$ which lies above the xy-plane. Let $\mathbf{F}$ be the vector field

$$\mathbf{F} = x\ln(1 + z)\,\hat{\boldsymbol{\imath}} + x(3 + y)\,\hat{\boldsymbol{\jmath}} + y\cos z\,\hat{\mathbf{k}}$$

Calculate

$$\iint_{\mathcal{S}} \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S$$

where $\hat{\mathbf{n}}$ is the upward normal on $\mathcal{S}$.

Q[8](*): Let $\mathcal{C}$ be the intersection of the paraboloid $z = 4 - x^2 - y^2$ with the cylinder $x^2 + (y - 1)^2 = 1$, oriented counterclockwise when viewed from high on the z-axis. Let $\mathbf{F} = xz\,\hat{\boldsymbol{\imath}} + x\,\hat{\boldsymbol{\jmath}} + yz\,\hat{\mathbf{k}}$. Find $\oint_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.

Q[9]: Let $\mathbf{F} = -ye^z\,\hat{\boldsymbol{\imath}} + x^3\cos z\,\hat{\boldsymbol{\jmath}} + z\sin(xy)\,\hat{\mathbf{k}}$, and let S be the part of the surface $z = (1 - x^2)(1 - y^2)$ that lies above the square $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$ in the xy-plane. Find the flux of $\boldsymbol{\nabla} \times \mathbf{F}$ upward through S.

Q[10]: Evaluate the integral $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$, in which $\mathbf{F} = (e^{x^2} - yz\,,\, \sin y - yz\,,\, xz + 2y)$ and C is the triangular path from $(1, 0, 0)$ to $(0, 1, 0)$ to $(0, 0, 1)$ to $(1, 0, 0)$.

Q[11](*): Let $\mathbf{F}(x, y, z) = -z\,\hat{\boldsymbol{\imath}} + x\,\hat{\boldsymbol{\jmath}} + y\,\hat{\mathbf{k}}$ be a vector field. Use Stokes' theorem to evaluate the line integral $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where C is the intersection of the plane $z = y$ and the ellipsoid $\frac{x^2}{4} + \frac{y^2}{2} + \frac{z^2}{2} = 1$, oriented counter-clockwise when viewed from high on the z-axis.

Q[12](*): Consider the vector field $\mathbf{F}(x, y, z) = z^2\,\hat{\boldsymbol{\imath}} + x^2\,\hat{\boldsymbol{\jmath}} + y^2\,\hat{\mathbf{k}}$ in $\mathbb{R}^3$.

(a) Compute the line integral $I_1 = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where C_1 is the curve consisting of three line segments, L_1 from $(2, 0, 0)$ to $(0, 2, 0)$, then L_2 from $(0, 2, 0)$ to $(0, 0, 2)$, finally L_3 from $(0, 0, 2)$ to $(2, 0, 0)$.

(b) A simple closed curve C_2 lies on the plane $E : x + y + z = 2$, enclosing a region R on the plane of area 3, and oriented in a counterclockwise direction as observed from the positive x-axis. Compute the line integral $I_2 = \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.

Q[13](∗): Let $C = C_1 + C_2 + C_3$ be the curve given by the union of the three parameterized curves

$$\begin{aligned}
\mathbf{r}_1(t) &= \big(2\cos t, 2\sin t, 0\big), & 0 \leqslant t \leqslant \pi/2 \\
\mathbf{r}_2(t) &= \big(0, 2\cos t, 2\sin t\big), & 0 \leqslant t \leqslant \pi/2 \\
\mathbf{r}_3(t) &= \big(2\sin t, 0, 2\cos t\big), & 0 \leqslant t \leqslant \pi/2
\end{aligned}$$

(a) Draw a picture of C. Clearly mark each of the curves C_1, C_2, and C_3 and indicate the orientations given by the parameterizations.

(b) Find and parameterize an oriented surface S whose boundary is C (with the given orientations).

(c) Compute the line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where

$$\mathbf{F} = \left(y + \sin(x^2)\,,\, z - 3x + \ln(1 + y^2)\,,\, y + e^{z^2}\right)$$

Q[14](∗): We consider the cone with equation $z = \sqrt{x^2 + y^2}$. Note that its tip, or vertex, is located at the origin $(0, 0, 0)$. The cone is oriented in such a way that the normal vectors point downwards (and away from the z axis). In the parts below, both S_1 and S_2 are oriented this way.

Let $\mathbf{F} = \big(- zy, zx, xy\cos(yz)\big)$.

(a) Let S_1 be the part of the cone that lies between the planes $z = 0$ and $z = 4$. Note that S_1 does not include any part of the plane $z = 4$. Use Stokes' theorem to determine the value of

$$\iint_{S_1} \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$$

Make a sketch indicating the orientations of S_1 and of the contour(s) of integration.

(b) Let S_2 be the part of the cone that lies below the plane $z = 4$ and above $z = 1$. Note that S_2 does not include any part of the planes $z = 1$ and $z = 4$. Determine the flux of $\boldsymbol{\nabla} \times \mathbf{F}$ across S_2. Justify your answer, including a sketch indicating the orientations of S_2 and of the contour(s) of integration.

Q[15](∗): Consider the curve C that is the intersection of the plane $z = x + 4$ and the cylinder $x^2 + y^2 = 4$, and suppose C is oriented so that it is traversed *clockwise* as seen from above.

Let $\mathbf{F}(x, y, z) = \big(x^3 + 2y\,,\, \sin(y) + z\,,\, x + \sin(z^2)\big)$.

Use Stokes' Theorem to evaluate the line integral $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.

Q[16](∗):

(a) Consider the vector field $\mathbf{F}(x,y,z) = (z^2, x^2, y^2)$ in $\mathbb{R}^3$. Compute the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve consisting of the three line segments, L_1 from $(2,0,0)$ to $(0,2,0)$, then L_2 from $(0,2,0)$ to $(0,0,2)$, and finally L_3 from $(0,0,2)$ to $(2,0,0)$.

(b) A simple closed curve C lies in the plane $x+y+z=2$. The surface this curve C surrounds inside the plane $x+y+z=2$ has area 3. The curve C is oriented in a counterclockwise direction as observed from the positive x-axis. Compute the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}$ is as in (a).

Q[17](*): Evaluate the line integral

$$\int_C \left(z + \frac{1}{1+z}\right) dx + xz\, dy + \left(3xy - \frac{x}{(z+1)^2}\right) dz$$

where C is the curve parameterized by

$$\mathbf{r}(t) = \left(\cos t\,,\, \sin t\,,\, 1 - \cos^2 t \sin t\right) \qquad 0 \leqslant t \leqslant 2\pi$$

Q[18](*): A simple closed curve C lies in the plane $x+y+z=1$. The surface this curve C surrounds inside the plane $x+y+z=1$ has area 5. The curve C is oriented in a clockwise direction as observed from the positive z-axis looking down at the plane $x+y+z=1$.

Compute the line integral of $\mathbf{F}(x,y,z) = (z^2, x^2, y^2)$ around C.

Q[19](*): Let C be the oriented curve consisting of the 5 line segments which form the paths from $(0,0,0)$ to $(0,1,1)$, from $(0,1,1)$ to $(0,1,2)$, from $(0,1,2)$ to $(0,2,0)$, from $(0,2,0)$ to $(2,2,0)$, and from $(2,2,0)$ to $(0,0,0)$. Let

$$\mathbf{F} = (-y + e^x \sin x)\,\hat{\boldsymbol{\imath}} + y^4\,\hat{\boldsymbol{\jmath}} + \sqrt{z}\tan z\,\hat{\mathbf{k}}$$

Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Q[20](*): Suppose the curve C is the intersection of the cylinder $x^2 + y^2 = 1$ with the surface $z = xy^2$, traversed clockwise if viewed from the positive z-axis, i.e. viewed "from above". Evaluate the line integral

$$\int_C (z + \sin z)\, dx + (x^3 - x^2 y)\, dy + (x\cos z - y)\, dz$$

Q[21](*): Evaluate $\iint_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$ where S is that part of the sphere $x^2+y^2+z^2 = 2$ above the plane $z=1$, $\hat{\mathbf{n}}$ is the upward unit normal, and

$$\mathbf{F}(x,y,z) = -y^2\,\hat{\boldsymbol{\imath}} + x^3\,\hat{\boldsymbol{\jmath}} + \left(e^x + e^y + z\right)\hat{\mathbf{k}}$$

Q[22](*): Let

$$\mathbf{F} = x\sin y\,\hat{\boldsymbol{\imath}} - y\sin x\,\hat{\boldsymbol{\jmath}} + (x-y)z^2\,\hat{\mathbf{k}}$$

Use Stokes' theorem to evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

along the path consisting of the straight line segments successively joining the points $P_0 = (0,0,0)$ to $P_1 = (\pi/2, 0, 0)$ to $P_2 = (\pi/2, 0, 1)$ to $P_3 = (0,0,1)$ to $P_4 = (0, \pi/2, 1)$ to $P_5 = (0, \pi/2, 0)$, and back to $(0,0,0)$.

Q[23](∗): Let

$$\mathbf{F} = \left(\frac{2z}{1+y} + \sin(x^2)\,,\ \frac{3z}{1+x} + \sin(y^2)\,,\ 5(x+1)(y+2) \right)$$

Let C be the oriented curve consisting of four line segments from $(0,0,0)$ to $(2,0,0)$, from $(2,0,0)$ to $(0,0,2)$, from $(0,0,2)$ to $(0,3,0)$, and from $(0,3,0)$ to $(0,0,0)$.

(a) Draw a picture of C. Clearly indicate the orientation on each line segment.

(b) Compute the work integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.

Q[24](∗): Evaluate $\displaystyle\iint_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$ where $\mathbf{F} = y\,\hat{\boldsymbol{\imath}} + 2z\,\hat{\boldsymbol{\jmath}} + 3x\,\hat{\mathbf{k}}$ and S is the surface $z = \sqrt{1-x^2-y^2}$, $z \geqslant 0$ and $\hat{\mathbf{n}}$ is a unit normal to S obeying $\hat{\mathbf{n}} \cdot \hat{\mathbf{k}} \geqslant 0$.

Q[25](∗): Let $\mathcal{S}$ be the curved surface below, oriented by the outward normal:

$$x^2 + y^2 + 2(z-1)^2 = 6, \qquad z \geqslant 0.$$

(E.g., at the high point of the surface, the unit normal is $\hat{\mathbf{k}}$.) Define

$$\mathbf{G} = \nabla \times \mathbf{F}, \qquad \text{where} \qquad \mathbf{F} = (xz - y^3 \cos z)\,\hat{\boldsymbol{\imath}} + x^3 e^z\,\hat{\boldsymbol{\jmath}} + xyze^{x^2+y^2+z^2}\,\hat{\mathbf{k}}.$$

Find $\iint_{\mathcal{S}} \mathbf{G} \cdot \hat{\mathbf{n}} \mathrm{d}S$.

Q[26](∗): Let C be a circle of radius R lying in the plane $x + y + z = 3$. Use Stokes' Theorem to calculate the value of

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

where $\mathbf{F} = z^2\hat{\boldsymbol{\imath}} + x^2\hat{\boldsymbol{\jmath}} + y^2\hat{\mathbf{k}}$. (You may use either orientation of the circle.)

Q[27]: Let S be the oriented surface consisting of the top and four sides of the cube whose vertices are $(\pm 1, \pm 1, \pm 1)$, oriented outward. If $\mathbf{F}(x,y,z) = (xyz, xy^2, x^2yz)$, find the flux of $\boldsymbol{\nabla} \times \mathbf{F}$ through S.

Q[28]: Let S denote the part of the spiral ramp (that is helicoidal surface) parametrized by

$$x = u\cos v,\ y = u \sin v,\ z = v \qquad 0 \leqslant u \leqslant 1,\ 0 \leqslant v \leqslant 2\pi$$

Let C denote the boundary of S with orientation specified by the upward pointing normal on S. Find

$$\int_C y\,\mathrm{d}x - x\,\mathrm{d}y + xy\,\mathrm{d}z$$

▸▸ Stage 3

Q[29]: Let C be the intersection of $x + 2y - z = 7$ and $x^2 - 2x + 4y^2 = 15$. The curve C is oriented counterclockwise when viewed from high on the z-axis. Let

$$\mathbf{F} = \left(e^{x^2} + yz\right)\hat{\boldsymbol{\imath}} + \left(\cos(y^2) - x^2\right)\hat{\boldsymbol{\jmath}} + \left(\sin(z^2) + xy\right)\hat{\mathbf{k}}$$

Evaluate $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.

Q[30](∗):

(a) Find the curl of the vector field $\mathbf{F} = \big(2 + x^2 + z\,,\, 0\,,\, 3 + x^2 z\big)$.

(b) Let C be the curve in $\mathbb{R}^3$ from the point $(0,0,0)$ to the point $(2,0,0)$, consisting of three consecutive line segments connecting the points $(0,0,0)$ to $(0,0,3)$, $(0,0,3)$ to $(0,1,0)$, and $(0,1,0)$ to $(2,0,0)$. Evaluate the line integral

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}$$

where $\mathbf{F}$ is the vector field from (a).

Q[31](∗):

(a) Let S be the bucket shaped surface consisting of the cylindrical surface $y^2 + z^2 = 9$ between $x = 0$ and $x = 5$, and the disc inside the yz-plane of radius 3 centered at the origin. (The bucket S has a bottom, but no lid.) Orient S in such a way that the unit normal points outward. Compute the flux of the vector field $\boldsymbol{\nabla}\times\mathbf{G}$ through S, where $\mathbf{G} = (x, -z, y)$.

(b) Compute the flux of the vector field $\mathbf{F} = (2 + z, xz^2, x\cos y)$ through S, where S is as in (a).

Q[32](∗): Let

$$\mathbf{F}(x,y,z) = \Big(\frac{y}{x} + x^{1+x^2}\,,\, x^2 - y^{1+y^2}\,,\, \cos^5(\ln z)\Big)$$

(a) Write down the domain D of $\mathbf{F}$.

(b) Circle the correct statement(s):

  (a) D is connected.

  (b) D is simply connected.

  (c) D is disconnected.

(c) Compute $\boldsymbol{\nabla}\times\mathbf{F}$.

(d) Let C be the square with corners $(3\pm 1, 3\pm 1)$ in the plane $z = 2$, oriented clockwise (viewed from above, i.e. down z-axis). Compute

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}$$

(e) Is $\mathbf{F}$ conservative?

Q[33](∗): A physicist studies a vector field $\mathbf{F}(x,y,z)$. From experiments, it is known that $\mathbf{F}$ is of the form

$$\mathbf{F}(x,y,z) = xz\,\hat{\boldsymbol{\imath}} + (axe^y z + byz)\,\hat{\boldsymbol{\jmath}} + (y^2 - xe^y z^2)\,\hat{\mathbf{k}}$$

for some real numbers a and b. It is further known that $\mathbf{F} = \boldsymbol{\nabla}\times\mathbf{G}$ for some differentiable vector field $\mathbf{G}$.

(a) Determine a and b.

(b) Evaluate the surface integral

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$$

where S is the part of the ellipsoid $x^2 + y^2 + \frac{1}{4}z^2 = 1$ for which $z \geqslant 0$, oriented so that its normal vector has a positive z-component.

Q[34](∗): Let C be the curve in the xy-plane from the point $(0,0)$ to the point $(5,5)$ consisting of the ten line segments consecutively connecting the points $(0,0)$, $(0,1)$, $(1,1)$, $(1,2)$, $(2,2)$, $(2,3)$, $(3,3)$, $(3,4)$, $(4,4)$, $(4,5)$, $(5,5)$. Evaluate the line integral

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

where

$$\mathbf{F} = y\,\hat{\boldsymbol{\imath}} + (2x - 10)\,\hat{\boldsymbol{\jmath}}$$

Q[35](∗): Let $\mathbf{F} = \big(\sin x^2\,,\, xz\,,\, z^2\big)$. Evaluate $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ around the curve C of intersection of the cylinder $x^2 + y^2 = 4$ with the surface $z = x^2$, traversed counter clockwise as viewed from high on the z-axis.

Q[36](∗): Explain how one deduces the differential form

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}$$

of Faraday's law from its integral form

$$\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{r} = -\frac{1}{c}\,\frac{\mathrm{d}}{\mathrm{d}t} \iint_S \mathbf{H} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$$

Q[37](∗): Let C be the curve given by the parametric equations:

$$x = \cos t,\ y = \sqrt{2}\sin t,\ z = \cos t,\ 0 \leqslant t \leqslant 2\pi$$

and let

$$\mathbf{F} = z\,\hat{\boldsymbol{\imath}} + x\,\hat{\boldsymbol{\jmath}} + y^3 z^3\,\hat{\mathbf{k}}$$

Use Stokes' theorem to evaluate

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

Q[38](∗): Use Stokes' theorem to evaluate

$$\oint_C z\,\mathrm{d}x + x\,\mathrm{d}y - y\,\mathrm{d}z$$

where C is the closed curve which is the intersection of the plane $x + y + z = 1$ with the sphere $x^2 + y^2 + z^2 = 1$. Assume that C is oriented clockwise as viewed from the origin.

Q[39](*): Let S be the part of the half cone

$$z = \sqrt{x^2 + y^2}, \quad y \geqslant 0,$$

that lies below the plane $z = 1$.

(a) Find a parametrization for S.

(b) Calculate the flux of the velocity field

$$\mathbf{v} = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} - 2z\,\hat{\mathbf{k}}$$

downward through S.

(c) A vector field $\mathbf{F}$ has curl $\boldsymbol{\nabla} \times \mathbf{F} = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} - 2z\,\hat{\mathbf{k}}$. On the xz-plane, the vector field $\mathbf{F}$ is constant with $\mathbf{F}(x, 0, z) = \hat{\boldsymbol{\jmath}}$. Given this information, calculate

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r},$$

where $\mathcal{C}$ is the half circle

$$x^2 + y^2 = 1,\ z = 1,\ y \geqslant 0$$

oriented from $(-1, 0, 1)$ to $(1, 0, 1)$.

Q[40]: Consider $\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS$ where S is the portion of the sphere $x^2 + y^2 + z^2 = 1$ that obeys $x + y + z \geqslant 1$, $\hat{\mathbf{n}}$ is the upward pointing normal to the sphere and $\mathbf{F} = (y - z)\hat{\boldsymbol{\imath}} + (z - x)\hat{\boldsymbol{\jmath}} + (x - y)\hat{\mathbf{k}}$. Find another surface S' with the property that $\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iint_{S'} (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, \mathrm{d}S$ and evaluate $\iint_{S'} (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, \mathrm{d}S$.

Chapter 5

TRUE / FALSE AND OTHER SHORT QUESTIONS

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

Q[1](*): True or false?

(a) $\nabla \cdot (\mathbf{a} \times \mathbf{r}) = 0$, where $\mathbf{a}$ is a constant vector in $\mathbb{R}^3$, and $\mathbf{r}$ is the vector field $\mathbf{r} = (x, y, z)$.

(b) $\nabla \times (\nabla f) = 0$ for all scalar fields f on $\mathbb{R}^3$ with continuous second partial derivatives.

(c) $\nabla \cdot (f\mathbf{F}) = \nabla(f) \cdot \mathbf{F} + f\nabla \cdot \mathbf{F}$, for every vector field $\mathbf{F}$ in $\mathbb{R}^3$ with continuous partial derivatives, and every scalar function f in $\mathbb{R}^3$ with continuous partial derivatives.

(d) Suppose $\mathbf{F}$ is a vector field with continuous partial derivatives in the region D, where D is $\mathbb{R}^3$ without the origin. If $\nabla \cdot \mathbf{F} > 0$ throughout D, then the flux of F through the sphere of radius 5 with center at the origin is positive.

(e) If a vector field $\mathbf{F}$ is defined and has continuous partial derivatives everywhere in $\mathbb{R}^3$, and it satisfies $\nabla \cdot \mathbf{F} = 0$, everywhere, then, for every sphere, the flux out of one hemisphere is equal to the flux into the opposite hemisphere.

(f) If $\mathbf{r}(t)$ is a twice continuously differentiable path in $\mathbb{R}^2$ with constant curvature κ, then $\mathbf{r}(t)$ parametrizes part of a circle of radius $1/\kappa$.

(g) The vector field $\mathbf{F} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ is conservative in its domain, which is $\mathbb{R}^2$, without the origin.

(h) If a vector field $\mathbf{F} = (P, Q)$ in $\mathbb{R}^2$ has $Q = 0$ everywhere in $\mathbb{R}^2$, then the line integral $\oint \mathbf{F} \cdot d\mathbf{r}$ is zero, for every simple closed curve in $\mathbb{R}^2$.

(i) If the acceleration and the speed of a moving particle in $\mathbb{R}^3$ are constant, then the motion is taking place along a spiral.

Q[2](*): True or false?

(a) $\nabla \times (\mathbf{a} \times \mathbf{r}) = 0$, where $\mathbf{a}$ is a constant vector in $\mathbb{R}^3$, and $\mathbf{r}$ is the vector field $\mathbf{r} = (x, y, z)$.

(b) $\nabla \cdot (\nabla f) = 0$ for all scalar fields f on $\mathbb{R}^3$ with continuous second partial derivatives.

(c) $\nabla(\nabla \cdot \mathbf{F}) = 0$ for every vector field $\mathbf{F}$ on $\mathbb{R}^3$ with continuous second partial derivatives.

(d) Suppose $\mathbf{F}$ is a vector field with continuous partial derivatives in the region D, where D is $\mathbb{R}^3$ without the origin. If $\nabla \cdot \mathbf{F} = 0$, then the flux of $\mathbf{F}$ through the sphere of radius 5 with center at the origin is 0.

(e) Suppose $\mathbf{F}$ is a vector field with continuous partial derivatives in the region D, where D is $\mathbb{R}^3$ without the origin. If $\nabla \times \mathbf{F} = \mathbf{0}$ then $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is zero, for every simple and smooth closed curve C in $\mathbb{R}^3$ which avoids the origin.

(f) If a vector field $\mathbf{F}$ is defined and has continuous partial derivatives everywhere in $\mathbb{R}^3$, and it satisfies $\nabla \cdot \mathbf{F} > 0$, everywhere, then, for every sphere, the flux *out* of one hemisphere is larger than the flux *into* the opposite hemisphere.

(g) If $\mathbf{r}(t)$ is a path in $\mathbb{R}^3$ with constant curvature κ, then $\mathbf{r}(t)$ parametrizes part of a circle of radius $1/\kappa$.

(h) The vector field $\mathbf{F} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, z\right)$ is conservative in its domain, which is $\mathbb{R}^3$, without the z-axis.

(i) If all flow lines of a vector field in $\mathbb{R}^3$ are parallel to the z-axis, then the circulation of the vector field around every closed curve is 0.

(j) If the speed of a moving particle is constant, then its acceleration is orthogonal to its velocity.

Q[3](*):

(a) True or false? If $\mathbf{r}(t)$ is the position at time t of an object moving in $\mathbb{R}^3$, and $\mathbf{r}(t)$ is twice differentiable, then $|\mathbf{r}''(t)|$ is the tangential component of its acceleration.

(b) Let $\mathbf{r}(t)$ is a smooth curve in $\mathbb{R}^3$ with unit tangent, normal and binormal vectors $\hat{\mathbf{T}}(t)$, $\hat{\mathbf{N}}(t)$, $\hat{\mathbf{B}}(t)$. Which two of these vectors span the plane normal to the curve at $\mathbf{r}(t)$?

(c) True or false? If $\mathbf{F} = P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}} + R\hat{\mathbf{k}}$ is a vector field on $\mathbb{R}^3$ such that P, Q, R have continuous first order derivatives, and if $\nabla \times \mathbf{F} = \mathbf{0}$ everywhere on $\mathbb{R}^3$, then $\mathbf{F}$ is conservative.

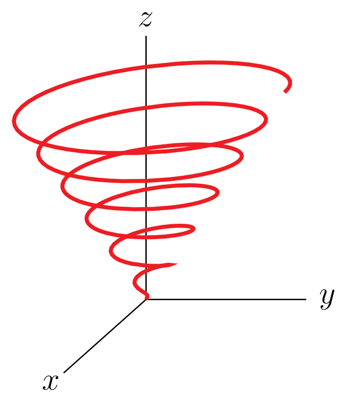
(d) True or false? If $\mathbf{F} = P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}} + R\hat{\mathbf{k}}$ is a vector field on $\mathbb{R}^3$ such that P, Q, R have continuous second order derivatives, then $\nabla \times (\nabla \cdot F) = 0$.

(e) True or false? If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ such that $|\mathbf{F}(x, y, z)| = 1$ for all x, y, z, and if S is the sphere $x^2 + y^2 + z^2 = 1$, then $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = 4\pi$.

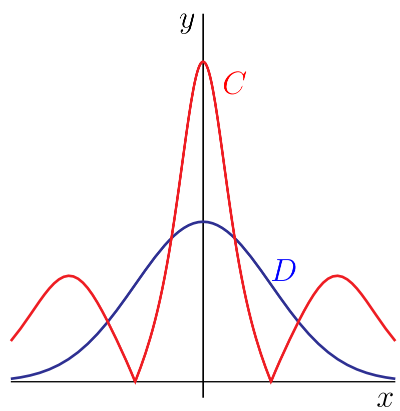
(f) True or false? Every closed surface S in $\mathbb{R}^3$ is orientable. (Recall that S is closed if it is the boundary of a solid region E.)

Q[4](*):

(a) In the curve shown below (a helix lying in the surface of a cone), is the curvature increasing, decreasing, or constant as z increases?



(b) Of the two functions shown below, one is a function $f(x)$ and one is its curvature $\kappa(x)$. Which is which?



(c) Let C be the curve of intersection of the cylinder $x^2 + z^2 = 1$ and the saddle $xz = y$. Parametrise C. (Be sure to specify the domain of your parametrisation.)

(d) Let H be the helical ramp (also known as a helicoid) which revolves around the z-axis in a clockwise direction viewed from above, beginning at the y-axis when $z = 0$, and rising 2π units each time it makes a full revolution. Let S be the the portion of H which lies outside the cylinder $x^2 + y^2 = 4$, above the $z = 0$ plane and below the $z = 5$ plane. Choose one of the following functions and give the domain on which the function you have chosen parametrizes S. (Hint: Only one of the following functions is possible.)

(a) $\mathbf{r}(u, v) = \big(u \cos v, u \sin v, u\big)$

(b) $\mathbf{r}(u, v) = \big(u \cos v, u \sin v, v\big)$

(c) $\mathbf{r}(u, v) = \big(u \sin v, u \cos v, u\big)$

(d) $\mathbf{r}(u, v) = \big(u \sin v, u \cos v, v\big)$

(e) Write down a parametrized curve of zero curvature and arclength 1. (Be sure to specify the domain of your parametrisation.)

(f) If $\nabla \cdot \mathbf{F}$ is a constant C on all of $\mathbb{R}^3$, and S is a cube of unit volume such that the flux outward through each side of S is 1, what is C?

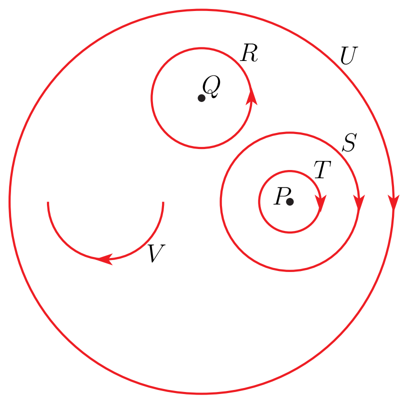
(g) Let

$$\mathbf{F}(x, y) = \big(ax + by\,,\, cx + dy\big)$$

Give the full set of a, b, c and d such that $\mathbf{F}$ is conservative.

(h) If $\mathbf{r}(s)$ has been parametrized by arclength (i.e. s is arclength), what is the arclength of $\mathbf{r}(s)$ between $s = 3$ and $s = 5$?

(i) Let $\mathbf{F}$ be a 2D vector field which is defined everywhere except at the points marked P and Q. Suppose that $\nabla \times \mathbf{F} = 0$ everywhere on the domain of $\mathbf{F}$. Consider the five curves R, S, T, U, and V shown in the picture.



Which of the following is necessarily true?

(1) $\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_T \mathbf{F} \cdot \mathrm{d}\mathbf{r}$

(2) $\int_R \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_S \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_T \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_U \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$

(3) $\int_R \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_S \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_T \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_U \mathbf{F} \cdot \mathrm{d}\mathbf{r}$

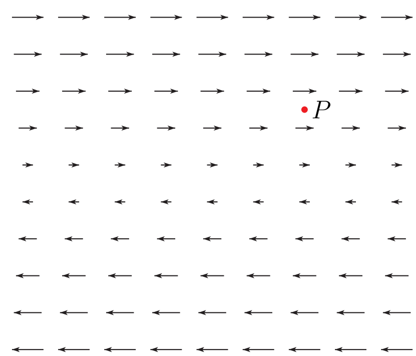
(4) $\int_U \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_R \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_S \mathbf{F} \cdot \mathrm{d}\mathbf{r}$

(5) $\int_V \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$

(j) Write down a 3D vector field $\mathbf{F}$ such that for all closed surfaces S, the volume enclosed by S is equal to

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$$

(k) Consider the vector field $\mathbf{F}$ in the xy-plane shown below. Is the $\hat{\mathbf{k}}^{\text{th}}$ component of $\nabla \times \mathbf{F}$ at P positive, negative or zero?



Q[5](∗): Say whether the following statements are true or false.

(a) If $\mathbf{F}$ is a 3D vector field defined on all of $\mathbb{R}^3$, and S_1 and S_2 are two surfaces with the same boundary, but $\iint_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S \neq \iint_{S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$, then $\nabla\cdot\mathbf{F}$ is not zero anywhere.

(b) If $\mathbf{F}$ is a vector field satisfying $\nabla\times\mathbf{F} = 0$ whose domain is not simply-connected, then $\mathbf{F}$ is not conservative.

(c) The osculating circle of a curve C at a point has the same unit tangent vector, unit normal vector, and curvature as C at that point.

(d) A planet orbiting a sun has period proportional to the cube of the major axis of the orbit.

(e) For any 3D vector field $\mathbf{F}$, $\nabla\cdot(\nabla\times\mathbf{F}) = 0$.

(f) A field whose divergence is zero everywhere in its domain has closed surfaces S in its domain.

(g) The gravitational force field is conservative.

(h) If $\mathbf{F}$ is a field defined on all of $\mathbb{R}^3$ such that $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = 3$ for some curve C, then $\nabla\times\mathbf{F}$ is non-zero at some point.

(i) The normal component of acceleration for a curve of constant curvature is constant.

(j) The curve defined by

$$\mathbf{r}_1(t) = \cos(t^4)\,\hat{\boldsymbol{\imath}} + 3t^4\hat{\boldsymbol{\jmath}}, \qquad -\infty < t < \infty,$$

is the same as the curve defined by

$$\mathbf{r}_2(t) = \cos t\,\hat{\boldsymbol{\imath}} + 3t\,\hat{\mathbf{k}}, \qquad -\infty < t < \infty$$

Q[6](∗): Which of the following statements are true (T) and which are false (F)? All real valued functions $f(x, y, z)$ and all vector fields $\mathbf{F}(x, y, z)$ have domain $\mathbb{R}^3$ unless specified otherwise.

(a) If f is a continuous real valued function and S a smooth oriented surface, then

$$\iint_S f\,\mathrm{d}S = -\iint_{-S} f\,\mathrm{d}S$$

where '$-S$' denotes the surface S but with the opposite orientation.

(b) Suppose the components of the vector field $\mathbf{F}$ have continuous partial derivatives. If $\iint_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = 0$ for every closed smooth surface, then $\mathbf{F}$ is conservative.

(c) Suppose S is a smooth surface bounded by a smooth simple closed curve C. The orientation of C is determined by that of S as in Stokes' theorem. Suppose the real valued function f has continuous partial derivatives. Then

$$\int_C f\, \mathrm{d}x = \iint_S \left(\frac{\partial f}{\partial z} \hat{\boldsymbol{\jmath}} - \frac{\partial f}{\partial y} \hat{\mathbf{k}} \right) \cdot \hat{\mathbf{n}}\, \mathrm{d}S$$

(d) Suppose the real valued function $f(x, y, z)$ has continuous second order partial derivatives. Then

$$(\boldsymbol{\nabla} f) \times (\boldsymbol{\nabla} f) = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} f)$$

(e) The curve parameterized by

$$\mathbf{r}(t) = \big(2 + 4t^3\,,\, -t^3\,,\, 1 - 2t^3\big) \qquad -\infty < t < \infty$$

has curvature $\kappa(t) = 0$ for all t.

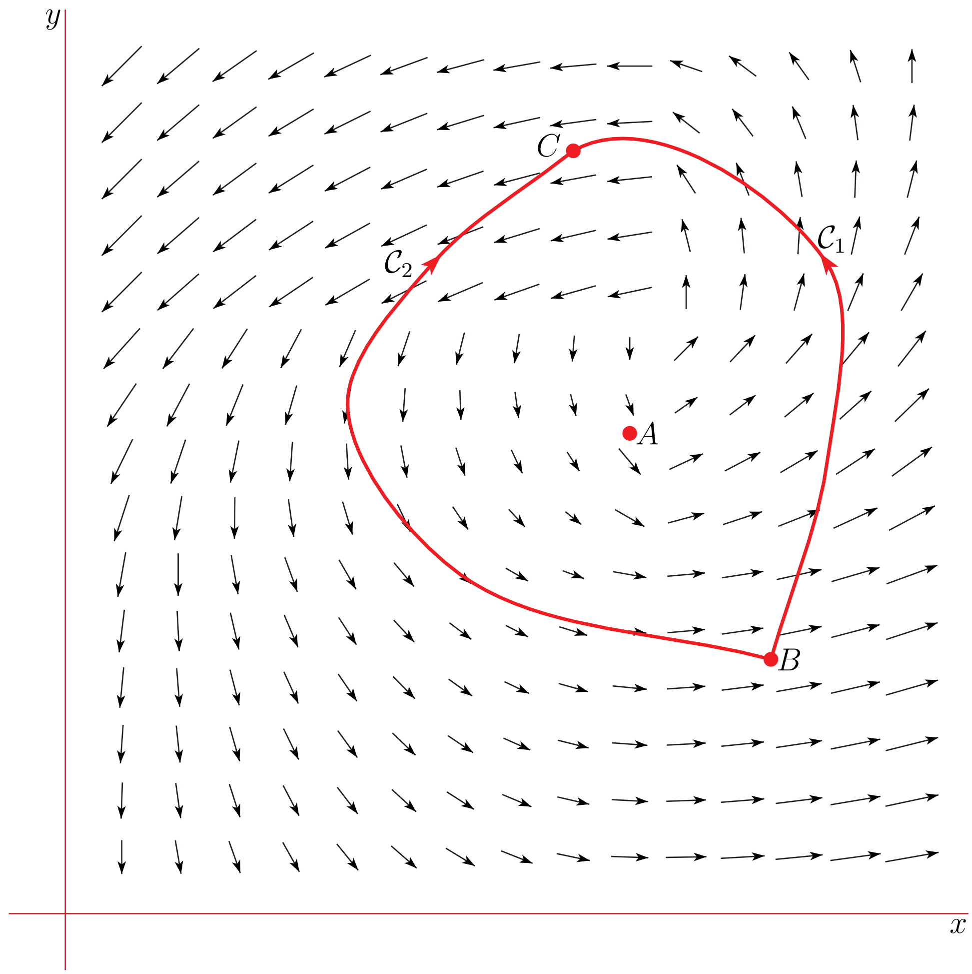
(f) If a smooth curve is parameterized by $\mathbf{r}(s)$ where s is arc length, then its tangent vector satisfies

$$|\mathbf{r}'(s)| = 1$$

(g) If S is the sphere $x^2 + y^2 + z^2 = 1$ and $\mathbf{F}$ is a constant vector field, then $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = 0$.

(h) There exists a vector field $\mathbf{F}$ whose components have continuous second order partial derivatives such that $\boldsymbol{\nabla} \times \mathbf{F} = (x, y, z)$.

Q[7](∗): The vector field $\mathbf{F} = P(x, y)\, \hat{\boldsymbol{\imath}} + Q(x, y)\, \hat{\boldsymbol{\jmath}}$ is plotted below.



In the following questions, give the answer that is best supported by the plot.

(a) The derivative P_y at the point labelled A is (a) positive, (b) negative, (c) zero, (d) there is not enough information to tell.

(b) The derivative Q_x at the point labelled A is (a) positive, (b) negative, (c) zero, (d) there is not enough information to tell.

(c) The curl of $\mathbf{F}$ at the point labelled A is (a) in the direction of $+\hat{\mathbf{k}}$ (b) in the direction of $-\hat{\mathbf{k}}$ (c) zero (d) there is not enough information to tell.

(d) The work done by the vector field on a particle travelling from point B to point C along the curve $\mathcal{C}_1$ is (a) positive (b) negative (c) zero (d) there is not enough information to tell.

(e) The work done by the vector field on a particle travelling from point B to point C along the curve $\mathcal{C}_2$ is (a) positive (b) negative (c) zero (d) there is not enough information to tell.

(f) The vector field $\mathbf{F}$ is (a) the gradient of some function f (b) the curl of some vector field $\mathbf{G}$ (c) not conservative (d) divergence free.

Q[8](*): Which of the following statements are true (T) and which are false (F)?

(a) The curve defined by

$$\mathbf{r}_1(t) = \cos(t^2)\,\hat{\boldsymbol{\imath}} + \sin(t^2)\,\hat{\boldsymbol{\jmath}} + 2t^2\,\hat{\mathbf{k}}, \qquad -\infty < t < \infty$$

is the same as the curve defined by

$$\mathbf{r}_2(t) = \cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}} + 2t\,\hat{\mathbf{k}}, \qquad -\infty < t < \infty$$

(b) The curve defined by

$$\mathbf{r}_1(t) = \cos(t^2)\,\hat{\boldsymbol{\imath}} + \sin(t^2)\,\hat{\boldsymbol{\jmath}} + 2t^2\,\hat{\mathbf{k}}, \qquad 0 \leqslant t \leqslant 1$$

is the same as the curve defined by

$$\mathbf{r}_2(t) = \cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}} + 2t\,\hat{\mathbf{k}}, \qquad 0 \leqslant t \leqslant 1$$

(c) If a smooth curve is parameterized by $\mathbf{r}(s)$ where s is arc length, then its tangent vector satisfies

$$|\mathbf{r}'(s)| = 1$$

(d) If $\mathbf{r}(t)$ defines a smooth curve C in space that has constant curvature $\kappa > 0$, then C is part of a circle with radius $1/\kappa$.

(e) If the speed of a moving object is constant, then its acceleration is orthogonal to its velocity.

(f) The vector field

$$\mathbf{F}(x, y, z) = \frac{-y}{x^2+y^2}\hat{\boldsymbol{\imath}} + \frac{x}{x^2+y^2}\hat{\boldsymbol{\jmath}} + z\hat{\mathbf{k}}$$

is conservative.

(g) Suppose the vector field $\mathbf{F}(x, y, z)$ is defined on an open domain and its components have continuous partial derivatives. If $\boldsymbol{\nabla} \times \mathbf{F} = 0$, then $\mathbf{F}$ is conservative.

(h) The region $D = \{\ (x, y) \mid x^2 + y^2 > 1\ \}$ is simply connected.

(i) The region $D = \{\ (x, y) \mid y - x^2 > 0\ \}$ is simply connected.

(j) If $\mathbf{F}$ is a vector field whose components have two continuous partial derivatives, then

$$\iint_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S = 0$$

when S is the boundary of a solid region E in $\mathbb{R}^3$.

Q[9](∗): Which of the following statements are true (T) and which are false (F)?

(a) If a smooth curve C is parameterized by $\mathbf{r}(s)$ where s is arc length, then the tangent vector $\mathbf{r}'(s)$ satisfies $|\mathbf{r}'(s)| = 1$.

(b) If $\mathbf{r}(t)$ defines a smooth curve C in space that has constant curvature $\kappa > 0$, then C is part of a circle with radius $1/\kappa$.

(c) Suppose $\mathbf{F}$ is a continuous vector field with open domain D. If

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$$

for every piecewise smooth closed curve C in D, then $\mathbf{F}$ is conservative.

(d) Suppose $\mathbf{F}$ is a vector field with open domain D, and the components of $\mathbf{F}$ have continuous partial derivatives. If $\boldsymbol{\nabla} \times \mathbf{F} = 0$ everywhere on D, then $\mathbf{F}$ is conservative.

(e) The curve defined by

$$\mathbf{r}_1(t) = \cos(t^2)\,\hat{\boldsymbol{\imath}} + \sin(t^2)\,\hat{\boldsymbol{\jmath}} + 2t^2\,\hat{\mathbf{k}}, \qquad -\infty < t < \infty$$

is the same as the curve defined by

$$\mathbf{r}_2(t) = \cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}} + 2t\,\hat{\mathbf{k}}, \qquad -\infty < t < \infty$$

(f) The curve defined by

$$\mathbf{r}_1(t) = \cos(t^2)\,\hat{\boldsymbol{\imath}} + \sin(t^2)\,\hat{\boldsymbol{\jmath}} + 2t^2\,\hat{\mathbf{k}}, \qquad 0 \leqslant t \leqslant 1$$

is the same as the curve defined by

$$\mathbf{r}_2(t) = \cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}} + 2t\,\hat{\mathbf{k}}, \qquad 0 \leqslant t \leqslant 1$$

(g) Suppose $\mathbf{F}(x, y, z)$ is a vector field whose components have continuous second order partial derivatives. Then $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times F) = 0$.

(h) Suppose the real valued function $f(x, y, z)$ has continuous second order partial derivatives. Then $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f) = 0$.

(i) The region $D = \{\ (x, y) \mid x^2 + y^2 > 1\ \}$ is simply connected.

(j) The region $D = \{\ (x, y) \mid y - x^2 > 0\ \}$ is simply connected.

Q[10](*): Let $\mathbf{F}$, $\mathbf{G}$ be vector fields, and f, g be scalar fields. Assume $\mathbf{F}$, $\mathbf{G}$, f, g are defined on all of $\mathbb{R}^3$ and have continuous partial derivatives of all orders everywhere. Mark each of the following as True (T) or False (F).

(a) If C is a closed curve and $\boldsymbol{\nabla} f = \mathbf{0}$, then $\int_C f\, \mathrm{d}s = 0$.

(b) If $\mathbf{r}(t)$ is a parametrization of a smooth curve C and the binormal $\mathbf{B}(t)$ is constant then C is a straight line.

(c) If $\mathbf{r}(t)$ is the position of a particle which travels with constant speed, then $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 0$.

(d) If C is a path from points A to B, then the line integral $\int_C \big(\mathbf{F} \times \mathbf{G}\big) \cdot \mathrm{d}\mathbf{r}$ is independent of the path C.

(e) The line integral $\int_C f\, \mathrm{d}s$ does not depend of the orientation of the curve C.

(f) If S is a parametric surface $\mathbf{r}(u,v)$ then a normal to S is given by

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial u}$$

(g) The surface area of the parametric surface S given by $\mathbf{r}(u,v) = x(u,v)\,\hat{\boldsymbol{\imath}} + y(u,v)\,\hat{\boldsymbol{\jmath}} + z(u,v)\,\hat{\mathbf{k}}$, $(u,v) \in D$, is given by

$$\iint_D \left(1 + \left(\tfrac{\partial z}{\partial u}\right)^2 + \left(\tfrac{\partial z}{\partial v}\right)^2\right)^{1/2} \mathrm{d}u\mathrm{d}v$$

(h) If $\mathbf{F}$ is the velocity field of an incompressible fluid then $\nabla \cdot \mathbf{F} = 0$.

(i) $\nabla \cdot \big(\mathbf{F} \times \mathbf{G}\big) = (\nabla \cdot \mathbf{F})\mathbf{G} + (\nabla \cdot \mathbf{G})\mathbf{F}$

Q[11](*): Say whether the following statements are true (T) or false (F). You may assume that all functions and vector fields are defined everywhere and have derivatives of all orders everywhere.

(a) The divergence of $\nabla \times \mathbf{F}$ is zero, for every $\mathbf{F}$.

(b) In a simply connected region, $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ depends only on the endpoints of C.

(c) If $\nabla f = 0$, then f is a constant function.

(d) If $\nabla \times \mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a constant vector field.

(e) If $\nabla \cdot \mathbf{F} = 0$, then $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = 0$ for every closed surface S.

(f) If $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$ for every closed curve C, then $\nabla \times \mathbf{F} = 0$.

(g) If $\mathbf{r}(t)$ is a path in three space with constant speed $|\mathbf{v}(t)|$, then the acceleration is perpendicular to the tangent vector, i.e. $\mathbf{a} \cdot \hat{\mathbf{T}} = 0$.

(h) If $\mathbf{r}(t)$ is a path in three space with constant curvature κ, then $\mathbf{r}(t)$ parameterizes part of a circle of radius $1/\kappa$.

(i) Let $\mathbf{F}$ be a vector field and suppose that S_1 and S_2 are oriented surfaces with the same boundary curve C, and C is given the direction that is compatible with the orientations of S_1 and S_2 . Then $\iint_{S1} \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iint_{S2} \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$.

(j) Let $A(t)$ be the area swept out by the trajectory of a planet from time $t = 0$ to time t. The $\frac{\mathrm{d}A}{\mathrm{d}t}$ is constant.

Q[12](*): Find the correct identity, if f is a function and $\mathbf{G}$ and $\mathbf{F}$ are vector fields. Select the true statement.

(a) $\nabla \cdot (f\mathbf{F}) = f\nabla \times (\mathbf{F}) + (\nabla f) \times \mathbf{F}$

(b) $\nabla \cdot (f\mathbf{F}) = f\nabla \cdot (\mathbf{F}) + \mathbf{F} \cdot \nabla f$

(c) $\nabla \times (f\mathbf{F}) = f\nabla \cdot (\mathbf{F}) + \mathbf{F} \cdot \nabla f$

(d) None of the above are true.

Q[13](*): True or False. Consider vector fields $\mathbf{F}$ and scalar functions f and g which are defined and smooth in all of three-dimensional space. Let $\mathbf{r} = (x, y, z)$ represent a variable point in space, and let $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ be a constant vector. Let Ω be a smoothly bounded domain with outer normal $\hat{\mathbf{n}}$. Which of the following are identites, always valid under these assumptions?

(a) $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} f = 0$

(b) $\mathbf{F} \times \boldsymbol{\nabla} f = f \boldsymbol{\nabla} \cdot \mathbf{F}$

(c) $\boldsymbol{\nabla}^2 f = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot f)$

(d) $\boldsymbol{\nabla} \times \boldsymbol{\nabla} f = \mathbf{0}$

(e) $(\boldsymbol{\nabla} \times f) + (\boldsymbol{\nabla} \times g) = \boldsymbol{\nabla} f \times \boldsymbol{\nabla} g$

(f) $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \mathbf{F} = 0$

(g) $\boldsymbol{\nabla} \cdot \frac{\mathbf{r}}{|\mathbf{r}|^2} = 0$ for $\mathbf{r} \neq \mathbf{0}$

(h) $\boldsymbol{\nabla} \times (\boldsymbol{\omega} \times \mathbf{r}) = \mathbf{0}$

(i) $\displaystyle\iiint_\Omega f \boldsymbol{\nabla} \cdot \mathbf{F}\, \mathrm{d}V = -\iiint_\Omega \boldsymbol{\nabla} f \cdot \mathbf{F}\, \mathrm{d}V + \iint_{\partial\Omega} f \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$

(j) $\displaystyle\iint_{\partial\Omega} f \hat{\mathbf{n}}\, \mathrm{d}S = -\iiint_\Omega \boldsymbol{\nabla} f\, \mathrm{d}V$

Q[14](*): Determine if the given statements are True or False. Provide a reason or a counterexample.

(a) A constant vector field is conservative on $\mathbb{R}^3$.

(b) If $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$ for all points in the domain of $\mathbf{F}$ then $\mathbf{F}$ is a constant vector field.

(c) Let $\mathbf{r}(t)$ be a parametrization of a curve C in $\mathbb{R}^3$. If $\mathbf{r}(t)$ and $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$ are orthogonal at all points of the curve C, then C lies on the surface of a sphere $x^2 + y^2 + z^2 = a^2$ for some $a > 0$.

(d) The curvature κ at a point on a curve depends on the orientation of the curve.

(e) The domain of a conservative vector field must be simply connected.

Q[15](*): Provide a short answer to each question.

(a) Compute $\boldsymbol{\nabla} \cdot \big(x^2 y\, \hat{\boldsymbol{\imath}} + e^y \sin x\, \hat{\boldsymbol{\jmath}} + e^{zx}\, \hat{\mathbf{k}}\big)$

(b) Compute $\boldsymbol{\nabla} \times \big(\cos x^2\, \hat{\boldsymbol{\imath}} - y^3 z\, \hat{\boldsymbol{\jmath}} + xz\, \hat{\mathbf{k}}\big)$

(c) Let

$$\mathbf{F} = \frac{x}{x^2 + y^2}\hat{\boldsymbol{\imath}} + \frac{y}{x^2 + y^2}\hat{\boldsymbol{\jmath}} + z^2\, \hat{\mathbf{k}}$$

and let D be the domain of $\mathbf{F}$. Consider the following four statments.

(I) D is connected

(II) D is disconnected

(III) D is simply connected

(IV) D is not simply connected

Choose one of the following:

(i) (II) and (III) are true

(ii) (I) and (III) are true

(iii) (I) and (IV) are true

(iv) (II) and (IV) are true

(v) Not enough information to determine

(d) True or False? If the speed of a particle is constant then the acceleration of the particle is zero. If your answer is **True**, provide a reason. If your answer is **False**, provide a counter example.

Q[16](*): Are each of the following statements True or False? Recall that $f \in C^k$ means that all derivatives of f up to order k exist and are continuous.

(a) $\boldsymbol{\nabla} \times (f\boldsymbol{\nabla} f) = \mathbf{0}$ for all C^2 scalar functions f in $\mathbb{R}^3$.

(b) $\boldsymbol{\nabla} \cdot (f\mathbf{F}) = \boldsymbol{\nabla} f \cdot \mathbf{F} + f\boldsymbol{\nabla} \cdot \mathbf{F}$ for all C^1 scalar functions f and C^1 vector fields $\mathbf{F}$ in $\mathbb{R}^3$.

(c) A smooth space curve C with constant curvature $\kappa = 0$ must be a part of a straight line.

(d) A smooth space curve C with constant curvature $\kappa \neq 0$ must be part of a circle of radius $1/\kappa$.

(e) If f is any smooth function defined in $\mathbb{R}^3$ and if C is any circle, then $\int_C \boldsymbol{\nabla} f \cdot \mathrm{d}\mathbf{r} = 0$.

(f) Suppose $\mathbf{F}$ is a smooth vector field in $\mathbb{R}^3$ and $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$ everywhere. Then, for every sphere, the flux *out* of one hemisphere is equal to the flux *into* the opposite hemisphere.

(g) Let $\mathbf{F}(x, y, z)$ be a continuously differentiable vector field which is defined for every (x, y, z). Then, $\iint_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = 0$ for any closed surface S. (A closed surface is a surface that is the boundary of a solid region.)

Q[17](*): True or false (reasons must be given):

(a) If a smooth vector field on $\mathbb{R}^3$ is curl free and divergence free, then its potential is harmonic. By definition, $\phi(x, y, z)$ is harmonic if $\big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\big)\phi(x, y, z) = 0$.

(b) If $\mathbf{F}$ is a smooth conservative vector field on $\mathbb{R}^3$, then its flux through any smooth closed surface is zero.

Q[18](*): The following statements may be true or false. Decide which. If true, give a proof. If false, provide a counter-example.

(a) If f is any smooth function defined in $\mathbb{R}^3$ and if C is any circle, then $\int_C \boldsymbol{\nabla} f \cdot \mathrm{d}\mathbf{r} = 0$.

(b) There is a vector field $\mathbf{F}$ that obeys $\boldsymbol{\nabla} \times \mathbf{F} = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}}$.

Q[19](∗): Short answers:

(a) Let S be the level surface $f(x,y,z) = 0$. Why is $\int_C \boldsymbol{\nabla} f \cdot \mathrm{d}\mathbf{r} = 0$ for any curve C on S?

(b) A point moving in space with position $\mathbf{r}(t)$ at time t satisfies the condition $\mathbf{a}(t) = f(t)\mathbf{r}(t)$ for all t for some real valued function f. Why is $\mathbf{v} \times \mathbf{r}$ a constant vector?

(c) Why is the trajectory of the point in (b) contained in a plane?

(d) Is the binormal vector, $\hat{\mathbf{B}}$, of a particle moving in space, always orthogonal to the unit tangent vector $\hat{\mathbf{T}}$ and unit normal $\hat{\mathbf{N}}$?

(e) If the curvature of the path of a particle moving in space is constant, is the acceleration zero when maximum speed occurs?

Q[20](∗): A region R is bounded by a simple closed curve $\mathcal{C}$. The curve $\mathcal{C}$ is oriented such that R lies to the left of $\mathcal{C}$ when walking along $\mathcal{C}$ in the direction of $\mathcal{C}$. Determine whether or not each of the following expressions is equal to the area of R. You must justify your conclusions.

(a) $\dfrac{1}{2}\displaystyle\int_C -y\,dx + x\,dy$

(b) $\dfrac{1}{2}\displaystyle\int_C -x\,dx + y\,dy$

(c) $\displaystyle\int_C y\,dx$

(d) $\displaystyle\int_C 3y\,dx + 4x\,dy$

Q[21](∗): Say whether each of the following statements is true or false and *explain why*.

(a) A moving particle has velocity and acceleration vectors that satisfy $|\mathbf{v}| = 1$ and $|\mathbf{a}| = 1$ at all times. Then the curvature of this particle's path is a constant.

(b) If $\mathbf{F}$ is any smooth vector field defined in $\mathbb{R}^3$ and if S is any sphere, then

$$\iint_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S = 0$$

Here $\hat{\mathbf{n}}$ is the outward normal to S.

(c) If $\mathbf{F}$ and $\mathbf{G}$ are smooth vector fields in $\mathbb{R}^3$ and if $\displaystyle\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_C \mathbf{G} \cdot \mathrm{d}\mathbf{r}$ for every circle C, then $\mathbf{F} = \mathbf{G}$.

Q[22](∗): Three quickies:

(a) A moving particle with position $\mathbf{r}(t) = (x(t), y(t), z(t))$ satisfies

$$\mathbf{a} = f(\mathbf{r}, \mathbf{v})\mathbf{r}$$

for some scalar-valued function f. Prove that $\mathbf{r} \times \mathbf{v}$ is constant.

(b) Calculate $\iint_{\mathcal{S}} (x\,\hat{\boldsymbol{\imath}} - y\,\hat{\boldsymbol{\jmath}} + z^2\,\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}} \mathrm{d}S$, where $\mathcal{S}$ is the boundary of any solid right circular cylinder of radius b with one base in the plane $z = 1$ and the other base in the plane $z = 3$.

(c) Let $\mathbf{F}$ and $\mathbf{G}$ be smooth vector fields defined in $\mathbb{R}^3$. Suppose that, for every circle C, we have $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S \mathbf{G} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$, where S is the oriented disk with boundary C. Prove that $\mathbf{G} = \boldsymbol{\nabla} \times \mathbf{F}$.

Part II

HINTS TO PROBLEMS

Hints for Exercises 1.1. — Jump to TABLE OF CONTENTS.

H-1: Draw sketches. Don't forget the range that the parameter runs over.

H-2: Find the value of t at which the three points occur on the curve.

H-3: The curve "crosses itself" when $(\sin t, t^2)$ gives the same coordinate for different values of t. When these crossings occur will depend on which crossing you're referring to, so your answers should all depend on t.

H-4: For part (b), find the position of P relative to the centre of the circle. Then combine your answer with part (a).

H-5: We aren't concerned with x, so we can eliminate it by solving one equation for x as a function of y and z and plugging the result into the other equation.

H-6: To determine whether the particle is rising or falling, we only need to consider its z-coordinate.

H-7: This is the setup from Lemma 1.1.3 in the CLP-4. The two quantities you're labelling are related, but different.

H-8: See the note just before Example 1.1.5.

H-9: To simplify your answer, remember: the cross product of **a** and **b** is a vector orthogonal to both **a** and **b**; the cross product of a vector with itself is zero; and two orthogonal vectors have dot product 0.

H-10: Evaluate $\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{r}(t)|^2$.

H-11: Just compute $|\mathbf{v}(t)|$. Note that $\left(e^{at}+e^{-at}\right)^2 = e^{2at}+2+e^{-2at}$.

H-12: To figure out what the path looks like, first concentrate on the x- and y-coordinates.

H-13: Review §1.5 of the CLP-4 text. The arc length should be positive.

H-14: From Lemma 1.1.3 in the CLP-4 text, we know the arclength from $t=0$ to $t=1$ will be

$$\int_0^1 \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\right| \mathrm{d}t$$

The notation looks a little confusing at first, but we can break it down piece by piece: $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)$ is a vector, whose components are functions of t. If we take its magnitude, we'll get one big function of t. That function is what we integrate. Before integrating it, however, we should simplify as much as possible.

H-16: $\mathbf{r}(t)$ is the position of the particle, so its acceleration is $r''(t)$.

H-17: Review §1.5 of the CLP-4 text.

H-18: Review §1.1 of the CLP-4 text.

H-19: (a) First parametrize $x^2+y^2=9$.

H-20: If you got the answer 0 in part (b), you dropped some absolute value signs.

H-22: The integral you get can be evaluated with a simple substitution. You may want to factor the integrand first.

H-23: (b) $\frac{1}{4x} + 1 + x$ is a perfect square.

(c), (d) Let

- $\mathbf{r}(x)$ be the position of the particle when its first coordinate is x,
- $\mathbf{R}(t)$ be the position of the particle at time t, and
- $x(t)$ be the x–coordinate of the particle at time t.

Then $\mathbf{R}(t) = \mathbf{r}\big(x(t)\big)$. We are told $|\mathbf{R}'(t)| = 9$ for all t.

H-24: Given the position of a particle, you can find its velocity.

H-25: If $\mathbf{r}(u)$ is the parametrization of $\mathcal{C}$ by u, then the position of the particle at time t is $\mathbf{R}(t) = \mathbf{r}\big(u(t)\big)$.

H-26: By Newton's law, $\mathbf{F} = m\mathbf{a}$.

H-27: Denote by $\mathbf{r}(x)$ the parametrization of C by x. If the x-coordinate of the particle at time t is $x(t)$, then the position of the particle at time t is $\mathbf{R}(t) = \mathbf{r}\big(x(t)\big)$. Also, though the particle is moving at a constant speed, it doesn't necessarily have a constant value of $\frac{\mathrm{dx}}{\mathrm{d}t}$.

H-28: The question is already set up as an xy-plane, with the camera at the origin, so the vector in the direction the camera is pointing is $\big(x(t), y(t)\big)$. Let θ be the angle the camera makes with the positive x-axis (due east). The tangent function gives a clean-looking relation between $\theta(t)$, $x(t)$, and $y(t)$.

H-29: Usng the Theorem of Pappus, the surface area and volume of this pipe are the same as that of a straight pipe with the same length and radius.

H-30: A helix can be parametrized by $\mathbf{r}(\theta) = a\cos\theta\,\hat{\boldsymbol{\imath}} + a\sin\theta\,\hat{\boldsymbol{\jmath}} + b\theta\,\hat{\mathbf{k}}$.

H-31: Define $\mathbf{u}(t) = e^{\alpha t}\frac{d\mathbf{r}}{dt}(t)$ and substitute $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t) = e^{-\alpha t}\mathbf{u}(t)$ into the given differential equation to find a differential equation for $\mathbf{u}$.

Hints for Exercises 1.2. — Jump to TABLE OF CONTENTS.

H-1: You're asked to find the arclength of the curve from $s = 1$ to $s = t$.

H-2: The arclength will be 0 at P.

H-3: $\mathbf{a}(t_0)$ and $\mathbf{b}(s_0)$ describe the same point on $\mathbf{R}$.

H-4: On your way to finding the relationship between t and arclength, you should realize that the curve has constant speed (with respect to t), though not constant velocity.

H-5: For which values of t is $|\mathbf{r}(t)| \leqslant 1$? Check the domain of t — we're not starting at zero.

H-6: Be careful with the domain.

H-7: Remember $\sqrt{x^2} = |x|$. You will need to consider cases for this one.

Hints for Exercises 1.3. — Jump to TABLE OF CONTENTS.

H-1: The curve is a circle, so you don't need to do any calculus.

H-2: Because $\mathbf{r}$ is a circle, you can parametrize it with respect to arclength without using an integral. You found κ in Question 1.

H-3: When t is large, does the spiral locally look like a circle of large radius, or small?

H-4: $\frac{\mathrm{d}s}{\mathrm{d}t} = |\mathbf{v}(t)| = |\mathbf{r}'(t)|$

H-5: $\hat{\mathbf{T}} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$

H-7: You can find the last two quantities by making use of the first three. Looking ahead, the formula list in Section 1.5 might come in handy.

H-8: We can calculate $\kappa = \dfrac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{\left|\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3\right|}$. We can also figure out what kind of a shape our curve is.

H-9: The maximum and minimum values of $\kappa(t)$ should be obvious from your formula for $\kappa(t)$.

H-11: For part (a), determine $\mathbf{r}(0)$, $\mathbf{r}(\pi)$, $\mathbf{r}(2\pi)$, $\mathbf{r}(3\pi)$, and $\mathbf{r}(4\pi)$, to help you map out the motion. Also visualize the thumbtack as the wheel moves.

For part (d), use the fact that you only care about $t = \pi$: where is this on your sketch? What does that mean about the direction of $\hat{\mathbf{N}}$?

H-12: You should find that $s = \theta$!

H-13: Since $\kappa(x)$ is never negative, $\kappa(x)$ is maximum when $\kappa^2(x)$ is maximum. The latter is easier to compute.

Hints for Exercises 1.4. — Jump to TABLE OF CONTENTS.

H-1: Use the right-hand rule to figure out how $\hat{\mathbf{B}}$ is oriented.

H-2: Speed is the norm of velocity. Does that fit this equation?

H-3: Review Example 1.4.4 and remember that positive torsion indicates "right-handed twisting." You shouldn't actually need to calculate anything.

H-4: (a) Show that the tangent vector $\hat{\mathbf{T}}(s)$ is a constant.

(b) Guess the plane. To do so, first show that the binormal $\hat{\mathbf{B}}(s)$ is a constant. Then show that $(\mathbf{r}(s) - \mathbf{r}(0)) \cdot \hat{\mathbf{B}}$ is a constant.

(c) Guess the circle. To do so, first show that $\mathbf{r}_c(s) = \mathbf{r}(s) + \frac{1}{\kappa(s)}\hat{\mathbf{N}}(s)$ is a constant.

H-5: It is not necessary to compute anything.

H-6: Both parts of this question make use of the quantity $\frac{\mathrm{d}s}{\mathrm{d}t}$.

H-7: $\tau(t) = \dfrac{(\mathbf{v}(t) \times \mathbf{a}(t)) \cdot \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}}{|\mathbf{v}(t) \times \mathbf{a}(t)|^2}$

H-8: Review §1.5 of the CLP-4 text.

H-9: The vector perpendicular to the plane containing the osculating circle is the binormal vector, $\hat{\mathbf{B}}$.

H-10: (a) The tangent vector of the curve is also a normal vector for the specified plane.

(b) Review §1.5 of the CLP-4 text.

H-11: Remember $\mathbf{a}(t) = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}(t)\,\hat{\mathbf{T}}(t) + \kappa(t)\big(\frac{\mathrm{d}s}{\mathrm{d}t}(t)\big)^2\hat{\mathbf{N}}(t)$. Remember also that $\hat{\mathbf{B}}$ is orthogonal to $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$, which are in the plane of C.

H-12: By Theorem 1.3.3 of the CLP-4 text, the tangential component of acceleration is $a_T(t) = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$

H-13: Use your answers to previous parts to calculate (d). Tangential and normal components of acceleration are defined just before Example 1.3.4 in the text.

H-14: (a) All points on the curve obey an equation that contains x's and y's, but no z's. There is a standard way to get a nice parametrization of this equation, that doesn't involve using square roots.

(b) You don't need to compute the constants for all points: only the given point.

H-15: For part (c), you only need to find $\hat{\mathbf{N}}$ at a point, which is easier than finding it for all t.

H-16: First parametrize $x^2 + y^2 = 1$ in the standard way. You don't need calculus for part (c).

H-17: Review §1.5 of the CLP-4 text.

H-18: Since $0 \leqslant t \leqslant 1$, you can simplify $|t| = t$.

H-19: For part (f), remember that you can write the equation of a plane easily once you know a point it passes through, and a vector normal to it. The plane should touch the curve when $t = 0$, and the plane should contain $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$.

H-20: It might be easier to find $\hat{\mathbf{B}}$ before you find $\hat{\mathbf{N}}$, then use the formula $\hat{\mathbf{N}}(t) = \hat{\mathbf{B}}(t) \times \hat{\mathbf{T}}(t)$.

H-21: The osculating plane at $\mathbf{r}(t_0)$ is the plane through $\mathbf{r}(t_0)$ with normal $\hat{\mathbf{B}}(t_0)$. Also, notice the points for parts (a) and (b) are not the same.

H-22: Since $t > 0$, we can simplify $\sqrt{t^2} = |t| = t$.

H-23: In this context, "distance travelled" means "arclength."

H-24: Use $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ to compute $\hat{\mathbf{B}}$.

H-25: (a) First find a parametrization $\big(x(\theta), y(\theta)\big)$ for $x^2 + y^2 = 1$.

H-26: You need to find the acceleration at $(1, 1, 1)$. Think about what strategies are available for computing the acceleration.

H-27: For part (a), $\mathbf{T}(t)$ will be a vector of the form $\mathbf{T}(t) = \frac{\langle 1, at, bt \rangle}{\sqrt{1+4t^2}}$ where a and b are nonzero constant real numbers.

For part (b), $\mathbf{N}(t)$ will be a vector of the form $\mathbf{N}(t) = \frac{\langle -4t, \alpha, \beta \rangle}{2\sqrt{1+4t^2}}$ where α and β are nonzero constant real numbers.

For part (e), $\kappa(t)$ will be a function of the form $\kappa(t) = \frac{\gamma}{(1+4t^2)^{3/2}}$, where γ is a positive constant real number.

H-28: Differentiate $\hat{\mathbf{N}} = \hat{\mathbf{B}} \times \hat{\mathbf{T}}$ with respect to s.

The vectors $\hat{\mathbf{N}}$, $\hat{\mathbf{B}}$, and $\hat{\mathbf{T}}$ form a right-handed triple. Sketch them (the same way you might sketch the x, y, and z axes) to figure out the signs of their cross products.

H-29: In part (b), note that $\mathbf{a}$ is the second derivative with respect to time (not θ). Exploit $\mathbf{a} = \frac{\mathrm{d}v}{\mathrm{d}t}\hat{\mathbf{T}} + v^2\kappa\hat{\mathbf{N}}$ to find what you're asked for.

H-30: For part (d), what is the relationship between the y- and z-components of the particle's position? How can you use that to find a plane containing the particle at all times t?

H-31: Rather than trying to wrangle trig identities, plug in $\theta = \pi$ as soon as you can for part (a). For part (c), remember that you need the chain rule if you want to make use of your previous derivatives.

Hints for Exercises 1.6. — Jump to TABLE OF CONTENTS.

H-1: Your differential is $\mathrm{d}s$, where s is arclength.

H-2: (a) You can parametrize the curve by $\mathbf{r}(\theta) = r(\theta)\,\cos\theta\,\hat{\boldsymbol{\imath}} + r(\theta)\,\sin\theta\,\hat{\boldsymbol{\jmath}}$, $\theta_1 \leqslant \theta \leqslant \theta_2$.

H-3: Following Definition 1.6.1, set $f(x, y, z) = \frac{xy}{z}$, $x(t) = \frac{2}{3}t^3$, $y(t) = \sqrt{3}t^2$, and $z(t) = 3t$.

H-4: Parametrize the circle in the usual way.

H-5: C can be parametrized as $(1 + t, 2 + 2t, 3 + 2t)$ for $0 \leqslant t \leqslant 1$.

H-7: Simplify! Also: $\frac{\mathrm{d}}{\mathrm{d}t}\{\operatorname{arcsec} t\} = \frac{1}{|t|\sqrt{t^2-1}}$.

H-8: Newton's law of motion is $\mathbf{F} = m\mathbf{a}$. The work done over a displacement $\mathrm{d}\mathbf{r}$ is $W = \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.

H-9: Sketch C and determine the normal vectors from the sketch. You can use x or y as the integration variable in your integrals.

H-10: (c) How is $x(t)^2 + y(t)^2$ related to $z(t)$?

(d) First, sketch $\big(x(t)\,,\,y(t)\big)$.

H-11: Remember $\bar{x} = \dfrac{\int_C x\rho\,\mathrm{d}s}{\int_C \rho\,\mathrm{d}s}$, etc. The integrals you evaluate should all be straightforward applications of the power rule.

Hints for Exercises 1.7. — Jump to TABLE OF CONTENTS.

H-1: Gravity pulls straight down, while the direction of the normal force depends on the curve of the wire. There is not enough information to know the magnitude of the forces, but you can approximate their directions.

H-2: This equation stems from $\mathbf{F} = m\mathbf{a}$. In that equation, $\mathbf{a}$ is what kind of derivative?

H-3: A thought experiment might help you avoid any calculations. If the wire were perfectly vertical or perfectly horizontal, what would $W\hat{\mathbf{N}}$ be?

H-4: The skater reaches their highest point when $|\mathbf{v}| = 0$.

H-5: The highest vertical height occurs just as the skateboarder's speed reduces to 0, at $y_S = \frac{E}{mg}$.

H-6: At the bottom of the culvert, all the skater's energy is kinetic, not potential. That is, in the equation $E = \frac{1}{2}m|\mathbf{v}|^2 + mgy$, we have $y = 0$.

H-7: Equation 1.7.2 tells us the normal force exerted by the track is $W\hat{\mathbf{N}}$, where $W = m\kappa|\mathbf{v}|^2 + mg\hat{\mathbf{k}} \cdot \hat{\mathbf{N}}$. Equation 1.3.3 part (c) says $\mathbf{a}(\theta) = \frac{\mathrm{d}^2 s}{\mathrm{d}\theta^2}\hat{\mathbf{T}} + \kappa\left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^2\hat{\mathbf{N}}$.

H-8: When $\theta = 13\pi/3$, $\frac{\mathrm{d}^2 s}{\mathrm{d}\theta^2} = 0$, which is handy for a quicker calculation.

Important equations: the normal force exerted by the track is $W\hat{\mathbf{N}}$, where $W = m\kappa|\mathbf{v}|^2 + mg\hat{\boldsymbol{\jmath}} \cdot \hat{\mathbf{N}}$ (Equation 1.7.2); $\quad \mathbf{a}(\theta) = \frac{\mathrm{d}^2 s}{\mathrm{d}\theta^2}\hat{\mathbf{T}} + \kappa\left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^2\hat{\mathbf{N}}$ (Equation 1.3.3, part (c)).

H-9: According to the equation in the text, the skiier will become airborne when:

$$|\mathbf{v}| > \sqrt{\frac{g}{\kappa}|\hat{\boldsymbol{\jmath}} \cdot \hat{\mathbf{N}}|}$$

So, we need $|\mathbf{v}|$ to be greater than $\sqrt{\frac{g}{\kappa}|\hat{\boldsymbol{\jmath}} \cdot \hat{\mathbf{N}}|}$ for *some* point on the curve inside the range $1/e \leqslant t \leqslant e$.

Note that g is given in metres per second, while the other quantities are in kilometres and hours.

H-10: There are now three forces acting on the bead: one parallel to $\hat{\boldsymbol{\jmath}}$ (exerted by gravity), one parallel to $\hat{\mathbf{N}}$ (exerted by the wire), and one parallel to $\hat{\mathbf{T}}$ (exerted by the jet pack).

Follow the reasoning in the sliding bead section of the text, focusing on the *tangential* forces.

H-11: If the snowmachine is moving at a constant speed, the tangential component of its acceleration is zero. Part (a) is similar to Question 10.

H-12: Follow the discussion in the text.

It's fine to leave part (b) pretty messy. Your answer for part (c) involves the root of a cubic function, but you don't need a high degree of accuracy to decide between the three options give.

Hints for Exercises 1.8. — Jump to TABLE OF CONTENTS.

H-2: r is allowed to be negative.

H-3: Compute, for each angle θ, the dot product $\hat{\mathbf{e}}_r(\theta) \cdot \hat{\mathbf{e}}_\theta(\theta)$.

H-5: The curve can be parametrized by $\mathbf{r}(\theta) = f(\theta)\big[\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}\big]$

Hints for Exercises 2.1. — Jump to TABLE OF CONTENTS.

H-1: Not all blanks represent a single interval.

H-2: Write down all coordinates where $\mathbf{v}(x,y)\cdot\hat{\imath} = 0$ or $\mathbf{v}(x,y)\cdot\hat{\jmath} = 0$, and look for a pattern.

H-3: If you know the speed and direction of an object, you can find its velocity.

H-4:

H-5: When the twig is at (x,y) it has velocity $\mathbf{v}(x,y)$.

H-6: Whenever the twig is on the y-axis, its velocity is parallel to the y-axis. So it remains on the y-axis for all time.

H-7: If you know the speed and direction of an object, you can find its velocity.

H-8: Set your face to be at the origin, $(0,0,0)$.

If A is "inversely proportional" to B, then there exists a constant α such that $AB = \alpha$. That way when $|B|$ goes up, $|A|$ goes down, and vice-versa.

H-9: Start with the regions where $\mathbf{v}(x,y)\cdot\hat{\imath}$ and $\mathbf{v}(x,y)\cdot\hat{\jmath}$ are positive and negative. As you move up/down/left/right, do the vectors get longer or shorter? More horizontal or more vertical?

H-10: $\mathbf{v}(x,y)\cdot\hat{\imath}$ is the distance from (x,y) to the origin, while $\mathbf{v}(x,y)\cdot\hat{\jmath}$ is the distance from (x,y) to the point $(1,1)$.

H-11: Factor $x^2 + xy = x(x+y)$ and $y^2 - xy = y(x-y)$. Chop the plane up into eight regions using the two coordinate axes and the lines $y = x$, $y = -x$.

H-12: What is the geometric interpretation of each summand?

H-13: (a), (c) Intrepret the vector field geometrically.

H-14: The constant G is the same for all masses, but M differs. The net force is the sum of three force vectors.

H-15: For part a., make a triangle with P as one of its vertices that is similar to the triangle made by the pole, the wall, and the ground. Its hypotenuse has length p; let its base be b and its height be h. Find a way to translate between (b, h) and (x, y).

For part b., use your answer from part a. Start by describing a point on a pole as its distance from the lower end of the pole, p. Then, consider $\frac{\mathrm{d}z}{\mathrm{d}t}$ and $\left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}\right)$ separately. If you're having a hard time simplifying your answer, note $\sqrt{x^2+y^2} = \sqrt{3}(1-z)$ for any point (x, y, z) on a pole when $H = 1$.

Hints for Exercises 2.2. — Jump to TABLE OF CONTENTS.

H-2: Express $x'(t)$ and $y'(t)$ purely in terms of $x(t)$ and $y(t)$.

H-3: Review §2.2 in the CLP-4 text.

Hints for Exercises 2.3. — Jump to TABLE OF CONTENTS.

H-1: Carefully consider the context that lead to each of these equations.

H-2: One of the three options will NEVER be true, for any $\mathbf{F}$.

H-3: Modify φ, the potential for $\mathbf{F}$.

H-4: a. If $\mathbf{F} + \mathbf{G}$ is conservative, what has to be true?
b. What if $\mathbf{F}$ and $\mathbf{G}$ are quite similar?
c. Find a potential for $\mathbf{F} + \mathbf{G}$.

H-5: Note that the domain is $D = \{\, (x, y) \mid x > 1 \,\}$. Compare to Example 2.3.14 in the text.

H-6: A potential does exist.

H-7: Recall $\frac{\mathrm{d}}{\mathrm{d}x} \ln |x| = \frac{1}{x}$.

H-8: Try the screening test, Theorem 2.3.9.

H-9: $\displaystyle\int \frac{x}{x^2+y^2+z^2}\,\mathrm{d}x$ can be evaluated by inspection, or with the substitution $u = x^2+y^2+z^2$.

H-11: For what values of the constants A and B does the vector field $\mathbf{F}$ pass the screening test $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$?

H-12: Review Example 2.1.2 in the CLP-4 text.

H-13: Following Example 2.3.3, the particle can never escape the region

$$\{\, (x, y, z) \mid \varphi(x, y, z) \geqslant -E \,\}$$

where E is the energy of the system.

H-14: Example 2.3.3 tells us $\frac{1}{2}m|\mathbf{v}(t)|^2 - \varphi\big(x(t), y(t), z(t)\big) = E$ is a constant quantity, provided $\mathbf{F}$ is conservative with potential $\varphi(x, y, z)$.

H-15: Find a potential φ. Notice f, g, and h are functions of one variable each — this simplifies things.

H-16: Write the points with curl $\mathbf{0}$ as multiples of a constant vector.

Hints for Exercises 2.4. — Jump to TABLE OF CONTENTS.

H-1: The top and bottom of the square can be easily paramerized using x as the parameter. The other two sides can be easily parameterized using y as the parameter.

H-2: Contrast Theorems 2.4.7 and 2.3.9.

H-3: Please don't do any computation, especially not to find C!

H-4: Review properties of conservative vector fields.

H-5: Review Theorem 2.4.6 in the text.

H-6: Review Theorem 2.4.6 in the text.

H-7: Part (d) is a hint.

H-8: The last part of the question is a huge hint.

H-11: Parametrize the curve using y as a parameter.

H-12: Use Theorems 2.4.6 and 2.4.7 in the CLP-4 text.

H-13: (a) Use Theorem 2.4.7 in the CLP-4 text.

(c) You may parametrize the curve using x as the parameter. Exploit the fact that, for the value of λ found in part (a), $\mathbf{F} + \lambda\mathbf{G}$ is conservative.

H-15: Parametrize the path using sines and cosines. The work done is $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$

H-16: Is $\mathbf{F}$ conservative?

H-17: Is $\mathbf{F} = xy\,\hat{\boldsymbol{\jmath}}$ conservative? Sketch C.

H-18: That the line integral is to be independent of path is a huge hint.

H-19: Note that

- $y = 0$ on the line segment from $(1, 0, 0)$ to $(0, 0, 1)$ and
- $x = 0$ on the line segment from $(0, 0, 1)$ to $(0, 1, 0)$ and
- $z = 0$ on the line segment from $(0, 1, 0)$ to $(1, 0, 0)$

H-20: That $\mathbf{F}$ is conservative should be a dead giveaway.

H-21: To calculate the integral, it might be easier to find a potential for $\mathbf{F}$ and use Theorem 2.4.2.

H-22: Your answer from (b) can help you in (c). Also, $\cos(1) = \cos(-1)$, because cosine is an even function.

H-23: Review §2.4.1 of the text.

H-24: Relate the integral of part (d) to the integral of part (c).

H-25: Write the integral of part (c) as $\int_C \mathbf{G} \cdot d\mathbf{r}$. What is the difference between **G** and **F**?

H-26: (d) How are **G** and **F** related?

H-27: (a) Start with $\frac{\partial f}{\partial z} = y^2 e^{yz}$.

(b) Use the result of part (a) to do part (b).

H-28: The integral in part (b) is path independent. That's a big hint.

H-29: Part (a) is a hint for part (b).

H-30: The three parts of this problem are closely related.

H-31: We can rewrite $x^2 + y^2 + z^2 = 2z$ as $x^2 + y^2 + (z-1)^2 = 1$.

H-32: (a) The curve can be easily parametrized by using x as a parameter.

(b) Don't evaluate the integral directly.

H-33: Refer to Example 1.4.4 for a parametrization of a helix.

H-34: (b) Parametrize each side of the square by arc length, and make use of the plentiful zeroes that arise.

H-35: Force is mass times acceleration, where acceleration is the second derivative of position, $\mathbf{r}(t)$, with respect to time, t. The work done by **F** between time a and time b is $\int_a^b \mathbf{F} \cdot d\mathbf{r}$.

H-36: Note that the curve goes from $(2,2)$ to $(1,1)$ — not the other way around.

For part (b), one possibility is to look for a path consisting of the line segment from $(2,2)$ to $(2,Y)$, followed by the line segment from $(2,Y)$ to $(1,Y)$, followed by the line segment from $(1,Y)$ to $(1,1)$, with Y being a parameter to be determined.

H-37: One possibility is to look for a path consisting of the line segment from $(0,0)$ to $(0,Y)$, followed by the line segment from $(0,Y)$ to $(2,Y)$, followed by the line segment from $(2,Y)$ to $(2,0)$, with Y being a parameter to be determined.

H-38: Is **F** conservative?

H-39: On S, note $z = 2 + x^2 - 3y^2$. Further, the vector field $\tilde{\mathbf{F}}(x,y,z) = z^2\,\hat{\mathbf{k}}$ is conservative (with potential $\frac{1}{3}z^3$), so $\int_{C_1} \tilde{\mathbf{F}} \cdot d\mathbf{r} = \int_{C_2} \tilde{\mathbf{F}} \cdot d\mathbf{r}$ for any two curves C_1 and C_2 from P_1 to P_2. Compare this to Questions 25 through 26.

H-40: Simplify the answer in part (a) as much as possible.

For part (c), start with $\frac{\partial f}{\partial y} = xe^{3x^2}$ and $\frac{\partial f}{\partial z} = x^2\cos(x^2 z)$.

For part (d), notice the difference between the given vector field and the conservative vector field of part (c). The resulting integral can be directly evaluated using methods from integral calculus.

H-41: For (b), remember $\frac{\mathrm{d}s}{\mathrm{d}t} = \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|$
Is the vector field of part (c) conservative?

H-42: For part (d), what is the difference between J and $\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$?
For part (e), many parts of the integral are zero: find as many as you can.

H-43: By Newton's law of motion, $m\mathbf{r}''(t) = \mathbf{F}(t)$.

Recall $\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$.

H-44: (a) Remember the arclength of the parametrized path $\mathbf{r}(t)$ from $t = a$ to $t = b$ is given by $\int_a^b |\mathbf{r}'(t)|\,\mathrm{d}t$. In this case, $|\mathbf{r}'(t)|$ can be simplified considerably.
(b) Remember $\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$.
(c) Gravity is conservative. Friction is not conservative.
(d) What are the tangential and normal components of acceleration?

Hints for Exercises 3.1. — Jump to TABLE OF CONTENTS.

H-1: Your answer will have the form $\mathbf{r}(x,y) = \psi_1(x,y)\hat{\pmb{\imath}} + \psi_2(x,y)\hat{\pmb{\jmath}} + \psi_3(x,y)\hat{\mathbf{k}}$.

H-3: First think about what properties $\mathbf{r}(u,v)$ has to have in order to be a parametrization.

H-4: First think about what properties $\mathbf{r}$ has to have in order to be a parametrization.

H-5: First think about what properties $\mathbf{r}(u,v)$ has to have in order to be a parametrization.

Hints for Exercises 3.2. — Jump to TABLE OF CONTENTS.

H-1: What are the tangent planes to the two surfaces at $(0,0,0)$?

H-2: Apply the chain rule to $G\big(\mathbf{r}(t)\big) = 0$.

H-4: To find a tangent vector to the curve of intersection of the surfaces $F(x,y,z) = 0$ and $G(x,y,z) = 0$ at (x_0,y_0,z_0), use Q[2] twice, once for the surface $F(x,y,z) = 0$ and once for the surface $G(x,y,z) = 0$.

H-5: To find a tangent vector to the curve of intersection of the surfaces $z = f(x,y)$ and $z = g(x,y)$ at (x_0,y_0,z_0), use Q[2] twice, once for the surface $z = f(x,y)$ and once for the surface $z = g(x,y)$.

H-10: Review §3.2 in the CLP-4 text.

H-11: Review §3.2 in the CLP-4 text.

H-13: Let (x,y,z) be a desired point. Then

- (x,y,z) must be on the surface and
- the normal vector to the surface at (x,y,z) must be parallel to the plane's normal vector.

H-14: First find a parametric equation for the normal line to S at (x_0, y_0, z_0). Then the requirement that $(0, 0, 0)$ lies on that normal line gives three equations in the four unknowns x_0, y_0, z_0 and t. The requirement that (x_0, y_0, z_0) lies on S gives a fourth equation. Solve this system of four equations.

H-15: Two (nonzero) vectors $\mathbf{v}$ and $\mathbf{w}$ are parallel if and only if there is a t such that $\mathbf{v} = t\,\mathbf{w}$. Don't forget that the point has to be on the hyperboloid.

H-16: (b) If $\mathbf{v}$ is tangent, at a point P, to the curve of intersection of the surfaces S_1 and S_2, then $\mathbf{v}$

- has to be tangent to S_1 at P, and so must be perpendicular to the normal vector to S_1 at P and
- has to be tangent to S_2 at P, and so must be perpendicular to the normal vector to S_2 at P.

H-17: The angle between the curve and the surface at P is $90°$ minus the angle between the curve and the normal vector to the surface at P.

H-18: At the highest and lowest points of the surface, the tangent plane is horizontal.

Hints for Exercises 3.3. — Jump to TABLE OF CONTENTS.

H-1: S is a very simple geometric object.

H-2: The triangle is part of the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

H-3: Flatten S out.

H-8: The total surface area of (b) (ii) can be determined without evaluating any integrals.

H-11: On S, (x, y) runs over the interior of $x^2 + y^2 = 2x$, or equivalently, the interior of $(x-1)^2 + y^2 = 1$.

H-12: See Example 3.1.5 of the CLP-4 text for a parametrization of the torus.

H-13: Call the part of the sphere in the first octant S. By definition, the centroid is $(\bar{x}, \bar{y}, \bar{z})$ with

$$\bar{x} = \frac{\iint_S x\,\mathrm{d}S}{\iint_S \mathrm{d}S} \qquad \bar{y} = \frac{\iint_S y\,\mathrm{d}S}{\iint_S \mathrm{d}S} \qquad \bar{z} = \frac{\iint_S z\,\mathrm{d}S}{\iint_S \mathrm{d}S}$$

The integrals will be easy if you use spherical coordinates. You can reduce the number of integrals evaluated by using symmetry.

H-14: Before parametrizing the cylinder, express $x^2 + y^2 = 2ay$ in cylindrical coordinates.

H-16: (a) The integral can be easily evaluated by using that the sphere has surface area $4\pi a^2$.

(c) Use cylindrcial coordinates for the top part of the cone.

H-20: Beware of signs. Note that $0 \leqslant z \leqslant 1$ on $\mathcal{S}$.

H-21: The z-coordinate of the centre of mass is the weighted average of the z-coordinate over the cone. Since a density has not been specified, we assume that it is a constant. We may take the density to be 1, so the z-coordinate of the centre of mass is $\iint_S z\,\mathrm{d}S / \iint_S \mathrm{d}S$.

H-22: Use cylindrcal coordinates. Note that because of the symmetry of the cone, only the z-component of the centre of mass requires an integral to be calculated. The z-coordinate of the centre of mass is the weighted average of the z-coordinate over the cone. That is $\bar{z} = \iint_S z\,\mathrm{d}S / \iint_S \mathrm{d}S$.

H-24: Review (3.3.2) in the CLP-4 text.

H-25: Don't be afraid to tweak spherical coordinates so as to fit the condition $x \geqslant \sqrt{y^2+z^2}$ well. To do so, first use a sketch to develop a geometric interpretation of $\frac{\sqrt{y^2+z^2}}{x}$.

H-26: The surface S may be parametrized by observing that, for each fixed y, $x^2+z^2 = \sin^2 y$ is a circle.

H-27: By symmetry, the centre of mass will lie on the z-axis. By definition, the z-coordinate of the centre of mass is the weighted average of z over S, which is

$$\bar{z} = \frac{\iint_S z\,\rho(x,y,z)\,\mathrm{d}S}{\iint_S \rho(x,y,z)\,\mathrm{d}S}$$

H-37: You can use the the cylindrical coordinates θ and z to parametrize the hyperboloid.

H-38: (a) Review §3.2 of the CLP-4 text.

(b) Review §3.3.1 of the CLP-4 text.

H-39: (a) Review §2.4.1 in the CLP-4 text.

(b) Use Lemma 2.3.6 of the CLP-4 text to show that the integrand is identically zero.

Hints for Exercises 4.1. — Jump to TABLE OF CONTENTS.

H-2: Compute $\boldsymbol{\nabla} \times \mathbf{F}$ for some simple vector fields.

H-3: For parts(a) and (b), write out the definitions of the left and right hand sides and observe that they are equal. Part (c) can be done easily by using other, simpler, vector identities.

H-6: (c) can be done efficiently by using (a) and (b).

H-12: (a) Find the magnitude and direction of the velocity vector. Then verify that $\boldsymbol{\Omega} \times \mathbf{r}$ has that magnitude and direction.

Hints for Exercises 4.2. — Jump to TABLE OF CONTENTS.

H-3: (b) The integral can be trivially evaluated by exploiting oddness and the fact that $\iiint_V \mathrm{d}V = \text{Volume}(V)$.

H-4: For part (a), use spherical coordinates.

H-5: (a) The integral is easier in polar coordinates.

(b) Since x is odd, $\iiint_V x\,\mathrm{d}V = 0$.

H-6: (a) The integral is easy in polar coordinates.

(b) The volume of the solid can be easily computed by decomposing the solid into thin horizontal pancakes. See Section 1.6 in the CLP-2 text.

H-7: The divergence theorem, of course.

H-8: It's easier to use the divergence theorem. But don't forget the base of the silo.

H-9: The divergence theorem, of course. The integral can be easily evaluated by using that, for any solid $\mathcal{V}$ in $\mathbb{R}^3$,

$$\iiint\limits_{\mathcal{V}} \mathrm{d}V = \text{Volume}(\mathcal{V}) \qquad \bar{x} = \frac{\iiint_{\mathcal{V}} x\,\mathrm{d}V}{\text{Volume}(\mathcal{V})} \qquad \bar{y} = \frac{\iiint_{\mathcal{V}} y\,\mathrm{d}V}{\text{Volume}(\mathcal{V})} \qquad \bar{z} = \frac{\iiint_{\mathcal{V}} z\,\mathrm{d}V}{\text{Volume}(\mathcal{V})}$$

where $(\bar{x}, \bar{y}, \bar{z})$ is the centroid of $\mathcal{V}$.

H-10: The complexity of **F** is a hint that the flux should not be evaluated directly.

H-11: The specified surface is not closed.

H-12: (a), (b), (c) Review warning 4.2.3 in the CLP-4 text.

(d) The divergence theorem can be used — with care.

(e) The equation can be made more understandable by completing a square.

H-13: (a) Use a suitable modification of spherical coordinate. Do not forget to specify the range of the parameters.

H-14: Don't evaluate the flux directly.

H-15: For practice, try doing this question twice — once using the divergence theorem and once using direct evaluation.

H-16: The question highlights that the vector field has divergence 0. Thta's a big hint.

H-17: As **F** looks complicated, it is probably wise not to try and evaluate the flux integral directly.

H-18: As **F** looks complicated, it is probably wise not to try and evaluate the flux integral directly.

H-19: The vector field **F** looks complicated. Try to avoid a direct evaluation of the flux integral.

H-20: The divergence of **F** is a lot simpler than **F** itself. By default, we want the outward flux.

H-21: The vector field **F** looks very complicated. That strongly suggests that we not evaluate the integral directly.

H-22: The divergence of $\mathbf{F}$ is a lot simpler than $\mathbf{F}$ itself.

H-23: Note that $\mathbf{F}(x, y, z)$ is not defined at $(x, y, z) = (0, 0, 0)$.

H-25: The surface S is not a closed surface.

H-29: The complexity of $\mathbf{F}$ is a hint that the flux should not be evaluated directly.

H-31: The flux can be calculated directly, but it is rather easier to calculate it using the Divergence Theorem.

H-32: Use that y is odd to easily evaluate some integrals.

H-34: (a) Use cylindrical coordinates.

(b) The volume of the V can be easily computed by decomposing V into thin horizontal washers. See Section 1.6 in the CLP-2 text.

H-35: Review the derivation of the heat equation in Section 4.2.1 of the CLP-4 text.

H-37: Make a judicious choice of parametrization.

H-38: Do not compute the integral directly.

H-39: Be careful about which normals to use in part (c). For practice, try to do part (c) in two different ways, with one being direct evaluation.

H-40: For part (b), do not evaluate the flux directly. In part (c), the flux can be related to the volume enclosed by the surface, and the centre of mass of the volume enclosed by the surface.

H-41: (b) We have several different methods for evaluating flux integrals. Think about what would be involved in applying each of them before settling on which one to use.

(c) Be sneaky — don't evaluate this integral directly.

H-42: For parts (b) and (c), write out carefully the integral that the divergence theorem gives you.

H-43: Note that $2^2 + 2(1^2) + 3(1)^2 = 9 < 16$ so that $(2, 1, 1)$ is inside S,
while $3^2 + 2(2^2) + 3(2)^2 = 29 > 16$ so that $(3, 2, 2)$ is outside S.

H-44: Review §4.2.2 in the CLP-4 text.

H-46: Consider very small a's.

H-47: Carefully draw a side view of S.

H-48: Both the divergence theorem and a vector identity in Theorem 4.1.4 of the CLP-4 text are useful.

H-49: x is an odd function.

H-50: You should be able to guess the centre of mass, $(\bar{x}, \bar{y})$ of the disk D. Then the integrals $\iint_D x \,\mathrm{d}x\mathrm{d}y$ and $\iint_D y \,\mathrm{d}x\mathrm{d}y$ can be found by using $\bar{x} = \frac{\iint_D x \,\mathrm{d}x\mathrm{d}y}{\iint_D \mathrm{d}x\mathrm{d}y}$ and $\bar{y} = \frac{\iint_D y \,\mathrm{d}x\mathrm{d}y}{\iint_D \mathrm{d}x\mathrm{d}y}$.

Hints for Exercises 4.3. — Jump to TABLE OF CONTENTS.

H-2: Let $\mathbf{r}(s) = x(s)\,\hat{\boldsymbol{\imath}} + y(s)\,\hat{\boldsymbol{\jmath}}$ be a counterclockwise parametrization of C by arc length. Then $\hat{\mathbf{T}}(s) = \mathbf{r}'(s) = x'(s)\,\hat{\boldsymbol{\imath}} + y'(s)\,\hat{\boldsymbol{\jmath}}$ is the forward pointing unit tangent vector to C at $\mathbf{r}(s)$ and $\hat{\mathbf{n}}(s) = \mathbf{r}'(s) \times \hat{\mathbf{k}} = y'(s)\,\hat{\boldsymbol{\imath}} - x'(s)\,\hat{\boldsymbol{\jmath}}$. To see that $\mathbf{r}'(s) \times \hat{\mathbf{k}}$ really is $\hat{\mathbf{n}}(s)$, note that $y'(s)\,\hat{\boldsymbol{\imath}} - x'(s)\,\hat{\boldsymbol{\jmath}}$

- has the same length, namely 1, as $\mathbf{r}'(s)$ (recall that $\mathbf{r}(s)$ is a parametrization by arc length),
- lies in the xy-plane and
- is perpendicular to $\mathbf{r}'(s)$. (Check that $\mathbf{r}'(s) \cdot \big[y'(s)\,\hat{\boldsymbol{\imath}} - x'(s)\,\hat{\boldsymbol{\jmath}}\big] = 0$.)
- Use the right hand rule to check that $\mathbf{r}'(s) \times \hat{\mathbf{k}}$ is $\hat{\mathbf{n}}$ rather than $-\hat{\mathbf{n}}$.

H-3: Use direct evaluation!

H-4: The functions $\frac{x}{x^2+y^2}$ and $\frac{-y}{x^2+y^2}$ are not defined, let alone continuous or differentiable, at $x = y = 0$.

H-5: For practice, evaluate this integral twice — once directly and once using Green's theorem.

H-6: The $\sin y^2$ and $\cos y^2$ in the integrand look hard to integrate. Try Green's theorem.

H-7: Don't do the integral directly.

H-8: Don't do the integral directly. Sketch the rectangle.

H-9: Do not compute the integral directly.

H-10: Don't do the integral directly. Sketch the triangle.

H-11: The integrand for direct evaluation looks complicated — don't evaluate this integral directly.

H-12: Direct evaluation is not the most efficient method available.

H-13: Green's theorem must be applied to a closed curve; note that the curve C is not closed.

Consider carefully the point $(0, 0)$ in your analysis.

You may use the fact that $\int \frac{\mathrm{d}t}{1+t^2} = \arctan(t) + C$.

H-14: If we were to try to evaluate this integral directly, then on the $y = x^2 - 4x + 3$ part of C, the integrand would contain $x^2 e^y = x^2 e^{x^2-4x+3}$. That looks hard to integrate, so try Green's theorem.

H-15: Beware the point $(0, 0)$.

H-18: It is possible to evaluate this integral by three different methods, one of them being direct evaluation (though it requires some ingenuity). Try to find all three.

H-20: Write $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} - A \oint_C \mathbf{G} \cdot \mathrm{d}\mathbf{r} = \oint_C (\mathbf{F} - A\mathbf{G}) \cdot \mathrm{d}\mathbf{r}$.

H-21: Note that $\mathbf{F}(x, y)$ is not defined at $(x, y) = (0, 0)$.

H-22: Note that $\mathbf{F}(x, y)$ is not defined at $(x, y) = (0, 0)$.

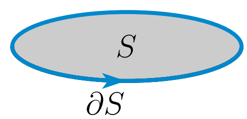
H-24: (a) All points on the curve obey an equation that contains x's and y's, but no z's.

(b) Exploit conservativeness as much as possible.

H-25: Use Green's theorem to convert the integral over C into an integral over the region R in the xy-plane whose boundary is C. Consider the sign of the integrand of the integral over R.

Hints for Exercises 4.4. — Jump to TABLE OF CONTENTS.

H-1: One approach is to first do



Then imagine slowly deforming the sketch to the get specified S's

H-2: Define the vector field $\mathbf{F}(x, y, z) = F_1(x, y)\,\hat{\imath} + F_2(x, y)\,\hat{\jmath}$.

H-3: First verify the vector identity $\boldsymbol{\nabla} \times [\phi\boldsymbol{\nabla}\psi + \psi\boldsymbol{\nabla}\phi] = \mathbf{0}$

H-4: To parametrize the curve $x^2 + y^2 = 1$, $z = y^2$, first parametrize the circle $x^2 + y^2 = 1$. That is, find $x(t)$ and $y(t)$ obeying $x(t^2) + y(t)^2 = 1$. Then set $z(t) = y(t)^2$.

H-5: Apply Stokes' theorem. Note that $\mathbf{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{\mathbf{k}}$ obeys $x(t) + y(t) + z(t) = 3$, for every t, and that $x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} = (1 + \cos t)\,\hat{\imath} + (1 + \sin t)\,\hat{\jmath}$ runs counterclockwise around the circle of radius 1 centered on $(1, 1)$.

H-6: The form of the integral should be quite suggestive.

H-7: The form of the integral should be quite suggestive.

H-8: What's the title of this section?

H-9: We are to evaluate a flux integral of the form $\iint_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$. Sure looks like one side of Stokes' theorem.

H-10: The vector field $\mathbf{F}$ looks too complicated for a direct evaluation of the line integral. So, try Stokes' theorem.

H-16: All three vertices of part (a) lie in the plane of part (b).

H-17: The curve C is the boundary of a surface. To guess the surface express the z component of $\mathbf{r}(t)$ in terms of the x and y components.

H-18: The fact that the surface is not completely specified is a big hint.

H-19: We are to evaluate the line integral of a complicated vector field around a relatively complicated closed curve. (Sketch it!) That certainly suggests that we should not try to evaluate the integral directly.

H-20: The integral looks messy. Compute the curl of $\mathbf{F}$ to help gauge if Stokes' theorem would be easier.

H-21: The form of the integrand is sugestive.

H-26: Let D be the disk in the plane $x+y+z=3$ whose boundary is C. Suppose that, as (x,y,z) runs over D, (x,y) runs over the ellipse D_{xy}. We are told that the area of D is πR^2, but we are not told the area of D'. So it is easier to deal with the integral $\iint_D \mathrm{d}S$ than with the integral $\iint_{D'} \mathrm{d}x\mathrm{d}y$.

H-29: Given the form of $\mathbf{F}$, direct evaluation looks hard.

The integral evaluations can be greatly simplified by using that the centroid $(\bar{x},\bar{y})$ of any region R in the xy-plane is

$$\bar{x} = \frac{\iint_R x\,\mathrm{d}x\,\mathrm{d}y}{\mathrm{Area}(R)} \qquad \bar{y} = \frac{\iint_R y\,\mathrm{d}x\,\mathrm{d}y}{\mathrm{Area}(R)}$$

H-30: Part (a) is a hint for part (b). Sketch the curve in part (b).

H-31: For practice, evaluate the flux of part (a) twice — once by direct evaluation and once using Stokes' theorem.

H-32: By definition, D is connected if any two points in D can be joined by a curve that lies completely in D.

By definition, D is simply connected if any simple closed curve in D can be shrunk to a point continuously in D.

H-33: Review §4.1.2 in the CLP-4 text.

H-34: Considering that there are ten line segments in C, it is probably not very efficient to use direct evaluation.

H-35: Direct evaluation looks hard.

H-36: Rewrite $\oint_C \mathbf{E}\cdot\mathrm{d}\mathbf{r}$ as a surface integral.

H-37: What is $x(t)^2+y(t)^2+z(t)^2=2$? How is $x(t)$ relatex to $z(t)$?

H-38: The intersection of the plane $x+y+z=1$ with the sphere $x^2+y^2+z^2=1$ is a circle. Use symmetry to guess the centre of the circle.

H-39: Sketch S.

H-40: You can avoid evaluating any integral by identifying S' as a simple geometric figure.

Hints for Exercises 5. — Jump to TABLE OF CONTENTS.

H-2: Read (d), (e), (f), (g), (h) very carefully.

H-3: Beware that in part (f) a surface is defined to be closed if and only if it is the boundary of a solid region E. Even though that is not the usual definition, it is be used in this question.

H-4: (b) In general, for which values of x is the curvature of $y = f(x)$ zero?

(c) First parametrize $x^2 + z^2 = 1$.

(d) First determine when $\mathbf{r}(u, v)$ has $z = 0$.

(e) What type of curve has curvature zero?

(f) What theorem relates the divergence of a vector field with flux integrals of the vector field?

(g) What is the screening test for conservativeness in two dimensions?

(h) What is the definition of "parametrized by arclength"?

(i) What theorem relates line integrals to curls?

(j) What theorem relates flux integrals to divergences?

(k) Use Stokes' theorem.

H-5: Read all of the statements very carefully. The details are critical.

(a) Note the word *anywhere*.

(b) If you have not learned about simply connected domains, skip this part. If you have, read the statement very carefully.

(d) If you have not learned about Kepler's three laws, skip this part.

(h) Read the statement very carefully. It does not specify that C is closed.

(i) Review §1.5 of the CLP-4 text.

H-8: Read all of the statements very carefully. The details are critical.

For part (d), note that the curve need not lie in a plane.

For part (g), note that the domain can have holes in it.

For parts (h) and (i), by definition, D is simply connected if any simply closed curve in D can be shrunk to a point continuously in D.

H-9: Read all of the statements very carefully. The details are critical.

For part (b), note that the curve need not lie in a plane.

For part (d), note that the domain can have holes in it.

For parts (i) and (j), by definition, D is simply connected if any simply closed curve in D can be shrunk to a point continuously in D.

H-10: Read all of the statements very carefully. The details are critical.

(a) The integral $\int_C f \, \mathrm{d}s = 0$ is not of the form $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.

(d) $\mathbf{F}$ and $\mathbf{G}$ can be any vector fields.

(e) Think about how $\int_C f \, \mathrm{d}s$ is defined.

(f) Look at $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial u}$ *very* closely.

(g) The integral is completely independent of $x(u,v)$ and $y(u,v)$.

H-11: Read all of the statements very carefully. The details are critical.

(b) Read the statement very carefully. "simply connected" plays no role here. The vector field $\mathbf{F}$ is not required to be conservative.

(e) Recall that S is closed when it is the boundary of a solid region V.

(g) Assume that the constant $|\mathbf{v}|$ is not zero.

(j) If you have not learned about Kepler's three laws, skip this part.

H-13: (g) Be careful. The power in the denominator is important.

(j) Beware the sign.

H-20: Review Corollary 4.3.5 in the CLP-4 text.

H-21:

(b) $\iint_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$ is a flux integral over the closed surface S.

(c) Consider $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} - \oint_C \mathbf{G} \cdot \mathrm{d}\mathbf{r} = \oint_C (\mathbf{F} - \mathbf{G}) \cdot \mathrm{d}\mathbf{r}$.

Part III

ANSWERS TO PROBLEMS

Answers to Exercises 1.1 — Jump to TABLE OF CONTENTS

A-1: (a) $\mathbf{r}(y) = \sqrt{a^2 - y^2}\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}}$, $0 \leqslant y \leqslant a$

(b) $\big(x(\phi), y(\phi)\big) = \big(a\sin\phi, -a\cos\phi\big)$, $\frac{\pi}{2} \leqslant \phi \leqslant \pi$

(c) $\big(x(s), y(s)\big) = \big(a\cos(\frac{\pi}{2} - \frac{s}{a}), a\sin(\frac{\pi}{2} - \frac{s}{a})\big)$, $0 \leqslant s \leqslant \frac{\pi}{2}a$

A-2: $(1, 25)$, $(-1/\sqrt{2}, 0)$, $(0, 25)$.

A-3: The curve crosses itself at all points $(0, (\pi n)^2)$ where n is an integer. It passes such a point twice, $2\pi n$ time units apart.

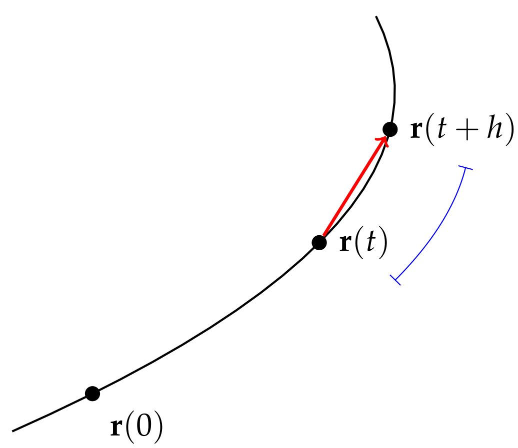
A-4: (a) $(a + a\theta, a)$ (b)$(a + a\theta + a\sin\theta, a + a\cos\theta)$

A-5: $z = -\frac{1}{2}\sqrt{1 - \frac{y^2}{2}} - \frac{y}{4}$

A-6: The particle is moving upwards from $t = 1$ to $t = 2$, and from $t = 3$ onwards. The particle is moving downwards from $t = 0$ to $t = 1$, and from $t = 2$ to $t = 3$.

The particle is moving faster when $t = 1$ than when $t = 3$.

A-7:



The red vector is $\mathbf{r}(t+h) - \mathbf{r}(t)$. The arclength of the segment indicated by the blue line is the (scalar) $s(t+h) - s(t)$.

Remark: as h approaches 0, the curve (if it's differentiable at t) starts to resemble a straight line, with the length of the vector $\mathbf{r}(t+h) - \mathbf{r}(t)$ approaching the scalar $s(t+h) - s(t)$. This step is crucial to understanding Lemma 1.1.3 in the CLP-4 text.

A-8: Velocity is a vector-valued quantity, so it has both a magnitude and a direction. Speed is a scalar — the magnitude of the velocity. It does not include a direction.

A-9: (c)

A-10: See the solution.

A-11: (d)

A-12: velocity $= -a\sin t\,\hat{\imath} + a\cos t\,\hat{\jmath} + c\,\hat{\mathbf{k}}$ speed $= \sqrt{a^2+c^2}$
acceleration $= -a\cos t\,\hat{\imath} - a\sin t\,\hat{\jmath}$

The path is a helix with radius a and with each turn having height $2\pi c$.

A-13: (a) $\hat{\mathbf{T}}(1) = \frac{(2,0,1)}{\sqrt{5}}$ (b) $\frac{1}{3}\left[5^{3/2}-8\right]$

A-14: 2

A-15: length $= \sqrt{a^2+b^2}\,T$

A-16: 1

A-17: (a) $20/3$ (b) $x(t) = -2\pi - 2t,\ y(t) = -2\pi t,\ z(t) = \pi^3/3 + \pi^2 t$

A-18: (a) $\mathbf{r}'(t) = \big(-3\sin t, 3\cos t, 4\big)$ (b) 5

A-19: (a) $x(\theta) = 3\cos\theta,\ y(\theta) = 3\sin\theta,\ z(\theta) = 6\cos\theta + 9\sin\theta,\ 0 \leqslant \theta \leqslant 2\pi$

(b) $s = \int_0^{2\pi} \sqrt{45 + 45\cos^2\theta - 108\sin\theta\cos\theta}\,d\theta$

A-20: (a) $\frac{1}{27}\big(10\sqrt{10}-1\big)$ (b) $\frac{2}{27}\big(10\sqrt{10}-1\big)$

A-21: $s(t) = \frac{t^3}{3} + \frac{t}{2}$

A-22: $\frac{8}{27}\left[\left(2+\frac{9}{4}b^m\right)^{3/2} - \left(2+\frac{9}{4}a^m\right)^{3/2}\right]$

A-23: (a) $\mathbf{r}(x) = x\,\hat{\imath} + \sqrt{x}\,\hat{\jmath} + \frac{2}{3}x^{3/2}\,\hat{\mathbf{k}}$ (b) 21 (c) $6\,\hat{\imath} + 3\,\hat{\jmath} + 6\,\hat{\mathbf{k}}$ (d) $-6\,\hat{\imath} - 12\,\hat{\jmath} + 12\,\hat{\mathbf{k}}$

A-24: $|t|$

A-25: (a) $\mathbf{r}(u) = u^3\,\hat{\imath} + 3u^2\,\hat{\jmath} + 6u\,\hat{\mathbf{k}}$ (b) 7 (c) 2 (d) 1

A-26: (a) $\mathbf{r}(t) = \big(\frac{\pi^2 t}{2} - \frac{t^3}{2}\big)\,\hat{\imath} + (t - \sin t)\,\hat{\jmath} + \left(\frac{1}{2}e^{2t} - t\right)\,\hat{\mathbf{k}}$ (b) $t = \pi$

(c) $-\pi^2\,\hat{\imath} + 2\,\hat{\jmath} + \big(e^{2\pi}-1\big)\,\hat{\mathbf{k}}$

A-27: (a) 21 (b) 6 (c) $2\hat{\imath} + 4\,\hat{\jmath} + 4\,\hat{\mathbf{k}}$ (d) $-\frac{8}{3}\big(2\hat{\imath} + \hat{\jmath} - 2\,\hat{\mathbf{k}}\big)$

A-28: $\frac{x(t)y'(t)-y(t)x'(t)}{x^2+y^2}$

A-29: Volume: 540π Surface area: 360π

A-30: $\dfrac{50}{\pi\sqrt{9+\frac{1}{400\pi^2}}} \approx 5.3$ cm

A-31: $\mathbf{r}(t) = \mathbf{r}_0 - \frac{e^{-\alpha t}-1}{\alpha}\mathbf{v}_0 + g\frac{1-\alpha t-e^{-\alpha t}}{\alpha^2}\hat{\mathbf{k}}$

Answers to Exercises 1.2 — Jump to TABLE OF CONTENTS

A-1: $t-1$

A-2: $\big(\sin(1/2), \cos(1/2), \sqrt{3}/2\big)$

A-3: A

A-4: (a) $\left({}^{3}/_{4}\,,\ -{}^{\sqrt{3}}/_{4}\,,\ -{}^{1}/_{2}\right)$ (b) $\mathbf{R}(s) = \left(2\sin^3({}^{s}/_{3}), 2\cos^3({}^{s}/_{3}), 3\sin({}^{s}/_{3})\cos({}^{s}/_{3})\right)$

A-5: (a) $\sqrt{2}$ (b) $\frac{s}{\sqrt{2}}\left(\cos\left(\ln\left(\frac{s}{\sqrt{2}}\right)\right), \sin\left(\ln\left(\frac{s}{\sqrt{2}}\right)\right)\right)$

A-6: $(\cos z,\ z\sin z,\ z)$ for $0 \leqslant z < \pi/2$. The curve is (the first quarter-turn of) a spiral, with width in the x-direction 2, and increasing width in the y-direction. The parameter z is the height, as well as a radian measure for the spiral.

A-7: $\mathbf{R}(s) =$
$$\begin{cases} \left(\frac{1}{2}\left[(2\sqrt{2}-3s)^{2/3}-1)\right], -\frac{1}{3}\left[(2\sqrt{2}-3s)^{2/3}-1\right]^{3/2}\right) & \text{when } s \leqslant \frac{1}{3}(2\sqrt{2}-1) \\ \left(\frac{1}{2}\left[(3s+2-2\sqrt{2})^{2/3}-1\right], \frac{1}{3}\left[(3s+2-2\sqrt{2})^{2/3}-1\right]^{3/2}\right) & \text{when } s > \frac{1}{3}(2\sqrt{2}-1) \end{cases}$$

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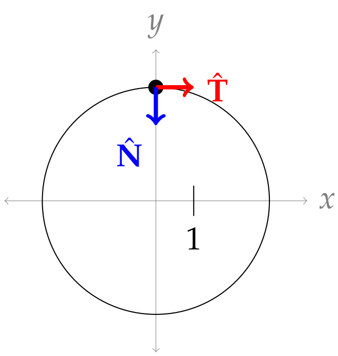


A-1:
$\rho = 3, \kappa = \frac{1}{3}$

A-2: $\hat{\mathbf{T}}(t) = (\cos t, -\sin t)$, $\hat{\mathbf{T}}(s) = (\cos(s/3), -\sin(s/3))$,
$\hat{\mathbf{N}}(t) = (-\sin t, -\cos t)$, $\hat{\mathbf{N}}(s) = (-\sin(s/3), -\cos(s/3))$

A-3: $\lim\limits_{t\to\infty} \kappa(t) = 0$

A-4: $\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{e^{2t} + 9 + \cos^2 t}$

A-5: $\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}t} = \frac{1}{\sqrt{2}}\big(-\sin t - \cos t\,,\ -\sin t + \cos t\big)$
$\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s} = \frac{1}{\sqrt{2}s}\left(-\sin\left(\ln\left(s/\sqrt{2}\right)\right) - \cos\left(\ln\left(s/\sqrt{2}\right)\right),\ -\sin\left(\ln\left(s/\sqrt{2}\right)\right) + \cos\left(\ln\left(s/\sqrt{2}\right)\right)\right)$

A-6: See the solution.

A-7:

A. $\mathbf{v}(t) = (e^t, 2t+1)$

B. $\mathbf{a}(t) = (e^t, 2)$

C. $\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{e^{2t} + (2t+1)^2}$

D. $\hat{\mathbf{T}}(t) = \left(\dfrac{e^t}{\sqrt{e^{2t}+(2t+1)^2}}, \dfrac{2t+1}{\sqrt{e^{2t}+(2t+1)^2}}\right)$

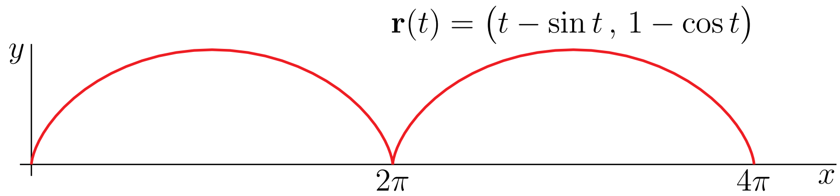
E. $\kappa(t) = \dfrac{e^t|1-2t|}{(e^{2t}+(2t+1)^2)^{3/2}}$

A-8: $\kappa(t) = \frac{1}{\sqrt{2}}$

A-9: $\kappa_{\max} = \frac{a}{b^2}$, $\kappa_{\min} = \frac{b}{a^2}$.

A-10: (a) $\kappa(0) = 2^{-3/2}$ (b) $(x+2)^2 + (y-3)^2 = 8$

A-11: (a)



(b) $\kappa(t) = \frac{1}{2^{3/2}\sqrt{1-\cos t}}$ (c) 4 (d) $(x-\pi)^2 + (y+2)^2 = 16$

A-12: $\kappa(s) = \pi s$

A-13: The maximum values occur at $(x, y) = \pm\left(1/\sqrt[4]{5}\,,\ \frac{1}{3}5^{-3/4}\right)$.

The limits $\lim_{x\to\pm\infty}\kappa(x) = 0$.

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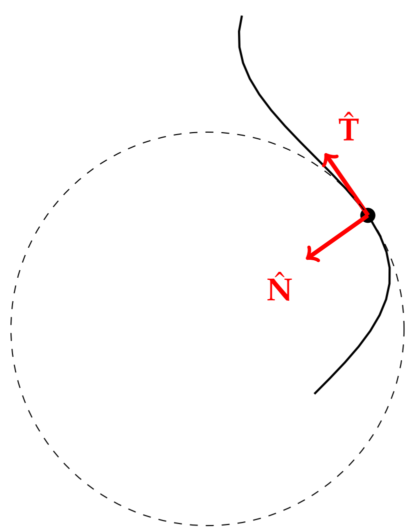
A-1:



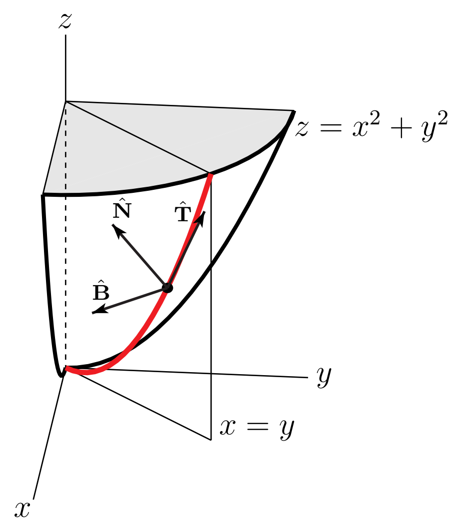
$\hat{\mathbf{B}}$ points out of the page (towards the reader).

A-2: arclength

A-3: $\mathbf{a}(t)$ and $\mathbf{b}(t)$ have negative torsion, $\mathbf{c}(t)$ has zero torsion.

A-4:

A-5: (a), (b)



(c) The torsion is zero.

A-6: (a) $\mathbf{r}'(t) = \left(e^t + e^{-t}\right)\hat{\imath} + \left(e^t - e^{-t}\right)\hat{\jmath} + 2\,\hat{\mathbf{k}}$, $\mathbf{r}''(t) = \left(e^t - e^{-t}\right)\hat{\imath} + \left(e^t + e^{-t}\right)\hat{\jmath}$,

$\kappa(t) = \frac{1}{2+e^{2t}+e^{-2t}}$

(b) $\sqrt{2}\left[e - \frac{1}{e}\right]$

A-7: $\frac{3}{181}$

A-8: $\hat{\mathbf{T}}(t) = \frac{\hat{\imath}+t\hat{\jmath}+t^2\,\hat{\mathbf{k}}}{\sqrt{1+t^2+t^4}}$ $\quad \hat{\mathbf{B}}(t) = \frac{t^2\,\hat{\imath}-2t\,\hat{\jmath}+\hat{\mathbf{k}}}{\sqrt{1+4t^2+t^4}}$ $\quad \hat{\mathbf{N}}(t) = \frac{-(t+2t^3)\,\hat{\imath}+(1-t^4)\,\hat{\jmath}+(2t+t^3)\hat{\mathbf{k}}}{\sqrt{1+t^2+t^4}\sqrt{1+4t^2+t^4}}$

$\kappa(t) = \frac{\sqrt{1+4t^2+t^4}}{[1+t^2+t^4]^{3/2}}$ $\quad \tau(t) = \frac{2}{1+4t^2+t^4}$

A-9: When $c = 0$, the plane is $z = 1$.
When $c = 1/5$, the plane is $(1/25)x + 3y - (30/e)z = -10$.

A-10: (a) $2x + y + 3z = 6$ (b) $\kappa(t) = \frac{2\sqrt{1+9t^2+9t^4}}{[1+4t^2+9t^4]^{3/2}}$

A-11: (a) 2 (b) $-\frac{\sqrt{3}}{2}\,\hat{\imath} - \frac{1}{2}\,\hat{\jmath} + \frac{\pi}{6}\,\hat{\mathbf{k}}$ (c) $\hat{\mathbf{B}} = \frac{1}{2\sqrt{2}}\,\hat{\imath} - \frac{\sqrt{3}}{2\sqrt{2}}\,\hat{\jmath} + \frac{1}{\sqrt{2}}\,\hat{\mathbf{k}}$

A-12: (a) $\mathbf{R}(t) = (-1, 0, \pi^2) + t(0, -1, 2\pi)$ (b) $a_T(t) = \frac{4t}{\sqrt{1+4t^2}}$

A-13: (a) $\sqrt{5}\,t$ (b) $\mathbf{a}_T(t) = \sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + 2\,\hat{\mathbf{k}}$ (c) $\mathbf{a}_N(t) = t\cos t\,\hat{\imath} - t\sin t\,\hat{\jmath}$

(d) $\kappa(t) = \frac{1}{5t}$

A-14: (a) $\mathbf{r}(\theta) = [-1 + 3\cos\theta]\,\hat{\imath} + 3\sin\theta\,\hat{\jmath} + [10 - 6\cos\theta]\,\hat{\mathbf{k}}$, $0 \leqslant \theta < 2\pi$

(b) At $(2, 0, 4)$, $\hat{\mathbf{T}} = \hat{\jmath}$, $\hat{\mathbf{N}} = \frac{-\hat{\imath}+2\hat{\mathbf{k}}}{\sqrt{5}}$, $\hat{\mathbf{B}} = \frac{2\hat{\imath}+\hat{\mathbf{k}}}{\sqrt{5}}$, $\kappa(0) = \frac{\sqrt{5}}{3}$

A-15: (a) $\hat{\mathbf{T}}(t) = \frac{t^2\,\hat{\imath}+\sqrt{2}\,t\,\hat{\jmath}+\hat{\mathbf{k}}}{t^2+1}$ (b) $\frac{\sqrt{2}}{(t^2+1)^2}$ (c) $\frac{4\,\hat{\imath}-3\sqrt{2}\,\hat{\jmath}-4\hat{\mathbf{k}}}{\sqrt{50}}$

A-16: (a) One possible parametrization is $\mathbf{r}(\theta) = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} + (1 - \cos\theta - \sin\theta)\,\hat{\mathbf{k}}$ with $0 \leqslant \theta \leqslant 2\pi$.

(b) $\kappa(\theta) = \frac{\sqrt{3}}{[2-\sin(2\theta)]^{3/2}}$

(c) maximum curvature $= \sqrt{3}$ at $\frac{\hat{\imath}}{\sqrt{2}} + \frac{\hat{\jmath}}{\sqrt{2}} + (1-\sqrt{2})\,\hat{\mathbf{k}}$ and $-\frac{\hat{\imath}}{\sqrt{2}} - \frac{\hat{\jmath}}{\sqrt{2}} + (1+\sqrt{2})\,\hat{\mathbf{k}}$
minimum curvature $= \frac{1}{3}$ at $-\frac{\hat{\imath}}{\sqrt{2}} + \frac{\hat{\jmath}}{\sqrt{2}} + \hat{\mathbf{k}}$ and $\frac{\hat{\imath}}{\sqrt{2}} - \frac{\hat{\jmath}}{\sqrt{2}} + \hat{\mathbf{k}}$

A-17: $\hat{\mathbf{T}}(t) = \frac{2t^2\,\hat{\imath}+2t\,\hat{\jmath}+\hat{\mathbf{k}}}{2t^2+1}$ $\quad \hat{\mathbf{N}}(t) = \frac{2t\,\hat{\imath}-(2t^2-1)\,\hat{\jmath}-2t\,\hat{\mathbf{k}}}{2t^2+1}$ $\quad \kappa(t) = \frac{2t}{(2t^2+1)^2}$

A-18: (a) $\frac{1}{3}\left[2^{3/2}-1\right]$ (b) $\frac{-\hat{\imath}-\hat{\jmath}}{\sqrt{2}}$ (c) $\frac{1}{2}$

A-19: (a) $\mathbf{v}(t) = \big(1\,,\,-1\,,\,t\big)$ (b) $\frac{\mathrm{d}s}{\mathrm{d}t}(t) = \sqrt{2+t^2}$ (c) $\mathbf{a}(t) = \big(0\,,\,0\,,\,1\big)$

(d) $\kappa(t) = \frac{\sqrt{2}}{[2+t^2]^{3/2}}$ (e) $\hat{\mathbf{N}}(t) = \frac{(-t\,,\,t\,,\,2)}{\sqrt{2(2+t^2)}}$ (f) $x+y=3$ (g) $(2,1,2)$

A-20: (a) $\hat{\mathbf{T}}(t) = \frac{t^2\,\hat{\imath}+\sqrt{2}t\,\hat{\jmath}+\hat{\mathbf{k}}}{t^2+1}$ (b) $\kappa(t) = \frac{\sqrt{2}}{(t^2+1)^2}$ (c) $\kappa(0) = \sqrt{2}$ (d) $\hat{\mathbf{N}}(0) = \hat{\jmath}$

(e) $\hat{\mathbf{B}}(0) = -\hat{\imath}$

A-21: (a) $x = 1-2t, y = -1+t, z = -1+3t$ (b) $3x-3y-z=-1$

A-22: (a) $\frac{\sqrt{5}\,\pi^2}{2}$ (b) $\kappa(t) = \frac{1}{5t}$

A-23: (a) 8 (b) $\hat{\mathbf{T}}(1) = \frac{1}{\sqrt{2}}(1,1,0), \hat{\mathbf{N}}(1) = (0,0,-1)$ (c) $\kappa(1) = \frac{1}{8}$

A-24: (a) $\frac{5}{2}$

(b) $\hat{\mathbf{T}}\big(\pi/6\big) = \frac{1}{5}\Big(-\frac{3}{2},\frac{3\sqrt{3}}{2},4\Big), \hat{\mathbf{N}}\big(\pi/6\big) = \frac{1}{2}\big(\sqrt{3},1,0\big), \hat{\mathbf{B}}\big(\pi/6\big) = \frac{1}{5}\big(-2,2\sqrt{3},-3\big)$

A-25: (a) $\mathbf{r}(\theta) = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} + \cos(2\theta)\,\hat{\mathbf{k}} \qquad 0 \leqslant \theta < 2\pi$ (b) $\frac{1}{5}$

(c) $z = \sqrt{2}\,x - \sqrt{2}\,y$

(d) radius $1/\kappa(\pi/4) = 5$ and centre $\big(-2\sqrt{2}\,,\,-2\sqrt{2}\,,\,0\big)$

A-26: $\frac{4}{9}(\hat{\imath} - 4\,\hat{\jmath} + \hat{\mathbf{k}})$

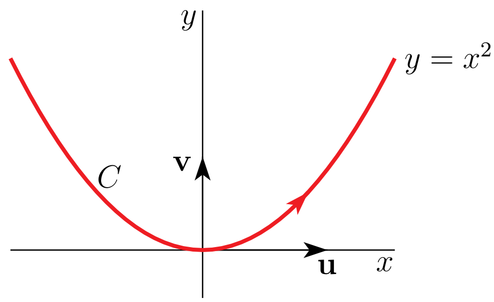
A-27: (a) $\hat{\mathbf{T}}(t) = \frac{\hat{\imath}+t\hat{\jmath}+\sqrt{3}t\hat{\mathbf{k}}}{\sqrt{1+4t^2}}$ (b) $\hat{\mathbf{N}}(t) = \frac{-4t\,\hat{\imath}+\hat{\jmath}+\sqrt{3}\hat{\mathbf{k}}}{2\sqrt{1+4t^2}}$ (c) 3 (d) $-\sqrt{3}y+z=0$

(e) $\kappa(t) = (1+4t^2)^{-3/2}$

(f) The curvature $\kappa(t)$ achieves its maximum value at $\mathbf{r}(0) = (0,0,0)$.

(g) The curvature never achieves a minimum.

(h) $\hat{\imath} = \frac{\mathbf{u}}{2}, \hat{\jmath} = \frac{\mathbf{v}-\sqrt{3}\,\mathbf{w}}{4}, \hat{\mathbf{k}} = \frac{\sqrt{3}\,\mathbf{v}+\mathbf{w}}{4}, \mathbf{r}(t) = t\,\mathbf{u} + t^2\,\mathbf{v}$



The curve $\big(a(t), b(t)\big) = (t, t^2)$ is the curve $y = x^2$. It is "curviest" at the origin, which is consistent with part (f). It becomes flatter and flatter as $|t|$ increases, but never achieves "perfect flatness", which is consistent with (g).

A-28: See the solution.

A-29: (a) $\hat{\mathbf{T}} = \frac{1}{\sqrt{6}}(0, 2, -\sqrt{2})$, $\hat{\mathbf{N}} = -\frac{1}{\sqrt{39}}(6, 1, \sqrt{2})$, $\hat{\mathbf{B}} = \frac{1}{\sqrt{13}}\big(-1, 2, 2\sqrt{2}\big)$, $\kappa = \frac{\sqrt{13}}{3\sqrt{3}} = \frac{\sqrt{39}}{9}$

(b) (i) $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{5\sqrt{2}}{\sqrt{3}}$ (ii) $\mathbf{v} = (0, \sqrt{2}, -1)$.

A-30: (a) $\mathbf{v}(t) = \big(-\sin t\,,\ \cos t\,,\ c\cos t\big)$, $\mathbf{a}(t) = \big(-\cos t\,,\ -\sin t\,,\ -c\sin t\big)$

(b) $v(t) = \sqrt{1 + c^2\cos^2 t}$ (c) $\frac{-c^2\sin t\cos t}{\sqrt{1+c^2\cos^2 t}}$

(d) The curve lies on the plane $z = cy$.

A-31: (a) $\frac{\sqrt{17}}{4}$ (b) $\frac{4}{\sqrt{17}}$ (c) (i) $4\sqrt{\pi}$ (ii) $\big(16\pi\,,\ -4\,,\ -4\pi\big)$ (iii) $4\sqrt{17}\,\pi$

Answers to Exercises 1.6 — Jump to TABLE OF CONTENTS

A-1: $\int_C \mathrm{d}s$

A-2: (a) See the solution. (b) 8

A-3: $\frac{4}{21\sqrt{3}}(2^7 - 1) + \frac{2}{5\sqrt{3}}(2^5 - 1)$

A-4: π kg

A-5: 26

A-6: (a) $\frac{5^{3/2}-1}{12}$ (b) $\frac{8-3^{3/2}}{3/2}$

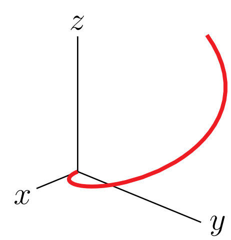
A-7: $\frac{1}{2}\ln 2$

A-8: (a) $\mathbf{r}(t) = t\,\hat{\pmb{\imath}} + \Big(1 + \frac{t^2}{2}\Big)\hat{\pmb{\jmath}} + \sin t\,\hat{\mathbf{k}}$ (b) $\mathbf{r}(\pi/2) = \frac{\pi}{2}\hat{\pmb{\imath}} + \Big(1 + \frac{\pi^2}{8}\Big)\hat{\pmb{\jmath}} + \hat{\mathbf{k}}$ (c) $\frac{\pi^2}{8} - \frac{1}{2}$

A-9: $2e^3$

A-10: (a) $\frac{1}{\sqrt{1+5\pi^2}}\big(-\hat{\pmb{\imath}} - \pi\,\hat{\pmb{\jmath}} + 2\pi\,\hat{\mathbf{k}}\big)$ (b) $\frac{1}{15}\big[(1+5\pi^2)^{3/2} - 1\big]$ (c) $z = x^2 + y^2$

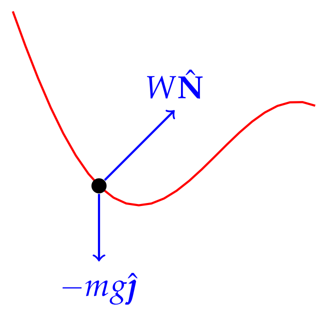
(d)



A-11: $\left(\frac{412}{55}, -\frac{92}{55}, \frac{4736}{693}\right)$

Answers to Exercises 1.7 — Jump to TABLE OF CONTENTS

A-1:



A-2: time

A-3: positive

A-4: $y = \frac{E}{mg}$ — just like a circular culvert (if the culvert is high enough).

A-5: 2940 J

A-6: at least $5\sqrt{9.8}$ m/s

A-7: $\left(-\frac{3}{\sqrt{2}} + 2.352\,,\ -\frac{5}{\sqrt{2}} + 3.92\,,\ -2\sqrt{2} + 3.136\right)$

A-8: $\sqrt{\frac{9.8}{\sqrt{6}}\left(100 + \frac{1}{\sqrt{2}}\right)} \approx 20$ m/s

A-9: $|\mathbf{v}| > 504$ kph

A-10: $U = mg\frac{\mathrm{d}y}{\mathrm{d}s}$

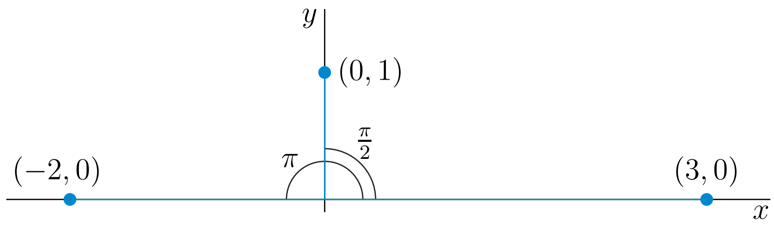
A-11: (a) $M = mg\hat{\boldsymbol{\jmath}} \cdot \hat{\mathbf{T}}$ (b) negative (c) $-\frac{1960}{\sqrt{3}} \approx -1131.6$ N

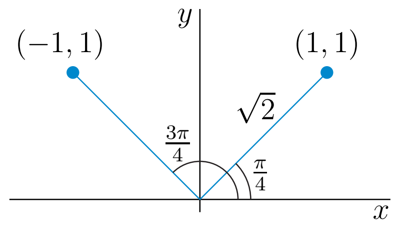
A-12: (a) $y_S = \frac{E}{mg}$ (b) $\frac{24(E-mgy_A)}{\left(9+7\left(\frac{y_A-3}{3}\right)^2\right)^{3/2}} = 4mg\left(\frac{\frac{y_A-3}{3}}{\sqrt{9+7\left(\frac{y_A-3}{3}\right)^2}}\right)$ (or equivalent)
(c) The skateboarder makes it up to the ceiling, but falls off rather than making it all the way around. Ouch.

A-13: (a), (b) See the solution. (c) $2\left[\frac{a^2+b^2}{gb}\pi\right]^{1/2}$

Answers to Exercises 1.8 — Jump to TABLE OF CONTENTS

A-1: The left hand sketch below contains the points, (x_1, y_1), (x_3, y_3), (x_5, y_5), that are on the axes. The right hand sketch below contains the points, (x_2, y_2), (x_4, y_4), that are not on the axes.





$r_1 = 3, \theta_1 = 0$ $\quad r_2 = \sqrt{2}, \theta_2 = \frac{\pi}{4}$ $\quad r_3 = 1, \theta_3 = \frac{\pi}{2}$ $\quad r_4 = \sqrt{2}, \theta_4 = \frac{3\pi}{4}$
$r_5 = 2, \theta_5 = \pi$

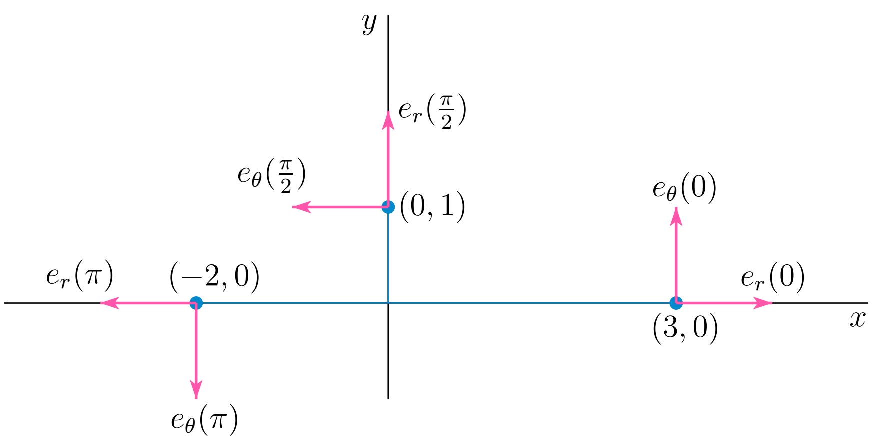
A-2: (a) $\big(r = 2\,,\ \theta = n\pi,\ n \text{ odd integer }\big)$ or $\big(r = -2\,,\ \theta = n\pi,\ n \text{ even integer }\big)$

(b) $\big(r = \sqrt{2}\,,\ \theta = \pi/4 + 2n\pi\big)$ or $\big(r = -\sqrt{2}\,,\ \theta = 5\pi/4 + 2n\pi\big)$, with n integer.

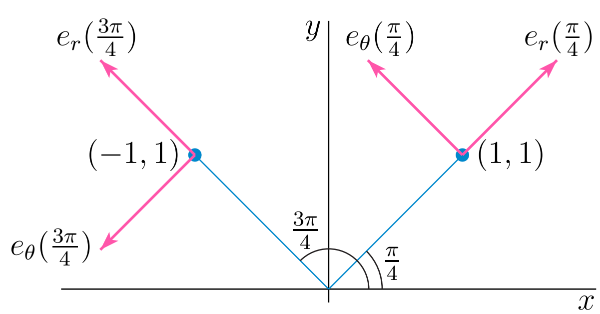
(c) $\big(r = \sqrt{2}\,,\ \theta = 5\pi/4 + 2n\pi\big)$ or $\big(r = -\sqrt{2}\,,\ \theta = \pi/4 + 2n\pi\big)$, with n integer.

A-3: (a) Both $\hat{\mathbf{e}}_r(\theta)$ and $\hat{\mathbf{e}}_\theta(\theta)$ have length 1. The angle between them is $\frac{\pi}{2}$. The cross product is $\hat{\mathbf{e}}_r(\theta) \times \hat{\mathbf{e}}_\theta(\theta) = \hat{\mathbf{k}}$.

(b) Here is a sketch of (x_i, y_i), $\hat{\mathbf{e}}_r(\theta_i)$, $\hat{\mathbf{e}}_\theta(\theta_i)$ for $i = 1, 3, 5$ (the points on the axes)



and here is a sketch (to a different scale) of (x_i, y_i), $\hat{\mathbf{e}}_r(\theta_i)$, $\hat{\mathbf{e}}_\theta(\theta_i)$ for $i = 2, 4$ (the points off the axes).



A-4: (a) $\leftrightarrow$ (E) (b) $\leftrightarrow$ (B) (c) $\leftrightarrow$ (F) (d) $\leftrightarrow$ (C) (e) $\leftrightarrow$ (A) (f) $\leftrightarrow$ (D)

A-5: $\kappa(\theta) = \frac{\left|f(\theta)^2+2f'(\theta)^2-f(\theta)f''(\theta)\right|}{[f(\theta)^2+f'(\theta)^2]^{3/2}}$

A-6: $\kappa(\theta) = \frac{3}{2^{3/2}a\sqrt{1-\cos\theta}} = \frac{3}{2\sqrt{2ar(\theta)}}$

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A-1: $\mathbf{v}(x,y)\cdot\hat{\boldsymbol{\imath}} \begin{cases} >0 & \text{when } \boxed{x>0} \\ =0 & \text{when } \boxed{x=0} \\ <0 & \text{when } \boxed{x<0} \end{cases}$ and $\mathbf{v}(x,y)\cdot\hat{\boldsymbol{\jmath}} \begin{cases} >0 & \text{when } \boxed{-2<x<2} \\ =0 & \text{when } \boxed{x\in\{-2,2\}} \\ <0 & \text{when } \boxed{x<-2 \text{ or } x>2} \end{cases}$ at least for (x,y) shown in the sketch.

A-2: $\mathbf{v}(x,y)\cdot\hat{\boldsymbol{\imath}} \begin{cases} >0 & \text{when } \boxed{y>-x} \\ =0 & \text{when } \boxed{y=-x} \\ <0 & \text{when } \boxed{y<-x} \end{cases}$ and $\mathbf{v}(x,y)\cdot\hat{\boldsymbol{\jmath}} \begin{cases} >0 & \text{when } \boxed{y<x} \\ =0 & \text{when } \boxed{y=x} \\ <0 & \text{when } \boxed{y>x} \end{cases}$ at least for (x,y) shown in the sketch.

A-3: $\mathbf{v}(x,y) = \frac{-y}{\sqrt{x^2+y^2}}(x,y)$

A-4: $P>0$ $Q>0$ $\frac{\partial Q}{\partial x}<0$ $\frac{\partial Q}{\partial y}>0$

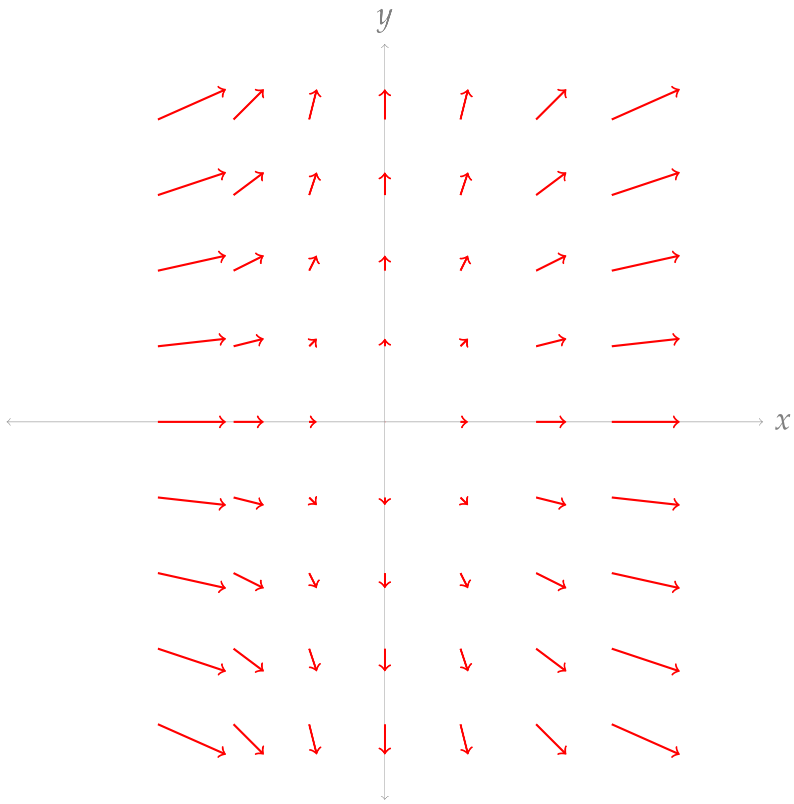
A-5: (a) $(1.01\,,\,1.01)$ (b) $(0\,,\,0)$ (c) $(0\,,\,0)$

A-6: $(0\,,\,-10)$

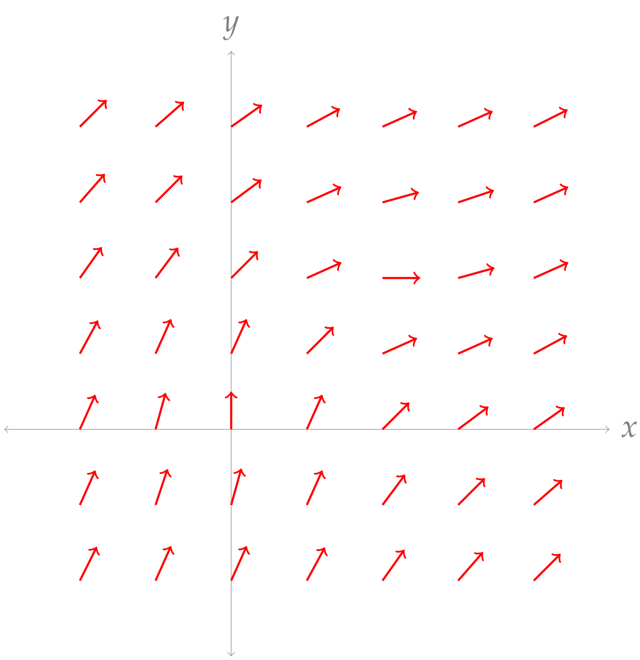
A-7: $\mathbf{v}(x,y) = \frac{-y}{\sqrt{x^2+y^2}}(x,y)$

A-8: If your face is at the origin, then $\mathbf{v}(x,y,z) = -\frac{\alpha}{x^2+y^2+z^2}(x,y,z)$ for some positive constant α.

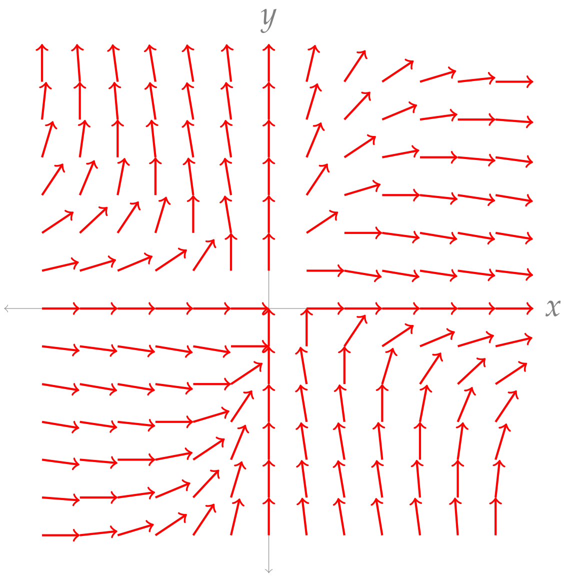
A-9:



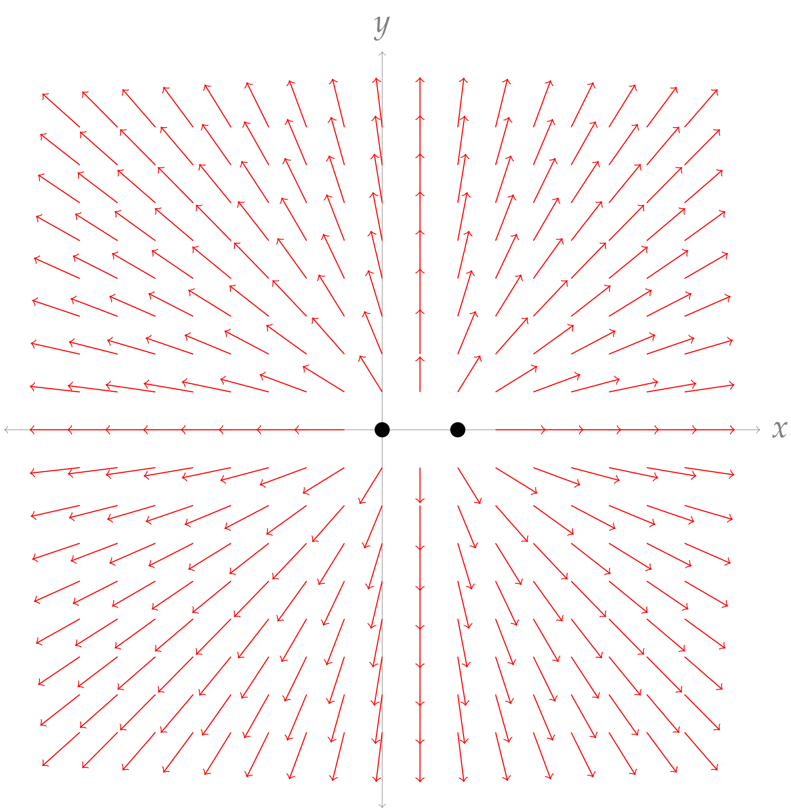
A-10:



A-11:

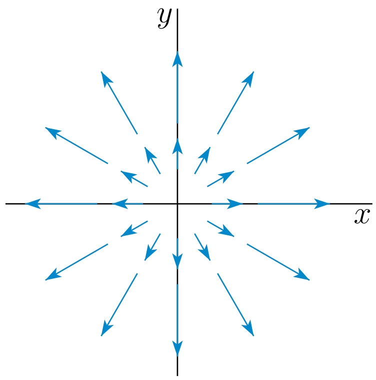


A-12:

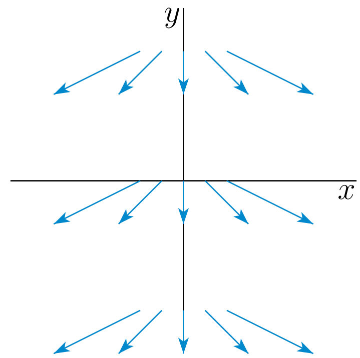


A-13:

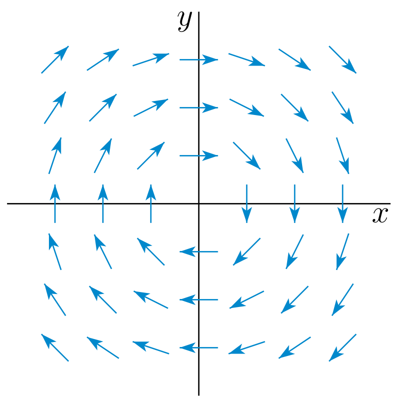
(a)



(b)



(c)



A-14: $\mathbf{f}(x,y) = \frac{-5G(x,y)}{(x^2+y^2)^{3/2}} + \frac{3G(2-x,3-y)}{((x-2)^2+(y-3)^2)^{3/2}} + \frac{7G(4-x,-y)}{((x-4)^2+y^2)^{3/2}}$

A-15: a. $\mathbf{v}(p) = \left(\left(1-\frac{p}{2}\right)\frac{1}{2\sqrt{3}}\,,\ -\frac{p}{4}\right)$ b. $\mathbf{V}(x,y,z) = \left(-\frac{x}{6}, -\frac{y}{6}, \frac{z}{2}\right)$ or equivalent

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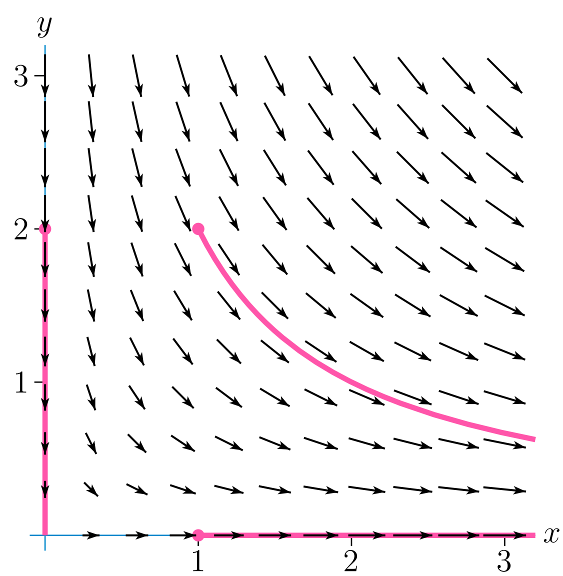
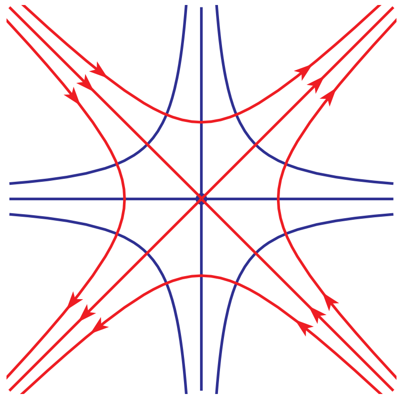
A-1:



A-2: $\mathbf{v}(x,y) = (-x-y\,,\ x-y)$

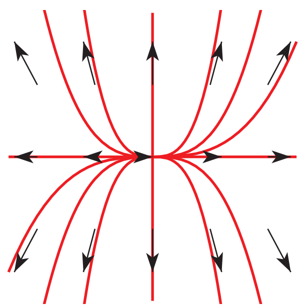
A-3: (a) $\frac{x^2}{2} = \frac{y^2}{2} + C$

(b)



A-4: $x = y^2, z = e^y$

A-5: The field lines are $y = C'x^3$ with C' a nonzero constant, as well as $x = 0$ and $y = 0$.



Answers to Exercises 2.3 — Jump to TABLE OF CONTENTS

A-1: In general, false.

A-2: a. C b. B c. C d. B

A-3: Let φ be a potential for $\mathbf{F}$. Define $\phi = \varphi + ax + by + cz$. Then $\nabla\phi = \nabla\varphi + (a, b, c) = \mathbf{F} + (a, b, c)$.

A-4:

a. If $\mathbf{F} + \mathbf{G}$ is *conservative* for any particular $\mathbf{F}$ and $\mathbf{G}$, then by definition, there exists a potential φ with $\mathbf{F} + \mathbf{G} = \nabla\varphi$.

   Since $\mathbf{F}$ is conservative, there also exists a potential ψ with $\mathbf{F} = \nabla\psi$.

   But now $\mathbf{G} = (\mathbf{F} + \mathbf{G}) - \mathbf{F} = \nabla\varphi - \nabla\psi = \nabla(\varphi - \psi)$. That means the function $(\varphi - \psi)$ is a potential for G. However, this is impossible: since $\mathbf{G}$ is non-conservative, no function with this property exists.

   So it is not possible that $\mathbf{F} + \mathbf{G}$ is conservative. It must be non-conservative.

b. Counterexample: if $\mathbf{F} = -\mathbf{G}$, then $\mathbf{F} + \mathbf{G} = \mathbf{0} = \nabla c$ for any constant c.

c. Since both fields are conservative, they both have potentials, say $\mathbf{F} = \nabla\varphi$ and $\mathbf{G} = \nabla\psi$. Then $\mathbf{F} + \mathbf{G} = \nabla\varphi + \nabla\psi = \nabla(\varphi + \psi)$. That is, $(\varphi + \psi)$ is a potential for $\mathbf{F} + \mathbf{G}$, so $\mathbf{F} + \mathbf{G}$ is conservative.

A-5: Yes, $\mathbf{F}$ is conservative on D. A potential is $\varphi(x, y) = \arctan \frac{y}{x}$.

A-6: $\varphi = \frac{1}{2}x^2 + xy - \frac{1}{2}y^2$

A-7: $\varphi = \ln|x| - \frac{x}{y}$

A-8: None exists: $\frac{\partial F_2}{\partial z} = \frac{1}{3}x^3$, while $\frac{\partial F_3}{\partial y} = \frac{1}{3}x^3 + 1$, so $\mathbf{F}$ fails the screening test, Theorem 2.3.9.

A-9: $\varphi = \frac{1}{2}\ln(x^2 + y^2 + z^2)$

A-10: (a) $\mathbf{F}$ is conservative with potential $\phi(x, y, z) = \frac{1}{2}x^2 - y^2 + \frac{3}{2}z^2 + C$ for any constant C.

(b) $\mathbf{F}$ is not conservative.

A-11: (a) $A = 2$, B is arbitrary.

(b) $\varphi(x, y, z) = xe^{(z^2)} + By^2z^3 + C$ for any constant C.

A-12: $\mathbf{v} = m\frac{x\hat{\imath} + y\hat{\jmath}}{x^2 + y^2}$ $\qquad \varphi = \frac{1}{2}m\ln(x^2 + y^2) + C$ for any constant C

A-13: It can never escape the sphere centred at the origin with radius $\sqrt{20}$.

A-14: $\sqrt{14}$

A-15: $\varphi = f^2(x) + g(y)h(z)$ is a potential for $\mathbf{F}$, so $\mathbf{F}$ is conservative.

A-16: The line through the origin in the direction of the vector $\langle 2, 1, 2\rangle$.

Answers to Exercises 2.4 — Jump to TABLE OF CONTENTS

A-1: $\frac{1}{6}$

A-2: a. A $\quad$ b. B $\quad$ c. A $\quad$ d. B

A-3: 0

A-4: 5

A-5: $a = 1$, $b = c = 0$

A-6: (a) Not conservative $\quad$ (b) Not conservative $\quad$ (c) Not conservative $\quad$ (d) Conservative

A-7: (a) The (largest possible) domain is $D = \{\, (x, y, z) \mid x^2 + y^2 \neq 0 \,\}$.

(b) $\nabla \times \mathbf{F} = \mathbf{0}$ on D $\quad$ (c) $\oint_C \mathbf{F} \cdot d\mathbf{r} = 4\pi$ $\quad$ (d) $\mathbf{F}$ is *not* conservative.

A-8: $9\frac{1}{2}$ for all paths from $(1, 0, -1)$ to $(0, -2, 3)$

A-9: $2(e - 1) + \frac{\pi^2}{2} + 3\pi$

A-10: (a) $-\frac{1}{4}$ (b) -1

A-11: $-\frac{40}{3}$

A-12: $a = 1,\ b = c = 0$

A-13: (a) $\lambda = -1$ (b) $\phi(x, y, z) = 2x^3yz^2 - xyz + y^2 + K$, for any constant K

(c) $e^2 + 2e - 2$

A-14: $\frac{7}{3}$

A-15: $\frac{1}{3}\left[1 - \frac{1}{2^{3/2}}\right] \approx 0.2155$

A-16: $\frac{\pi^3}{8} + \frac{\pi^2}{4} - 1$

A-17: $-\frac{2}{3}$

A-18: The line integral is independent of path because it is of the form $\int_C \mathbf{F} \cdot d\mathbf{r}$ with $\mathbf{F}$ being a conservative field. The value of the integral is $1 + \frac{\pi}{2}$.

A-19: $\frac{1}{2}$

A-20: πe^π

A-21: (a) $\alpha = 1, \beta = \gamma$ (b) $e^e - \beta(e + 1)$

A-22: (a) $\mathbf{0}$ (b) Yes. In fact $\mathbf{F} = \boldsymbol{\nabla} f$ with $f = \sin x + 2y - \cos y + e^z$. (c) -4

A-23: (a) $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$. $\mathbf{F}$ is conservative. (b) $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi^2$

A-24: (a) $a = -1, b = 3$ (b) $f(x, y, z) = xye^x + yz^3 + C$ works for any constant C

(c) $\pi e^\pi - 2$ (d) $\pi e^\pi - \frac{32}{15}$

A-25: (a) $A = 2, B = 3$ (b) $\varphi(x, y, z) = xy^2e^{3z} + x^2y^3$ is one allowed scalar potential.

(c) $6 + e - 2[e - 1] = 8 - e \approx 5.2817$

A-26: (a) $a = \pi,\ b = 3$ (b) $\varphi(x, y, z) = x^2 \sin(\pi y) - xe^z - 3ye^z + C$ for any constant C

(c) -8 (d) $-\frac{13}{2}$

A-27: (a) $f(x, y, z) = ye^{yz} + y\cos^2 x + C$ works for any constant C

(b) $2e^\pi - e^{-\pi^2} - 1$

A-28: (a) $\mathbf{0}$.

(b) $\mathbf{F}$ is conservative with potential $\varphi(x, y, z) = x^2 + y^2 + z^2$. So the integral is $\varphi(a_1, a_2, a_3) - \varphi(0, 0, 0) = \mathbf{a} \cdot \mathbf{a}$.

A-29: (a) $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$ (b) $\frac{\pi e}{2} - 1$

A-30: (a), (b) $f(x, y) = y\sin(x^2) + \cos(y) + C$ is a potential for any constant C. Because $\mathbf{F}$ has a potential, it is conservative.

(c) $-1 - \frac{\pi}{2}\sin(1)$

A-31: (a) $p = 2$, $m = 2$, $n = 2$, but $q \in \mathbb{R}$ is completely free   (b) $4q$

A-32: (a) $\frac{2}{3}\big[14^{3/2} - 1\big] \approx 34.26$ (b) $\sin 1 + \frac{3}{2} \approx 2.3415$

A-33: $2\pi + \frac{1}{3}$

A-34: (a) 18   (b) $3 - e$

A-35: (a) $\mathbf{r}(t) = t\,\hat{\boldsymbol{\imath}} + \big(1 + \frac{t^2}{2}\big)\,\hat{\boldsymbol{\jmath}} + \sin t\,\hat{\mathbf{k}}$ (b) $\mathbf{r}_1 = \frac{\pi}{2}\,\hat{\boldsymbol{\imath}} + \big(1 + \frac{\pi^2}{8}\big)\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}$ (c) $\frac{\pi^2}{8} - \frac{1}{2}$

A-36: (a) -5

(b) One possibility is the path consisting of the line segment from $(2, 2)$ to $(2, -3)$, followed by the line segment from $(2, -3)$ to $(1, -3)$, followed by the line segment from $(1, -3)$ to $(1, 1)$.

Another possibility is the path from $(2, 2)$ to $(1, 1)$ along the parabola $27x^2 - 80x + 54$.

A-37: One possibility is the path consisting of the line segment from $(0, 0)$ to $(0, 1)$, followed by the line segment from $(0, 1)$ to $(2, 1)$, followed by the line segment from $(2, 1)$ to $(2, 0)$.

Another possibility is the path tracing out the half ellipse $\Big(\cos t + 1\,,\ \frac{4}{\pi}\sin t\Big)$, with t running from π to 0.

A-38: See the solution.

A-39: $a = 4$

A-40: (a) $\boldsymbol{\nabla} \times \mathbf{F} = [-(b+2)x\cos(x^2z) + (b+2)x^3z\sin(x^2z)]\,\hat{\boldsymbol{\jmath}} + (6-a)x^2e^{3x^2}\,\hat{\mathbf{k}}$

(b) $a = 6$, $b = -2$ (c) $f(x, y, z) = xye^{3x^2} + \sin(x^2z) + C$ for any constant C

(d) $\frac{1}{3}e^3 + \sin 1 - \frac{1}{3}$

A-41: (a) $\frac{23}{15} = 1.5\dot{3}$ (b) $\frac{2}{3}\big[14^{3/2} - 1\big] \approx 34.26$ (c) $\sin 1 + \frac{3}{2} \approx 2.3415$

A-42: (a) $A = -4$, $B = -2$ (b) $\varphi(x, y, z) = -x^4y^2z + yz^3 + C$ with C being an arbitrary constant.   (c) -2 (d) $-\frac{37}{24} \approx -1.5417$ (e) $\frac{1}{2}$

A-43: (a) $\mathbf{v}(t) = \big(t^2\,,\ t^3\,,\ -t^2\big)$ (b) $\mathbf{r}(t) = \Big(\frac{t^3}{3} + 1\,,\ \frac{t^4}{4} + 2\,,\ -\frac{t^3}{3} + 3\Big)$ (c) $\kappa(t) = \frac{\sqrt{2}}{t^2(2+t^2)^{3/2}}$

(d) $2T^4 + T^6$

A-44: (a) 8   (b) $\frac{1}{8}$ (c) $-\frac{16}{5}(3^5 - 1) \approx -774.4$ (d) $\Big(0, 0, -\frac{9}{8}\Big)$

Answers to Exercises 3.1 — Jump to TABLE OF CONTENTS

A-1: $\mathbf{r}(x, y) = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + (e^{x+1} + xy)\hat{\mathbf{k}}$

A-2: parabolic bowl

A-3: (a) No   (b) Yes   (c) Yes   (d) Yes   (e) No

A-4: (a) No.   (b) Yes.   (c) No.   (d) Yes.   (e) Yes.

A-5: (a) No (b) Yes (c) Yes

A-6: (a) A, F (b) B, E (c) G, J (d) H, L

A-7: (a) $(x, y, z) = (2 + \frac{1}{\sqrt{2}} \cos\theta\ ,\ 2 + \frac{1}{\sqrt{2}} \cos\theta\ ,\ 4 + \sin\theta), 0 \leqslant \theta \leqslant 2\pi$.

Answers to Exercises 3.2 — Jump to TABLE OF CONTENTS

A-1: Yes. The plane $z = 0$ is the tangent plane to both surfaces at $(0, 0, 0)$.

A-2: See the solution.

A-3:

$$\begin{aligned}
&(x - x_0\,,\, y - y_0\,,\, z - z_0) = t\big(-f_x(x_0, y_0)\,,\, -f_y(x_0, y_0)\,,\, 1\big) \qquad \text{or} \\
&x = x_0 - t f_x(x_0, y_0) \quad y = y_0 - t f_y(x_0, y_0) \quad z = f(x_0, y_0) + t
\end{aligned}$$

A-4: The normal plane is $\mathbf{n} \cdot (x - x_0\,,\, y - y_0\,,\, z - z_0) = 0$, where the normal vector $\mathbf{n} = \nabla F(x_0, y_0, z_0) \times \nabla G(x_0, y_0, z_0)$.

A-5: Tangent line is

$$\begin{aligned}
x &= x_0 + t\big[g_y(x_0, y_0) - f_y(x_0, y_0)\big] \\
y &= y_0 + t\big[f_x(x_0, y_0) - g_x(x_0, y_0)\big] \\
z &= z_0 + t\big[f_x(x_0, y_0) g_y(x_0, y_0) - f_y(x_0, y_0) g_x(x_0, y_0)\big]
\end{aligned}$$

A-6: $2x + y + 9z = 2$

A-7: $2x + y + z = 6$

A-8: $z = -\frac{3}{4}x - \frac{3}{2}y + \frac{11}{4}$

A-9: (a) $2ax - 2ay + z = -a^2$ (b) $a = \frac{1}{2}$.

A-10: $x + 3y - 2z = 1$

A-11: $y = 2x - 2$

A-12: The tangent plane is $\frac{8}{25}x - \frac{6}{25}y - z = -\frac{8}{5}$.

The normal line is $(x, y, z) = (-1, 2, \frac{4}{5}) + t(\frac{8}{25}\,,\, -\frac{6}{25}\,,\, -1)$.

A-13: $\pm(1, 0, -2)$

A-14: $\big(\frac{1}{\sqrt{2}}\,,\, -1\,,\, -\frac{1}{2}\big)$ and $\big(-\frac{1}{\sqrt{2}}\,,\, -1\,,\, -\frac{1}{2}\big)$

A-15: $\pm\big(\frac{1}{2}, -1, -1\big)$

A-16: (a) $(1, 0, 3)$ (b) $(3, 3, -1)$ (c) $\mathbf{r}(t) = (1, 1, 3) + t(3, 3, -1)$

A-17: 49.11° (to two decimal places)

A-18: The horizontal tangent planes are $z = 0$, $z = e^{-1}$ and $z = -e^{-1}$. The largest and smallest values of z are e^{-1} and $-e^{-1}$, respectively.

Answers to Exercises 3.3 — Jump to TABLE OF CONTENTS

A-1: $ab\sqrt{1+\tan^2\theta} = ab\sec\theta$

A-2: (a) $\frac{1}{2}\sqrt{a^2b^2+a^2c^2+b^2c^2}$

(b) See the solution.

A-3: $\frac{\pi a h}{2}$

A-4: $\frac{116}{15}\pi$

A-5: $\frac{\pi}{6}\left[\left(1+4a^2\right)^{3/2}-1\right]$

A-6: $5\sqrt{2}\pi$

A-7: $\frac{4}{15}\left[9\sqrt{3}-8\sqrt{2}+1\right]$

A-8: (a) $F(x,y) = \sqrt{1+f_x(x,y)^2+f_y(x,y)^2}$ (b) (i) $\int_0^{2\pi} \mathrm{d}\theta \int_0^1 \mathrm{d}r\, \frac{2r}{\sqrt{4-r^2}}$ (ii) $\frac{32\pi}{3}$

A-9: $255\sqrt{2}\pi \approx 1132.9$

A-10: $a^2[\pi-2]$

A-11: $\sqrt{2}\,\pi$

A-12: $(2\pi)^2 Rr$

A-13: $\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$

A-14: $16a^2$

A-15: $\frac{\pi}{2}a^3\sqrt{a^2+b^2}$

A-16: (a) $4\pi a^{2n+3}$ (b) $3abc$ (c) $\frac{\pi}{3}$

A-17: (a) $\frac{8}{3}$ (b) $\frac{16}{3}$

A-18: (a) $\frac{1}{6}$ (b) $\frac{1}{2}$

A-19: $\frac{8}{27}\left[\left(\frac{13}{4}\right)^{3/2}-1\right]$

A-20: 9π

A-21: (a) $\mathbf{r}(\theta,z) = \frac{2}{3}(3-z)\cos\theta\,\hat{\boldsymbol{\imath}} + \frac{2}{3}(3-z)\sin\theta\,\hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}} \qquad 0 \leqslant \theta < 2\pi, \quad 0 \leqslant z \leqslant 3.$

(b) 1

A-22: $\left(0,0,{}^{a}/_{3}\right)$

A-23: 2π

A-24: $-\frac{14}{3}$

A-25: $\frac{\sqrt{2}\,\pi}{4}$

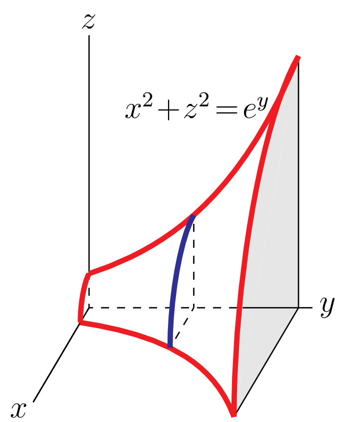
A-26: $\frac{16}{3}\pi$

A-27: $\big(0, 0, {}^2/_3\big)$

A-28: 4

A-29: $\frac{81}{16}$

A-30: (a) $\mathbf{r}(Y,\theta) = e^Y \sin\theta\,\hat{\boldsymbol{\imath}} + Y\,\hat{\boldsymbol{\jmath}} + e^Y\cos\theta\,\hat{\mathbf{k}} \qquad 0 \leqslant Y \leqslant 1,\ 0 \leqslant \theta \leqslant 2\pi$



(b) $\frac{2\pi}{3}\Big[(1+e^2)^{3/2} - 2^{3/2}\Big]$ (c) $\pi\big(1-e^2\big)$

A-31: 12π

A-32: $\frac{\sqrt{5}\,\pi}{8}$

A-33: $-20\,\pi$

A-34: 3

A-35: 2π

A-36: 2π

A-37: 192π

A-38: (a) $x+y+z = 1+\pi/4$ (b) $\frac{2\pi}{3}\big[2\sqrt{2}-1\big]$

A-39: (a) Yes. See the solution for the explanation. (b) See the solution for the proof.

A-40: (a) (i) $\mathbf{r}(u,v) = \Big(u\,,\ v\,,\ \frac{1}{3}(16-2u-4v)\Big)\,\hat{\mathbf{k}} \qquad u \geqslant 0,\ v \geqslant 0,\ u+2v \leqslant 8$

(a) (ii) $\mathbf{r}(u,v) = \big(4\cos u\sin v\,,\ 4\sin u\sin v\,,\ 4\cos v\big) \qquad 0 \leqslant u \leqslant 2\pi,\ 0 \leqslant v \leqslant \frac{\pi}{4}$

(a) (iii) $\mathbf{r}(u,v) = \big(u\,,\ v\,,\ \sqrt{1+u^2+v^2}\big) \qquad u^2+v^2 \leqslant 99$

or $\mathbf{r}(u,v) = \big(u\cos v\,,\ u\sin v\,,\ \sqrt{1+u^2}\big) \qquad 0 \leqslant v \leqslant 2\pi,\ 0 \leqslant u \leqslant \sqrt{99}$

(b) $32\pi\Big[1-\frac{1}{\sqrt{2}}\Big]$

A-41: (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}+\frac{2}{3}$

A-42: $-\frac{5}{6}$

Answers to Exercises 4.1 — Jump to TABLE OF CONTENTS

A-1: (a) A (b) B (c) C (d) A (e) B

A-2: No.

A-3:

A-4: (a) $\nabla \cdot \mathbf{F} = 3$, $\nabla \times \mathbf{F} = \mathbf{0}$

(b) $\nabla \cdot \mathbf{F} = y^2 - z^2 + x^2$, $\nabla \times \mathbf{F} = 2yz\,\hat{\imath} - 2xz\,\hat{\jmath} - 2xy\,\hat{\mathbf{k}}$

(c) $\nabla \cdot \mathbf{F} = \frac{1}{\sqrt{x^2+y^2}}$, $\nabla \times \mathbf{F} = \mathbf{0}$

(d) $\nabla \cdot \mathbf{F} = 0$, $\nabla \times \mathbf{F} = \frac{\hat{\mathbf{k}}}{\sqrt{x^2+y^2}}$

A-5: (a) $\frac{2}{r}$ (b) $\big(xe^{xy} - 2x\big)\,\hat{\imath} + y\big(1 - e^{xy}\big)\,\hat{\jmath} + z\,\hat{\mathbf{k}}$

A-6: (a) $k = -3$ (b) $k = 2$ (c) $k = -2$

A-7: (a) 3 (b) $2\mathbf{r}$ (c) $-2\mathbf{a}$ (d) $\frac{2}{r}$

A-8: (a) $a = -3$ (b) $a = 4$ (c) $a = 12$

A-9: (a) $\mathbf{F}$ cannot have a vector potential.

(b) Two solutions are $\mathbf{A} = \frac{1}{2}(z^2 - y^2)x\hat{\imath} - \frac{1}{2}yz^2\hat{\jmath}$ and $\mathbf{A} = \frac{1}{2}xz^2\hat{\imath} + \frac{1}{2}(x^2 - z^2)y\hat{\jmath}$.

A-10: (a) $D = \big\{\, (x, y, z) \;\big|\; x^2 + z^2 \neq 0 \,\big\}$ (b) $\nabla \times \mathbf{F} = \mathbf{0}$ on D (c) $\nabla \cdot \mathbf{F} = 1$ on D

(d) $\mathbf{F}$ is not conservative on the domain D of part (a).

A-11: (a) $\alpha = \beta = -1$

(b) Any function of the form $g(x, y, z) = xyz + w(z)$ will work.

A-12: (a) See the solution

(b) $\nabla \times (\mathbf{\Omega} \times \mathbf{r}) = 2\mathbf{\Omega}$ $\nabla \cdot (\mathbf{\Omega} \times \mathbf{r}) = 0$ (c) 1095km/hr

A-13: See the solution.

Answers to Exercises 4.2 — Jump to TABLE OF CONTENTS

A-1: See the solution.

A-2: See the solution.

A-3: (a), (b) $\frac{8\pi}{3}$

A-4: (a), (b) $\frac{4}{3}\pi a^3$

A-5: (a) $-\frac{81}{4}\pi$ (b) $2|V|$ (c) $2|V| + \frac{81}{4}\pi$

A-6: (a), (b) 2π

A-7: (a) z (b) 0

A-8: π

A-9: $[3+3x_0-y_0]\,V$

A-10: $-\pi$

A-11: $\frac{16}{3}\pi$

A-12: (a) $\boldsymbol{\nabla}\cdot\mathbf{F}(x,y,z)=0$ except at $(x,y,z)=(0,0,0)$, where $\mathbf{F}$ is not defined.

(b) 4π (c) No. (d) 4π (e) 0

A-13: (a) $\mathbf{r}(\theta,\varphi)=\sin\varphi\cos\theta\,\hat{\boldsymbol{\imath}}+2\sin\varphi\sin\theta\,\hat{\boldsymbol{\jmath}}+2\cos\varphi\,\hat{\mathbf{k}} \qquad 0\leqslant\theta<2\pi,\ 0\leqslant\varphi\leqslant\pi$

(b) 16π (c) 16π, again

A-14: 40π

A-15: 24π

A-16: $\frac{\pi}{2}$

A-17: $\frac{3}{2}\pi$

A-18: $\frac{32}{3}\pi$

A-19: 3π

A-20: $\frac{26}{3}$

A-21: 2π

A-22: (a) $\frac{64}{3}\pi$ (b) $\frac{128}{3}\pi$

A-23: (a) $\boldsymbol{\nabla}\cdot\mathbf{F}=0$ if $(x,y,z)\neq\mathbf{0}$ and is not defined if $(x,y,z)=\mathbf{0}$.

(b) 4π (c) 0

(d) The flux integrals $\iint_{S_1}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ and $\iint_{S_2}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ are different, because the one point, $(0,0,0)$, where $\boldsymbol{\nabla}\cdot\mathbf{F}$ fails to be well-defined and zero, is contained inside S_1 but is not contained inside S_2.

A-24: 72

A-25: (a) 3 (b) -14π

A-26: $-\frac{4\pi}{5}\,3^{5/2}$

A-27: (a) 0 (b) $\frac{625}{2}\pi$

A-28: 4π

A-29: π

A-30: 12π

A-31: $\frac{188}{15}\pi\approx 39.37$

A-32: 5π

A-33: $\left[\frac{\pi}{6} - \frac{1}{3}\right] a^3$

A-34: (a) $-8\sqrt{2}\pi$ (b) $8\sqrt{2}\pi$ (c) $16\sqrt{2}\pi$

A-35: See the solution.

A-36: See the solution.

A-37: (a), (b) 36π

A-38: $\frac{e}{4}$

A-39: (a) $\frac{1}{\sqrt{3}}(-1,-1,1)$ (b) $-\frac{27\pi}{2}$ (c) $-\frac{81\pi}{2}$

A-40: (a) $\boldsymbol{\nabla} \cdot \mathbf{F} = 2 + 2z$ (b) $\pi \frac{23}{6} 5^3 = 479\frac{1}{6}\pi$

(c) Let S be an oriented surface that encloses a solid V and has outward pointing normal. If $\bar{z} = -\frac{9}{2|V|} - 1$, where $|V|$ is the volume of V and $\bar{z}$ is the z-component of the centroid (i.e. centre of mass with constant density) of V, then $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = -9$. One surface which obeys this condition is the unit cube (with outward normal) centred on $\left(0, 0, -\frac{11}{2}\right)$.

A-41: (a) $\frac{\sqrt{5}\pi}{8}$ (b) 20 (c) 18

A-42: (a) $\frac{\pi}{4} + \frac{\pi(b+d)}{6}$

(b) $\iint_{\sigma_1 \cup \sigma_3} \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$ is zero if and only if $d = -b$.

(c) $\iint_{\sigma_1 \cup \sigma_3} \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$ is zero for all a, b, c, d.

A-43: (a) 4π (b) 0.

A-44: See the solution.

A-45: See the solution.

A-46: $\frac{3}{4}$

A-47: $9\pi a^3 + 9\pi a^2$

A-48: See the solution.

A-49: (a) 0 (b) $\frac{15}{2}\pi$

A-50: $30 + 24\pi$

Answers to Exercises 4.3 — Jump to TABLE OF CONTENTS

A-1: See the solution.

A-2: See the solution.

A-3: (a) 1 (b) 1 (c) 0

A-4: See the solution.

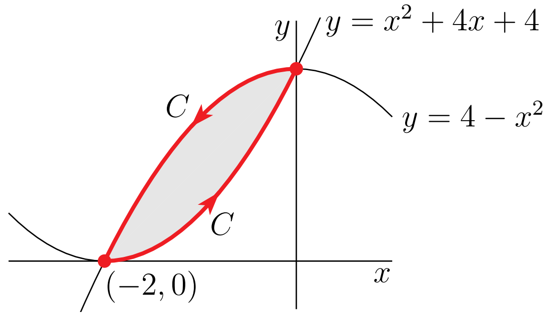
A-5: -54

A-6: 9

A-7: $\frac{32}{3}\pi$

A-8: -6

A-9: (a)



(b) $-\frac{8}{3}$

A-10: $-\frac{1}{3}$

A-11: 54

A-12: $\frac{10}{3}$

A-13: (a) $-\frac{\pi}{2}$ (b) $\frac{3\pi}{2}$ (c) No.

A-14: 54

A-15: (a) $I_2 = 0$ (b) $I_3 = \pi$ (c) $I_4 = \pi$

A-16: (a) $Q_x - P_y = 0$ except at $(0, 0)$ where it is not defined.

(b) -2π (c) No. (d) 0 (e) -2π

A-17: (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4} + \frac{2}{3}$

A-18: $\frac{3\pi}{2}$

A-19: $\frac{3\pi}{8}$

A-20: $A = -2$

A-21: $-\pi$

A-22: $\oint_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = 0$ and $\oint_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} = 2\pi$

A-23: (a) $\frac{1}{2} + \frac{1}{12}\left[5^{3/2} - 1\right] \approx 1.3484$ (b) $\frac{3}{4}$

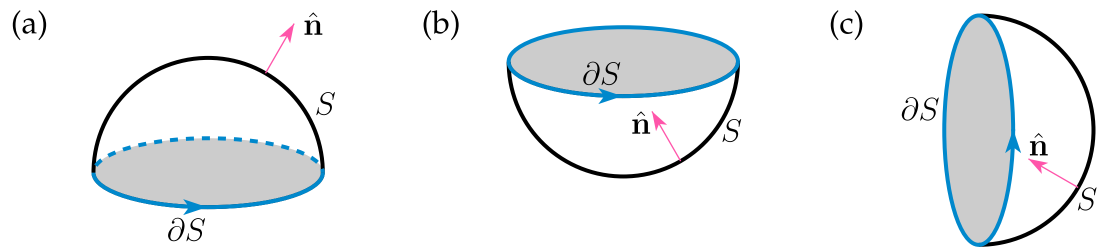
A-24: (a) The projection of the curve on the xy-plane (i.e. the top view of the curve) is a circle. See the solution for more details.

(b) (i) 0 (b) (ii) 0

A-25: $6x^2 + 3y^2 = 1$

Answers to Exercises 4.4 — Jump to TABLE OF CONTENTS

A-1:



A-2: See the solution.

A-3: See the solution.

A-4: (a) 2π (b) 2π

A-5: π

A-6: 8π

A-7: 12π

A-8: π

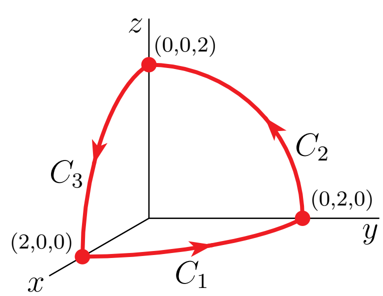
A-9: 8

A-10: 1

A-11: 4π

A-12: (a) 8 (b) $4\sqrt{3}$

A-13: (a)



(b) $S = \left\{ (x,y,z) \,\middle|\, x^2+y^2+z^2 = 4,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0 \right\}$ with

$$\mathbf{r}(\theta,\varphi) = 2\cos\theta\sin\varphi\,\hat{\boldsymbol{\imath}} + 2\sin\theta\sin\varphi\,\hat{\boldsymbol{\jmath}} + 2\cos\varphi\,\hat{\mathbf{k}}, \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2},\ 0 \leqslant \varphi \leqslant \frac{\pi}{2}$$

and

$$\hat{\mathbf{n}} = \cos\theta\sin\varphi\,\hat{\boldsymbol{\imath}} + \sin\theta\sin\varphi\,\hat{\boldsymbol{\jmath}} + \cos\varphi\,\hat{\mathbf{k}} = \frac{1}{2}\,\mathbf{r}(\theta,\varphi)$$

(c) -4π

A-14: (a) -128π, (b) -126π

A-15: 4π

A-16: (a) 8 (b) $4\sqrt{3}$

A-17: $5\pi/4$

A-18: $-\frac{10}{\sqrt{3}}$

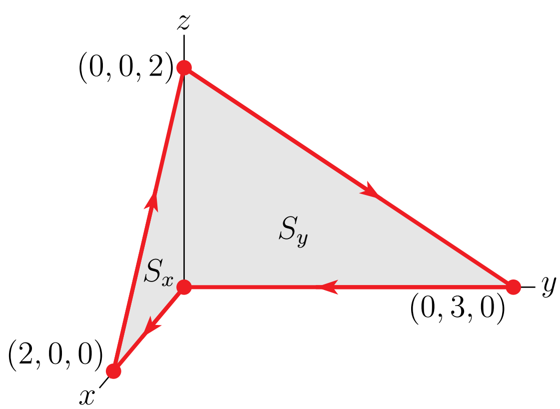
A-19: -2

A-20: $-\pi$

A-21: $\frac{3\pi}{4}$

A-22: $\frac{\pi}{3}$

A-23: (a)



(b) $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 10$

A-24: $-\pi$

A-25: 24π

A-26: $2\sqrt{3}\pi R^2$

A-27: $\frac{4}{3}$

A-28: -2π

A-29: 24π

A-30: (a) $\nabla \times \mathbf{F} = (1 - 2xz)\,\hat{\boldsymbol{\jmath}}$ (b) $20/3$

A-31: (a) -18π (b) -18π

A-32: (a) $D = \{\, (x, y, z) \mid x > 0,\ y > 0,\ z > 0 \,\}$

(b) The domain D is both connected and simply connected.

(c) $\nabla \times \mathbf{F} = \big(2x - 1/x\big)\hat{\mathbf{k}}$

(d) $2 \ln 2 - 24$

(e) No. $\mathbf{F}$ is not conservative.

A-33: (a) $a = 2$, $b = -1$ (b) $\frac{\pi}{4}$

A-34: -15

A-35: 12π

A-36: Rewrite $\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{r}$ as a surface integral. For the details, see the solution.

A-37: $\sqrt{2}\pi$

A-38: $\frac{2\pi}{3\sqrt{3}}$

A-39: (a) One possible parametrization is $\mathbf{r}(r,\theta) = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath} + r\,\hat{\mathbf{k}}$ with $0 \leqslant r \leqslant 1$, $0 \leqslant \theta \leqslant \pi$.

(b) π

A-40: $-\frac{4}{\sqrt{3}}\pi$

Answers to Exercises 5 — Jump to TABLE OF CONTENTS

A-1: (a) True (b) True (c) True (d) False (e) True

(f) That depends. If $\kappa = 0$, the curve is part of a straight line. If $\kappa > 0$ it is part of a circle of radius $\frac{1}{\kappa}$.

(g) False. (h) False. (i) False.

A-2: (a) False (b) False (c) False (d) False (e) True (f) True

(g) False (h) False (i) False (j) True

A-3: (a) False. (b) $\hat{\mathbf{N}}(t)$, $\hat{\mathbf{B}}(t)$ (c) True. (d) False. (e) False. (f) True.

A-4: (a) decreasing (b) $f(x)$ is D

(c) $\mathbf{r}(\theta) = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\mathbf{k}} + \sin\theta\,\cos\theta\,\hat{\jmath}$, $0 \leqslant \theta < 2\pi$

(d) We want parametrisation (d) with domain $|u| \geqslant 2$, $0 \leqslant v \leqslant 5$.

(e) One possible answer is $\mathbf{r}(t) = t\,\hat{\imath}$, $0 \leqslant t \leqslant 1$.

(f) $C = 6$ (g) $\{\ (a,b,c,d)\ \big|\ a,b,c,d$ all real and $b = c\ \}$ (h) 2

(i) (1) True (2) False (3) False (4) False (5) False

(j) Any vector field whose divergence is 1 everywhere will work. One such vector field is $\mathbf{F} = x\,\hat{\imath}$.

(k) negative

A-5: (a) false (b) false (c) true (d) false

(e) true, assuming that the second derivatives of the vector field exist and are continuous.

(f) silly, but true (g) true (h) false (i) false (j) false

A-6: (a) False (b) False (c) True (d) True (e) True (f) True (g) True (h) False

A-7: (a) $P_y < 0$ (b) $Q_x > 0$ (c) $\boldsymbol{\nabla} \times \mathbf{F}$ is in the direction of $+\hat{\mathbf{k}}$ at A

(d) $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} > 0$ (e) $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} < 0$ (f) $\mathbf{F}$ is not conservative

A-8: (a) False (b) True (c) True (d) False (e) True (f) False (g) False (h) False (i) True (j) True

A-9: (a) True (b) False (c) True (d) False (e) False (f) True (g) True (h) False (i) False (j) True

A-10: (a) False. (b) False. (c) True. (d) False. (e) True. (f) False.

(g) False. (h) True. (i) False.

A-11: (a) True (b) False (c) True (d) False (e) True (f) True (g) True (h) False (i) False (j) True

A-12: (b)

A-13: (a) False. (b) False. (c) False. (d) True. (e) False. (f) True. (g) False. (h) False. (i) True. (j) False.

A-14: (a) True (b) False (c) True, assuming that $\mathbf{r}(t)$ is not indentically $\mathbf{0}$.

(d) False (e) False

A-15:

(a) $2xy + e^y \sin x + xe^{xz}$ (b) $y^3\,\hat{\boldsymbol{\imath}} - z\,\hat{\boldsymbol{\jmath}}$ (c) (iii) (d) False.

A-16: (a) True (b) True (c) True (d) False (e) True (f) True (g) True

A-17: (a) True (b) False

A-18: (a) True (b) False

A-19: (a), (b), (c) See the solution. (d) Yes (d) No

A-20: (a) yes (b) no (c) no (d) yes

A-21: (a) True (b) True (c) False

A-22: (a), (c) See the solution. (b) $8\pi b^2$

Part IV

SOLUTIONS TO PROBLEMS

Solutions to Exercises 1.1 — Jump to TABLE OF CONTENTS

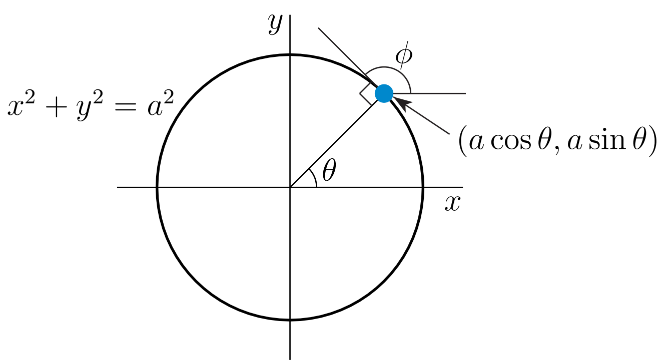
S-1: (a) Since, on the specified part of the circle, $x = \sqrt{a^2 - y^2}$ and y runs from 0 to a, the parametrization is $\mathbf{r}(y) = \sqrt{a^2 - y^2}\,\hat{\imath} + y\,\hat{\jmath}$, $0 \leqslant y \leqslant a$.

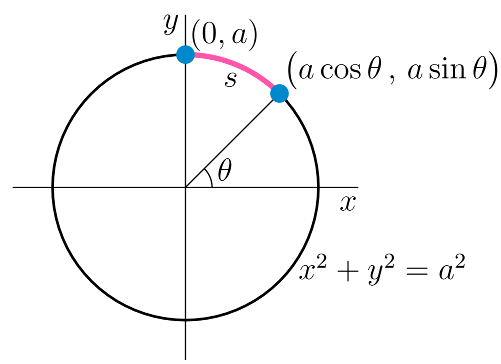
(b) Let θ be the angle between

- the radius vector from the origin to the point $(a\cos\theta, a\sin\theta)$ on the circle and
- the positive x-axis.

The tangent line to the circle at $(a\cos\theta, a\sin\theta)$ is perpendicular to the radius vector and so makes angle $\phi = \frac{\pi}{2} + \theta$ with the positive x axis. (See the figure on the left below.) As $\theta = \phi - \frac{\pi}{2}$, the desired parametrization is

$$\big(x(\phi), y(\phi)\big) = \big(a\cos(\phi - \tfrac{\pi}{2}), a\sin(\phi - \tfrac{\pi}{2})\big) = \big(a\sin\phi, -a\cos\phi\big),\ \tfrac{\pi}{2} \leqslant \phi \leqslant \pi$$





(c) Let θ be the angle between

- the radius vector from the origin to the point $(a\cos\theta, a\sin\theta)$ on the circle and
- the positive x-axis.

The arc from $(0, a)$ to $(a\cos\theta, a\sin\theta)$ subtends an angle $\frac{\pi}{2} - \theta$ and so has length $s = a\big(\frac{\pi}{2} - \theta\big)$. (See the figure on the right above.) Thus $\theta = \frac{\pi}{2} - \frac{s}{a}$ and the desired parametrization is

$$\big(x(s), y(s)\big) = \left(a\cos\left(\frac{\pi}{2} - \frac{s}{a}\right)\ ,\ a\sin\left(\frac{\pi}{2} - \frac{s}{a}\right)\right),\ 0 \leqslant s \leqslant \frac{\pi}{2}a$$

S-2: We can find the time at which the curve hits a given point by considering the two equations that arise from the two coordinates. For the y-coordinate to be 0, we must have $(t-5)^2 = 0$, i.e. $t = 5$. So, the point $(-1/\sqrt{2}, 0)$ happens when $t = 5$.

Similarly, for the y-coordinate to be 25, we need $(t-5)^2 = 25$, so $(t-5) = \pm 5$. When $t = 0$, the curve hits $(1, 25)$; when $t = 10$, the curve hits $(0, 25)$.

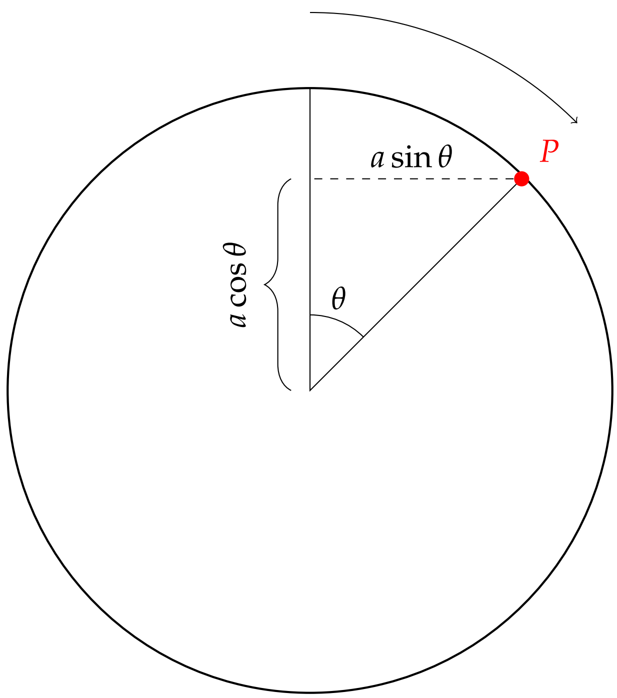
So, in order, the curve passes through the points $(1, 25)$, $(-1/\sqrt{2}, 0)$, and $(0, 25)$.

S-3: The curve "crosses itself" when the same coordinates occur for different values of t, say t_1 and t_2. So, we want to know when $\sin t_1 = \sin t_2$ and also $t_1^2 = t_2^2$. Since t_1 and t_2 should be different, the second equation tells us $t_2 = -t_1$. Then the first equation tells us $\sin t_1 = \sin t_2 = \sin(-t_1) = -\sin t_1$. That is, $\sin t_1 = -\sin t_1$, so $\sin t_1 = 0$. That happens whenever $t_1 = \pi n$ for an integer n.

So, the points at which the curve crosses itself are those points $(0, (\pi n)^2)$ where n is an integer. It passes such a point at times $t = \pi n$ and $t = -\pi n$. So, the curve hits this point $2\pi n$ time units apart.

S-4: (a) Pretend that the circle is a spool of thread. As the circle rolls, it dispenses the thread along the ground. When the circle rolls θ radians, it dispenses the arc length θa of thread and the circle advances a distance θa. So the centre of the circle has moved θa units to the right from its starting point, $x = a$. The centre of the circle always has y-coordinate a. So, after rolling θ radians, the centre of the circle is at position $\mathbf{c}(\theta) = (a + a\theta, a)$.

(b) Now, let's consider the position of P on the circle, after the circle has rolled θ radians.



From the diagram, we see that P is $a\cos\theta$ units above the centre of the circle, and $a\sin\theta$ units to the right of it. So, the position of P is $(a + a\theta + a\sin\theta, a + a\cos\theta)$.

Remark: this type of curve is known as a *cycloid*.

S-5: We aren't concerned with x, so we can eliminate it by solving for it in one equation, and plugging that into the other. Since C lies on the plane, $x = -y - z$, so:

$$\begin{aligned} 1 &= x^2 - \frac{1}{4}y^2 + 3z^2 = (-y-z)^2 - \frac{1}{4}y^2 + 3z^2 \\ &= \frac{3}{4}y^2 + 4z^2 + 2yz \end{aligned}$$

Completing the square,

$$1 = \frac{1}{2}y^2 + \left(2z + \frac{y}{2}\right)^2$$
$$1 - \frac{y^2}{2} = \left(2z + \frac{y}{2}\right)^2$$

Since y is small, the left hand is close to 1 and the right hand side is close to $(2z)^2$. So $(2z^2) \approx 1$. Since z is negative, $z \approx -\frac{1}{2}$ and $2z + \frac{y}{2} < 0$. Also, $1 - \frac{y^2}{2}$ is positive, so it has a real square root.

$$-\sqrt{1 - \frac{y^2}{2}} = 2z + \frac{y}{2}$$
$$-\frac{1}{2}\sqrt{1 - \frac{y^2}{2}} - \frac{y}{4} = z$$

S-6: To determine whether the particle is rising or falling, we only need to consider its z-coordinate: $z(t) = (t-1)^2(t-3)^2$. Its derivative with respect to time is $z'(t) = 4(t-1)(t-2)(t-3)$. This is positive when $1 < t < 2$ and when $3 < t$, so the particle is increasing on $(1,2) \cup (3,\infty)$ and decreasing on $(0,1) \cup (2,3)$.

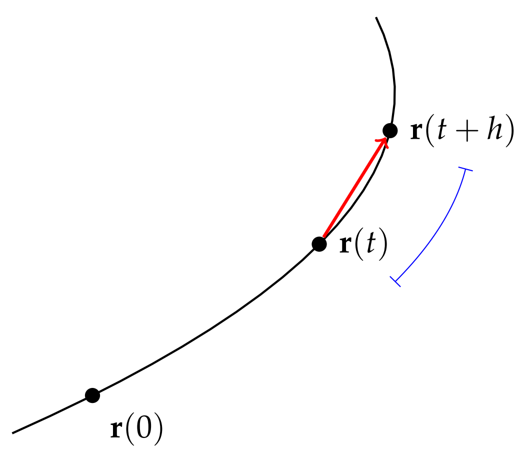
If $\mathbf{r}(t)$ is the position of the particle at time t, then its speed is $|\mathbf{r}'(t)|$. We differentiate:

$$\mathbf{r}'(t) = -e^{-t}\,\hat{\boldsymbol{\imath}} - \frac{1}{t^2}\,\hat{\boldsymbol{\jmath}} + 4(t-1)(t-2)(t-3)\hat{\mathbf{k}}$$

So, $\mathbf{r}(1) = -\frac{1}{e}\,\hat{\boldsymbol{\imath}} - 1\,\hat{\boldsymbol{\jmath}}$ and $\mathbf{r}(3) = -\frac{1}{e^3}\,\hat{\boldsymbol{\imath}} - \frac{1}{9}\,\hat{\boldsymbol{\jmath}}$. The absolute value of every component of $\mathbf{r}(1)$ is greater than or equal to that of the corresponding component of $\mathbf{r}(3)$, so $|\mathbf{r}(1)| > |\mathbf{r}(3)|$. That is, the particle is moving more swiftly at $t = 1$ than at $t = 3$.

Note: We could also compute the sizes of both vectors directly: $|\mathbf{r}'(1)| = \sqrt{\left(\frac{1}{e}\right)^2 + (-1)^2}$, and $|\mathbf{r}'(3)| = \sqrt{\left(\frac{1}{e^3}\right)^2 + \left(-\frac{1}{9}\right)^2}$.

S-7:



The red vector is $\mathbf{r}(t+h) - \mathbf{r}(t)$. The arclength of the segment indicated by the blue line is the (scalar) $s(t+h) - s(t)$.

Remark: as h approaches 0, the curve (if it's differentiable at t) starts to resemble a straight line, with the length of the vector $\mathbf{r}(t+h) - \mathbf{r}(t)$ approaching the scalar $s(t+h) - s(t)$. This step is crucial to understanding Lemma 1.1.3 in the CLP-4 text.

S-8: Velocity is a vector-valued quantity, so it has both a magnitude and a direction. Speed is a scalar — the magnitude of the velocity. It does not include a direction.

S-9: By the product rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[(\mathbf{r}\times\mathbf{r}')\cdot\mathbf{r}''\big] = (\mathbf{r}'\times\mathbf{r}')\cdot\mathbf{r}'' + (\mathbf{r}\times\mathbf{r}'')\cdot\mathbf{r}'' + (\mathbf{r}\times\mathbf{r}')\cdot\mathbf{r}'''$$

The first term vanishes because $\mathbf{r}'\times\mathbf{r}' = \mathbf{0}$. The second term vanishes because $\mathbf{r}\times\mathbf{r}''$ is perpendicular to $\mathbf{r}''$. So

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[(\mathbf{r}\times\mathbf{r}')\cdot\mathbf{r}''\big] = (\mathbf{r}\times\mathbf{r}')\cdot\mathbf{r}'''$$

which is (c).

S-10: we are told that $\mathbf{r}(t)\perp\mathbf{r}'(t)$, so that $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$, for all t. Consequently

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{r}(t)|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\big[\mathbf{r}(t)\cdot\mathbf{r}(t)\big] = 2\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$$

So $|\mathbf{r}(t)|^2$ is a constant, say A, independent of time and $\mathbf{r}(t)$ always lies on the sphere of radius $\sqrt{A}$ centred on the origin.

S-11: We have

$$\mathbf{v}(t) = \mathbf{r}'(t) = 5\sqrt{2}\,\hat{\imath} + 5e^{5t}\,\hat{\jmath} + 5e^{-5t}\,\hat{\mathbf{k}}$$

and hence

$$|\mathbf{v}(t)| = |\mathbf{r}'(t)| = 5\big|\sqrt{2}\,\hat{\imath} + e^{5t}\,\hat{\jmath} + e^{-5t}\,\hat{\mathbf{k}}\big| = 5\sqrt{2 + e^{10t} + e^{-10t}}$$

Since $2 + e^{10t} + e^{-10t} = \big(e^{5t} + e^{-5t}\big)^2$, that's (d).

S-12: We are told that

$$\mathbf{r}(t) = a\cos t\,\hat{\imath} + a\sin t\,\hat{\jmath} + ct\,\hat{\mathbf{k}}$$

So, by definition,

$$\begin{aligned}
\text{velocity} &= \mathbf{v}(t) = \mathbf{r}'(t) = -a\sin t\,\hat{\imath} + a\cos t\,\hat{\jmath} + c\,\hat{\mathbf{k}}\\
\text{speed} &= \frac{\mathrm{d}s}{\mathrm{d}t}(t) = |\mathbf{r}'(t)| = \sqrt{a^2 + c^2}\\
\text{acceleration} &= \mathbf{a}(t) = \mathbf{r}''(t) = -a\cos t\,\hat{\imath} - a\sin t\,\hat{\jmath}
\end{aligned}$$

As t runs over an interval of length 2π, (x, y) traces out a circle of radius a and z increases by $2\pi c$. The path is a helix with radius a and with each turn having height $2\pi c$.

S-13: (a) Since $\mathbf{r}'(t) = (2t, 0, t^2)$, the specified unit tangent at $t = 1$ is

$$\hat{\mathbf{T}}(1) = \frac{(2, 0, 1)}{\sqrt{5}}$$

(b) We are to find the arc length between $\mathbf{r}(0)$ and $\mathbf{r}(-1)$. As $\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{4t^2 + t^4}$, the

$$\text{arc length} = \int_{-1}^{0} \sqrt{4t^2 + t^4}\,\mathrm{d}t$$

The integrand is even, so

$$\text{arc length} = \int_0^1 \sqrt{4t^2 + t^4}\,\mathrm{d}t = \int_0^1 t\sqrt{4 + t^2}\,\mathrm{d}t = \Big[\tfrac{1}{3}(4 + t^2)^{3/2}\Big]_0^1 = \tfrac{1}{3}\big[5^{3/2} - 8\big]$$

S-14: By Lemma 1.1.3 in the CLP-4 text, the arclength of $\mathbf{r}(t)$ from $t = 0$ to $t = 1$ is $\int_0^1 \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\right| \mathrm{d}t$. We'll calculate this in a few pieces to make the steps clearer.

$$\begin{aligned}
\mathbf{r}(t) &= \left(t, \sqrt{\frac{3}{2}}t^2, t^3\right)\\
\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t) &= \left(1, \sqrt{6}t, 3t^2\right)\\
\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\right| &= \sqrt{1^2 + (\sqrt{6}t)^2 + (3t^2)^2} = \sqrt{1 + 6t^2 + 9t^4} = \sqrt{(3t^2 + 1)^2} = 3t^2 + 1\\
\int_0^1 \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\right| \mathrm{d}t &= \int_0^1 \left(3t^2 + 1\right) \mathrm{d}t = 2
\end{aligned}$$

S-15: Since

$$\begin{aligned}
x'(t) &= a\big[\cos^2 t - \sin^2 t\big] = a\cos 2t\\
y'(t) &= 2a\sin t\cos t = a\sin 2t\\
z'(t) &= b
\end{aligned}$$

we have

$$\frac{\mathrm{d}s}{\mathrm{d}t}(t) = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = \sqrt{a^2 + b^2}$$

As the speed $\frac{\mathrm{d}s}{\mathrm{d}t}(t)$ is constant, the length is just $\frac{\mathrm{d}s}{\mathrm{d}t}\,T = \sqrt{a^2 + b^2}\,T$.

S-16: Since $\mathbf{r}(t)$ is the position of the particle, its acceleration is $r''(t)$.

$$\begin{aligned}
\mathbf{r}(t) &= (t + \sin t, \cos t)\\
\mathbf{r}'(t) &= (1 + \cos t, -\sin t)\\
\mathbf{r}''(t) &= (-\sin t, -\cos t)\\
|\mathbf{r}''(t)| &= \sqrt{\sin^2 t + \cos^2 t} = 1
\end{aligned}$$

The magnitude of acceleration is constant, but its *direction* is changing, since $\mathbf{r}''(t)$ is a vector with changing direction.

S-17: (a) The speed is

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}t}(t) = |\mathbf{r}'(t)| &= \left|\big(2\cos t - 2t\sin t\,,\, 2\sin t + 2t\cos t\,,\, t^2\big)\right| \\ &= \sqrt{\big(2\cos t - 2t\sin t\big)^2 + \big(2\sin t + 2t\cos t\big)^2 + t^4} \\ &= \sqrt{4 + 4t^2 + t^4} \\ &= 2 + t^2\end{aligned}$$

so the length of the curve is

$$\text{length } = \int_0^2 \frac{\mathrm{d}s}{\mathrm{d}t}\,\mathrm{d}t = \int_0^2 (2+t^2)\,\mathrm{d}t = \left[2t + \frac{t^3}{3}\right]_0^2 = \frac{20}{3}$$

(b) A tangent vector to the curve at $\mathbf{r}(\pi) = \big(-2\pi\,,\,0\,,\,{}^{\pi^3}/_3\big)$ is

$$\mathbf{r}'(\pi) = \big(2\cos\pi - 2\pi\sin\pi\,,\, 2\sin\pi + 2\pi\cos\pi\,,\,\pi^2\big) = (-2\,,\,-2\pi\,,\,\pi^2)$$

So parametric equations for the tangent line at $\mathbf{r}(\pi)$ are

$$\begin{aligned} x(t) &= -2\pi - 2t \\ y(t) &= -2\pi t \\ z(t) &= {}^{\pi^3}/_3 + \pi^2 t \end{aligned}$$

S-18: (a) As $\mathbf{r}(t) = \big(3\cos t, 3\sin t, 4t\big)$, the velocity of the particle is

$$\mathbf{r}'(t) = \big(-3\sin t, 3\cos t, 4\big)$$

(b) As $\frac{\mathrm{d}s}{\mathrm{d}t}$, the rate of change of arc length per unit time, is

$$\frac{\mathrm{d}s}{\mathrm{d}t}(t) = |\mathbf{r}'(t)| = \big|\big(-3\sin t, 3\cos t, 4\big)\big| = 5$$

the arclength of its path between $t = 1$ and $t = 2$ is

$$\int_1^2 \mathrm{d}t\,\frac{\mathrm{d}s}{\mathrm{d}t}(t) = \int_1^2 \mathrm{d}t\; 5 = 5$$

S-19: (a) We can parametrize the circle $x^2+y^2=9$ as $x(\theta)=3\cos\theta$, $y(\theta)=3\sin\theta$ with θ running from 0 to 2π. As $z=2x+3y$, the ellipse can be parametrized by

$$x(\theta)=3\cos\theta,\ y(\theta)=3\sin\theta,\ z(\theta)=2x(\theta)+3y(\theta)=6\cos\theta+9\sin\theta,\ 0\leqslant\theta\leqslant 2\pi$$

(b) As

$$\begin{aligned}\frac{ds}{d\theta}&=\sqrt{x'(\theta)^2+y'(\theta)^2+z'(\theta)^2}\\&=\sqrt{9\sin^2\theta+9\cos^2\theta+36\sin^2\theta+81\cos^2\theta-108\sin\theta\cos\theta}\\&=\sqrt{45+45\cos^2\theta-108\sin\theta\cos\theta}\end{aligned}$$

the circumference is

$$s=\int_0^{2\pi}\sqrt{45+45\cos^2\theta-108\sin\theta\cos\theta}\,d\theta$$

S-20: (a) As

$$\mathbf{r}'(t)=-\sin t\cos^2 t\,\hat{\imath}+\sin^2 t\cos t\,\hat{\imath}+3\sin^2 t\cos t\,\hat{\mathbf{k}}=\sin t\cos t\big(-\cos t\,\hat{\imath}+\sin t\,\hat{\jmath}+3\sin t\,\hat{\mathbf{k}}\big)$$

$$\frac{ds}{dt}(t)=|\sin t\cos t|\sqrt{\cos^2 t+\sin^2 t+9\sin^2 t}=|\sin t\cos t|\sqrt{1+9\sin^2 t}$$

the arclength from $t=0$ to $t=\frac{\pi}{2}$ is

$$\begin{aligned}\int_0^{\pi/2}\frac{ds}{dt}(t)\,dt&=\int_0^{\pi/2}\sin t\cos t\sqrt{1+9\sin^2 t}\,dt\\&=\frac{1}{18}\int_1^{10}\sqrt{u}\,du\qquad\text{with } u=1+9\sin^2 t,\ du=18\sin t\cos t\,dt\\&=\frac{1}{18}\Big[\frac{2}{3}u^{3/2}\Big]_1^{10}\\&=\frac{1}{27}\big(10\sqrt{10}-1\big)\end{aligned}$$

(b) The arclength from $t=0$ to $t=\pi$ is

$$\begin{aligned}\int_0^{\pi}\frac{ds}{dt}(t)\,dt&=\int_0^{\pi}|\sin t\cos t|\sqrt{1+9\sin^2 t}\,dt\qquad\text{Don't forget the absolute value signs!}\\&=2\int_0^{\pi/2}|\sin t\cos t|\sqrt{1+9\sin^2 t}\,dt=2\int_0^{\pi/2}\sin t\cos t\sqrt{1+9\sin^2 t}\,dt\end{aligned}$$

since the integrand is invariant under $t\to\pi-t$. So the arc length from $t=0$ to $t=\pi$ is just twice the arc length from part (a), namely $\frac{2}{27}\big(10\sqrt{10}-1\big)$.

S-21: Since

$$\begin{aligned}
\mathbf{r}(t) &= \frac{t^3}{3}\,\hat{\boldsymbol{\imath}} + \frac{t^2}{2}\,\hat{\boldsymbol{\jmath}} + \frac{t}{2}\,\hat{\mathbf{k}} \\
\mathbf{r}'(t) &= t^2\,\hat{\boldsymbol{\imath}} + t\,\hat{\boldsymbol{\jmath}} + \frac{1}{2}\,\hat{\mathbf{k}} \\
\frac{\mathrm{d}s}{\mathrm{d}t}(t) = |\mathbf{r}'(t)| &= \sqrt{t^4 + t^2 + \frac{1}{4}} = \sqrt{\Big(t^2 + \frac{1}{2}\Big)^2} = t^2 + \frac{1}{2}
\end{aligned}$$

the length of the curve is

$$s(t) = \int_0^t \frac{\mathrm{d}s}{\mathrm{d}t}(u)\,\mathrm{d}u = \int_0^t \Big(u^2 + \frac{1}{2}\Big)\,\mathrm{d}u = \frac{t^3}{3} + \frac{t}{2}$$

S-22: Since

$$\begin{aligned}
\mathbf{r}(t) &= t^m\,\hat{\boldsymbol{\imath}} + t^m\,\hat{\boldsymbol{\jmath}} + t^{3m/2}\,\hat{\mathbf{k}} \\
\mathbf{r}'(t) &= mt^{m-1}\,\hat{\boldsymbol{\imath}} + mt^{m-1}\,\hat{\boldsymbol{\jmath}} + \frac{3m}{2}t^{3m/2-1}\,\hat{\mathbf{k}} \\
\frac{\mathrm{d}s}{\mathrm{d}t} = |\mathbf{r}'(t)| &= \sqrt{2m^2t^{2m-2} + \frac{9m^2}{4}t^{3m-2}} = mt^{m-1}\sqrt{2 + \frac{9}{4}t^m}
\end{aligned}$$

the arc length is

$$\begin{aligned}
\int_a^b \frac{\mathrm{d}s}{\mathrm{d}t}(t)\,\mathrm{d}t &= \int_a^b mt^{m-1}\sqrt{2 + \frac{9}{4}t^m}\,\mathrm{d}t \\
&= \frac{4}{9}\int_{2+\frac{9}{4}a^m}^{2+\frac{9}{4}b^m} \sqrt{u}\,\mathrm{d}u \qquad \text{with } u = 2 + \frac{9}{4}t^m,\ \mathrm{d}u = \frac{9m}{4}t^{m-1} \\
&= \frac{4}{9}\Big[\frac{2}{3}u^{3/2}\Big]_{2+\frac{9}{4}a^m}^{2+\frac{9}{4}b^m} \\
&= \frac{8}{27}\Big[\Big(2 + \frac{9}{4}b^m\Big)^{3/2} - \Big(2 + \frac{9}{4}a^m\Big)^{3/2}\Big]
\end{aligned}$$

S-23: (a) Since $y = \sqrt{x}$ and $z = \frac{2}{3}xy = \frac{2}{3}x^{3/2}$,

$$\mathbf{r}(x) = x\,\hat{\boldsymbol{\imath}} + \sqrt{x}\,\hat{\boldsymbol{\jmath}} + \frac{2}{3}x^{3/2}\,\hat{\mathbf{k}}$$

For the remaining parts of this problem we will also need

$$\begin{aligned}
\mathbf{r}'(x) &= \hat{\boldsymbol{\imath}} + \frac{1}{2\sqrt{x}}\,\hat{\boldsymbol{\jmath}} + \sqrt{x}\,\hat{\mathbf{k}} \\
\mathbf{r}''(x) &= -\frac{1}{4x^{3/2}}\,\hat{\boldsymbol{\jmath}} + \frac{1}{2\sqrt{x}}\,\hat{\mathbf{k}} \\
\frac{\mathrm{d}s}{\mathrm{d}x} = |\mathbf{r}'(x)| &= \sqrt{1 + \frac{1}{4x} + x} = \sqrt{\Big(\frac{1}{2\sqrt{x}} + \sqrt{x}\Big)^2} = \frac{1}{2\sqrt{x}} + \sqrt{x} \\
\frac{\mathrm{d}s}{\mathrm{d}x}(1) &= \frac{3}{2}
\end{aligned}$$

(b)

$$\int_C ds = \int_0^9 \frac{\mathrm{d}s}{\mathrm{d}x}\,\mathrm{d}x = \int_0^9 \left(\frac{1}{2\sqrt{x}} + \sqrt{x}\right)\,\mathrm{d}x = \left[\sqrt{x} + \frac{2}{3}x^{3/2}\right]_0^9 = 3 + 18 = 21$$

(c) Denote by

- $\mathbf{r}(x)$ the position of the particle when its first coordinate is x,
- $\mathbf{R}(t)$ the position of the particle at time t,
- $x(t)$ the x–coordinate of the particle at time t, and
- $s(x)$ the arc length of the curve from the origin to $\mathbf{r}(x)$.

We are told that $|\mathbf{R}'(t)| = 9$ for all t. So

$$\begin{aligned}\mathbf{R}(t) = \mathbf{r}\big(x(t)\big) &\implies \mathbf{R}'(t) = \mathbf{r}'\big(x(t)\big)\frac{\mathrm{d}x}{\mathrm{d}t}(t)\\ &\implies 9 = |\mathbf{R}'(t)| = \frac{\mathrm{d}s}{\mathrm{d}x}\big(x(t)\big)\frac{\mathrm{d}x}{\mathrm{d}t}(t) = \left(\frac{1}{2\sqrt{x(t)}} + \sqrt{x(t)}\right)\frac{\mathrm{d}x}{\mathrm{d}t}(t)\end{aligned}$$

In particular, if the particle is at $(1, 1, \frac{2}{3})$ at time 0, then $x(0) = 1$ and

$$9 = \left(\frac{1}{2\sqrt{1}} + \sqrt{1}\right)\frac{\mathrm{d}x}{\mathrm{d}t}(0) \implies \frac{\mathrm{d}x}{\mathrm{d}t}(0) = 6$$

so that

$$\mathbf{R}'(0) = \mathbf{r}'(1)\frac{\mathrm{d}x}{\mathrm{d}t}(0) = \left(\hat{\boldsymbol{\imath}} + \frac{1}{2}\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\right)6 = 6\,\hat{\boldsymbol{\imath}} + 3\,\hat{\boldsymbol{\jmath}} + 6\,\hat{\mathbf{k}}$$

(d) By the product and chain rules,

$$\mathbf{R}'(t) = \mathbf{r}'\big(x(t)\big)\frac{\mathrm{d}x}{\mathrm{d}t}(t) \implies \mathbf{R}''(t) = \mathbf{r}''\big(x(t)\big)\left(\frac{\mathrm{d}x}{\mathrm{d}t}(t)\right)^2 + \mathbf{r}'\big(x(t)\big)\frac{\mathrm{d}^2x}{\mathrm{d}t^2}(t)$$

We saw in part (c) that $9 = |\mathbf{R}'(t)| = \left(\frac{1}{2\sqrt{x(t)}} + \sqrt{x(t)}\right)\frac{\mathrm{d}x}{\mathrm{d}t}(t)$ so that

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = 9\left(\frac{1}{2\sqrt{x(t)}} + \sqrt{x(t)}\right)^{-1}$$

Differentiating that gives

$$\frac{d^2x}{dt^2}(t) = -9\left(\frac{1}{2\sqrt{x(t)}} + \sqrt{x(t)}\right)^{-2}\left(-\frac{1}{4x(t)^{3/2}} + \frac{1}{2\sqrt{x(t)}}\right)\frac{\mathrm{d}x}{\mathrm{d}t}(t)$$

In particular, when $t = 0$, $x(0) = 1$ and $\frac{\mathrm{d}x}{\mathrm{d}t}(0) = 6$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}(0) = -9\left(\frac{3}{2}\right)^{-2}\left(\frac{1}{4}\right)6 = -6$$

so

$$\begin{aligned}\mathbf{R}''(0) &= \mathbf{r}''(1)\ \big(6\big)^2 + \mathbf{r}'(1)\big(-6\big) = 36\left(-\frac{1}{4}\hat{\boldsymbol{\jmath}} + \frac{1}{2}\hat{\mathbf{k}}\right) - 6\left(\hat{\boldsymbol{\imath}} + \frac{1}{2}\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\right)\\ &= -6\,\hat{\boldsymbol{\imath}} - 12\,\hat{\boldsymbol{\jmath}} + 12\,\hat{\mathbf{k}}\end{aligned}$$

S-24: Given the position of the particle, we can find its velocity:

$$\mathbf{v}(t) = \mathbf{r}'(t) = (\cos t, -\sin t, 1)$$

Applying the given formula,

$$\mathbf{L}(t) = \mathbf{r} \times \mathbf{v} = (\sin t, \cos t, t) \times (\cos t, -\sin t, 1).$$

Solution 1: We can first compute the cross product, then differentiate:

$$\begin{aligned}\mathbf{L}(t) &= (\cos t + t\sin t)\hat{\boldsymbol{\imath}} + (t\cos t - \sin t)\hat{\boldsymbol{\jmath}} - \hat{\mathbf{k}}\\ \mathbf{L}'(t) &= t\cos t\,\hat{\boldsymbol{\imath}} - t\sin t\,\hat{\boldsymbol{\jmath}}\\ |\mathbf{L}'(t)| &= \sqrt{t^2(\sin^2 t + \cos^2 t)} = \sqrt{t^2} = |t|\end{aligned}$$

Solution 2: Using the product rule:

$$\begin{aligned}\mathbf{L}'(t) &= \mathbf{r}'(t) \times \mathbf{v}(t) + \mathbf{r}(t) \times \mathbf{v}'(t)\\ &= \underbrace{\mathbf{r}'(t) \times \mathbf{r}'(t)}_{0} + \mathbf{r}(t) \times \mathbf{v}'(t)\\ &= (\sin t, \cos t, t) \times (-\sin t, -\cos t, 0)\\ &= t\cos t\,\hat{\boldsymbol{\imath}} - t\sin t\,\hat{\boldsymbol{\jmath}}\\ |\mathbf{L}'(t)| &= \sqrt{t^2\cos^2 t + t^2\sin t^2} = |t|\end{aligned}$$

S-25: (a) Since $z = 6u$, $y = \frac{z^2}{12} = 3u^2$ and $x = \frac{yz}{18} = u^3$,

$$\mathbf{r}(u) = u^3\,\hat{\boldsymbol{\imath}} + 3u^2\,\hat{\boldsymbol{\jmath}} + 6u\,\hat{\mathbf{k}}$$

(b)

$$\begin{aligned}\mathbf{r}'(u) &= 3u^2\,\hat{\boldsymbol{\imath}} + 6u\,\hat{\boldsymbol{\jmath}} + 6\,\hat{\mathbf{k}}\\ \mathbf{r}''(u) &= 6u\,\hat{\boldsymbol{\imath}} + 6\,\hat{\boldsymbol{\jmath}}\\ \frac{\mathrm{d}s}{\mathrm{d}u}(u) &= |\mathbf{r}'(u)| = \sqrt{9u^4 + 36u^2 + 36} = 3\big(u^2 + 2\big)\end{aligned}$$

$$\int_{\mathcal{C}} ds = \int_0^1 \frac{\mathrm{d}s}{\mathrm{d}u}\,\mathrm{d}u = \int_0^1 3\big(u^2 + 2\big)\,\mathrm{d}u = \big[u^3 + 6u\big]_0^1 = 7$$

(c) Denote by $\mathbf{R}(t)$ the position of the particle at time t. Then

$$\mathbf{R}(t) = \mathbf{r}\big(u(t)\big) \implies \mathbf{R}'(t) = \mathbf{r}'\big(u(t)\big)\frac{\mathrm{d}u}{\mathrm{d}t}$$

In particular, if the particle is at $(1, 3, 6)$ at time t_1, then $u(t_1) = 1$ and

$$6\,\hat{\boldsymbol{\imath}} + 12\,\hat{\boldsymbol{\jmath}} + 12\,\hat{\mathbf{k}} = \mathbf{R}'(t_1) = \mathbf{r}'(1)\frac{\mathrm{d}u}{\mathrm{d}t}(t_1) = \big(3\,\hat{\boldsymbol{\imath}} + 6\,\hat{\boldsymbol{\jmath}} + 6\,\hat{\mathbf{k}}\big)\frac{\mathrm{d}u}{\mathrm{d}t}(t_1)$$

which implies that $\frac{\mathrm{d}u}{\mathrm{d}t}(t_1) = 2$.

(d) By the product and chain rules,

$$\mathbf{R}'(t) = \mathbf{r}'\big(u(t)\big)\frac{\mathrm{d}u}{\mathrm{d}t} \implies \mathbf{R}''(t) = \mathbf{r}''\big(u(t)\big)\Big(\frac{\mathrm{d}u}{\mathrm{d}t}\Big)^2 + \mathbf{r}'\big(u(t)\big)\frac{\mathrm{d}^2u}{\mathrm{d}t^2}$$

In particular,

$$\begin{aligned}
27\,\hat{\boldsymbol{\imath}} + 30\,\hat{\boldsymbol{\jmath}} + 6\,\hat{\mathbf{k}} = \mathbf{R}''(t_1) &= \mathbf{r}''(1)\Big(\frac{\mathrm{d}u}{\mathrm{d}t}(t_1)\Big)^2 + \mathbf{r}'(1)\frac{\mathrm{d}^2u}{\mathrm{d}t^2}(t_1)\\
&= \big(6\,\hat{\boldsymbol{\imath}} + 6\,\hat{\boldsymbol{\jmath}}\big)2^2 + \big(3\,\hat{\boldsymbol{\imath}} + 6\,\hat{\boldsymbol{\jmath}} + 6\,\hat{\mathbf{k}}\big)\frac{\mathrm{d}^2u}{\mathrm{d}t^2}(t_1)
\end{aligned}$$

Simplifying

$$3\,\hat{\boldsymbol{\imath}} + 6\,\hat{\boldsymbol{\jmath}} + 6\,\hat{\mathbf{k}} = \big(3\,\hat{\boldsymbol{\imath}} + 6\,\hat{\boldsymbol{\jmath}} + 6\,\hat{\mathbf{k}}\big)\frac{\mathrm{d}^2u}{\mathrm{d}t^2}(t_1) \implies \frac{\mathrm{d}^2u}{\mathrm{d}t^2}(t_1) = 1$$

S-26: (a) According to Newton,

$$m\mathbf{r}''(t) = \mathbf{F}(t) \qquad \text{so that} \qquad \mathbf{r}''(t) = -3t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}} + 2e^{2t}\,\hat{\mathbf{k}}$$

Integrating once gives

$$\mathbf{r}'(t) = -3\frac{t^2}{2}\,\hat{\boldsymbol{\imath}} - \cos t\,\hat{\boldsymbol{\jmath}} + e^{2t}\,\hat{\mathbf{k}} + \mathbf{c}$$

for some constant vector $\mathbf{c}$. We are told that $\mathbf{r}'(0) = \mathbf{v}_0 = \frac{\pi^2}{2}\,\hat{\boldsymbol{\imath}}$. This forces $\mathbf{c} = \frac{\pi^2}{2}\,\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \hat{\mathbf{k}}$ so that

$$\mathbf{r}'(t) = \left(\frac{\pi^2}{2} - \frac{3t^2}{2}\right)\hat{\boldsymbol{\imath}} + (1 - \cos t)\,\hat{\boldsymbol{\jmath}} + \big(e^{2t} - 1\big)\,\hat{\mathbf{k}}$$

Integrating a second time gives

$$\mathbf{r}(t) = \left(\frac{\pi^2 t}{2} - \frac{t^3}{2}\right)\hat{\boldsymbol{\imath}} + (t - \sin t)\,\hat{\boldsymbol{\jmath}} + \left(\frac{1}{2}e^{2t} - t\right)\hat{\mathbf{k}} + \mathbf{c}$$

for some (other) constant vector $\mathbf{c}$. We are told that $\mathbf{r}(0) = \mathbf{r}_0 = \frac{1}{2}\,\hat{\mathbf{k}}$. This forces $\mathbf{c} = \mathbf{0}$ so that

$$\mathbf{r}(t) = \left(\frac{\pi^2 t}{2} - \frac{t^3}{2}\right)\hat{\boldsymbol{\imath}} + (t - \sin t)\,\hat{\boldsymbol{\jmath}} + \left(\frac{1}{2}e^{2t} - t\right)\hat{\mathbf{k}}$$

(b) The particle is in the plane $x = 0$ when

$$0 = \left(\frac{\pi^2 t}{2} - \frac{t^3}{2}\right) = \frac{t}{2}(\pi^2 - t^2) \iff t = 0, \pm\pi$$

So the desired time is $t = \pi$.

(c) At time $t = \pi$, the velocity is

$$\begin{aligned}
\mathbf{r}'(\pi) &= \left(\frac{\pi^2}{2} - \frac{3\pi^2}{2}\right)\hat{\boldsymbol{\imath}} + (1 - \cos\pi)\,\hat{\boldsymbol{\jmath}} + \left(e^{2\pi} - 1\right)\hat{\mathbf{k}} \\
&= -\pi^2\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} + \left(e^{2\pi} - 1\right)\hat{\mathbf{k}}
\end{aligned}$$

S-27: (a) Parametrize C by x. Since $y = x^2$ and $z = \frac{2}{3}x^3$,

$$\begin{aligned}
\mathbf{r}(x) &= x\,\hat{\boldsymbol{\imath}} + x^2\,\hat{\boldsymbol{\jmath}} + \frac{2}{3}x^3\,\hat{\mathbf{k}} \\
\mathbf{r}'(x) &= \hat{\boldsymbol{\imath}} + 2x\,\hat{\boldsymbol{\jmath}} + 2x^2\,\hat{\mathbf{k}} \\
\mathbf{r}''(x) &= 2\,\hat{\boldsymbol{\jmath}} + 4x\,\hat{\mathbf{k}} \\
\frac{\mathrm{d}s}{\mathrm{d}x} &= |\mathbf{r}'(x)| = \sqrt{1 + 4x^2 + 4x^4} = 1 + 2x^2
\end{aligned}$$

and

$$\int_C \mathrm{d}s = \int_0^3 \frac{\mathrm{d}s}{\mathrm{d}x}\,\mathrm{d}x = \int_0^3 \left(1 + 2x^2\right)\mathrm{d}x = \left[x + \frac{2}{3}x^3\right]_0^3 = 21$$

(b) The particle travelled a distance of 21 units in $\frac{7}{2}$ time units. This corresponds to a speed of $\frac{21}{7/2} = 6$.

(c) Denote by $\mathbf{R}(t)$ the position of the particle at time t. Then

$$\mathbf{R}(t) = \mathbf{r}\big(x(t)\big) \implies \mathbf{R}'(t) = \mathbf{r}'\big(x(t)\big)\frac{\mathrm{d}x}{\mathrm{d}t}$$

By parts (a) and (b) and the chain rule

$$6 = \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}s}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = (1 + 2x^2)\frac{\mathrm{d}x}{\mathrm{d}t} \implies \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{6}{1 + 2x^2}$$

In particular, the particle is at $\left(1, 1, \frac{2}{3}\right)$ at $x = 1$. At this time $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{6}{1+2\times 1} = 2$ and

$$\mathbf{R}' = \mathbf{r}'(1)\frac{\mathrm{d}x}{\mathrm{d}t} = (\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} + 2\,\hat{\mathbf{k}})2 = 2\hat{\boldsymbol{\imath}} + 4\,\hat{\boldsymbol{\jmath}} + 4\,\hat{\mathbf{k}}$$

(d) By the product and chain rules,

$$\mathbf{R}'(t) = \mathbf{r}'\big(x(t)\big)\frac{\mathrm{d}x}{\mathrm{d}t} \implies \mathbf{R}''(t) = \mathbf{r}''\big(x(t)\big)\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \mathbf{r}'\big(x(t)\big)\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

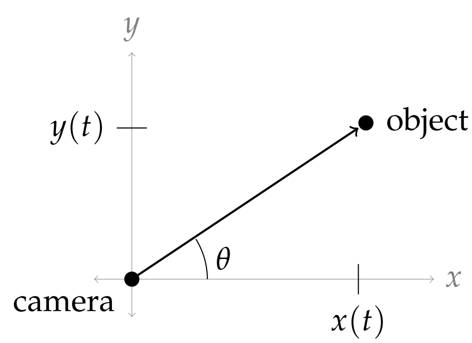
Applying $\frac{\mathrm{d}}{\mathrm{d}t}$ to $6 = \big(1+2x(t)^2\big)\frac{\mathrm{d}x}{\mathrm{d}t}(t)$ gives

$$0 = 4x\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + (1+2x^2)\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

In particular, when $x = 1$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 2$, $0 = 4 \times 1\big(2\big)^2 + (3)\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ gives $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{16}{3}$ and

$$\mathbf{R}'' = \big(2\hat{\boldsymbol{\jmath}} + 4\,\hat{\mathbf{k}}\big)\big(2\big)^2 - \big(\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 2\,\hat{\mathbf{k}}\big)\frac{16}{3} = -\frac{8}{3}\big(2\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - 2\,\hat{\mathbf{k}}\big)$$

S-28: The question is already set up as an xy-plane, with the camera at the origin, so the vector in the direction the camera is pointing is $\big(x(t), y(t)\big)$. Let θ be the angle the camera makes with the positive x-axis (due east). The camera, the object, and the due-east direction (positive x-axis) make a right triangle.



$$\tan\theta = \frac{y}{x}$$

Differentiating implicitly with respect to t:

$$\sec^2\theta\,\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{xy' - yx'}{x^2}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \cos^2\theta\left(\frac{xy'-yx'}{x^2}\right) = \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2\left(\frac{xy'-yx'}{x^2}\right) = \frac{xy'-yx'}{x^2+y^2}$$

S-29: Using the Theorem of Pappus, we can calculate the surface area and volume of a pipe with the same length and radius as this pipe. So, we need to find the length of the pipe, L.

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = (\sqrt{2t}, t, 1)$$

$$\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right| = \sqrt{2t + t^2 + 1} = |t+1|$$

$$L = \int_0^{10}(t+1)\mathrm{d}t = 60$$

A pipe with radius 3 and length 60 has surface area $60(2\pi \cdot 3) = 360\pi$ and volume $60(\pi \cdot 3^2) = 540\pi$.

S-30: In general a helix can be parametrized by

$$\mathbf{r}(\theta) = a\cos\theta\,\hat{\boldsymbol{\imath}} + a\sin\theta\,\hat{\boldsymbol{\jmath}} + b\theta\,\hat{\mathbf{k}}$$

Our first task is to determine a and b. The radius of the helix is 3 cm, so $a = 3$ cm. After 10 turns (i.e. $\theta = 20\pi$) the height, $b\theta$, is 1 cm. So $b(20\pi) = 1$ and $b = \frac{1}{20\pi}$cm/rad. Thus $\mathbf{r}(\theta) = 3\cos\theta\,\hat{\boldsymbol{\imath}} + 3\sin\theta\,\hat{\boldsymbol{\jmath}} + \frac{1}{20\pi}\theta\hat{\mathbf{k}}$.

With each full turn of the helix (i.e. each increase of θ by 2π) the height of the helix increases by $2\pi b = \frac{1}{10}$cm. So if we can determine the length of wire in one full turn of the helix, we can easily determine how many turns the helix goes through in total, and from that we can determine the total height of the helix.

As $\mathbf{r}'(\theta) = -3\sin\theta\,\hat{\boldsymbol{\imath}} + 3\cos\theta\,\hat{\boldsymbol{\jmath}} + \frac{1}{20\pi}\hat{\mathbf{k}}$ we have $\frac{ds}{d\theta} = \left|\mathbf{r}'(\theta)\right| = \sqrt{9 + \frac{1}{400\pi^2}}$. So the length of one full turn of the helix is

$$\int_0^{2\pi} \sqrt{9 + \frac{1}{400\pi^2}}\,\mathrm{d}\theta = 2\pi\sqrt{9 + \frac{1}{400\pi^2}}$$

and 1000cm of wire generates

$$\frac{1000}{2\pi\sqrt{9 + \frac{1}{400\pi^2}}} = \frac{500}{\pi\sqrt{9 + \frac{1}{400\pi^2}}}$$

turns. Each turn adds $\frac{1}{10}$cm to the height, so the total height is

$$\frac{500}{\pi\sqrt{9 + \frac{1}{400\pi^2}}} \cdot \frac{1}{10} = \frac{50}{\pi\sqrt{9 + \frac{1}{400\pi^2}}} \approx 5.3 \text{ cm}$$

Remark. We can check that this answer is reasonable by taking advantage of the fact that each coil adds only a very small height (relative to the radius). So we expect the length of one coil to be *about* the same as the circumference of a circle of the same radius, namely 6π. If we were making actual circles of the wire, there would be $\frac{1000}{6\pi}$ of them. Stacking up at 10 per centimetre, this would make a pile of height $\frac{1000}{6\pi \cdot 10}$cm. Since this number is also approximately 5.3cm, we feel our result is reasonable.

S-31: Define $\mathbf{u}(t) = e^{\alpha t}\frac{d\mathbf{r}}{dt}(t)$. Then

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}(t) &= \alpha e^{\alpha t}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t) + e^{\alpha t}\frac{d^2\mathbf{r}}{dt^2}(t) \\
&= \alpha e^{\alpha t}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t) - g e^{\alpha t}\hat{\mathbf{k}} - \alpha e^{\alpha t}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t) \\
&= -g e^{\alpha t}\hat{\mathbf{k}}
\end{aligned}$$

Integrating both sides of this equation from $t = 0$ to $t = T$ gives

$$\mathbf{u}(T) - \mathbf{u}(0) = -g\frac{e^{\alpha T} - 1}{\alpha}\hat{\mathbf{k}}$$

so that

$$\mathbf{u}(T) = \mathbf{u}(0) - g\frac{e^{\alpha T} - 1}{\alpha}\hat{\mathbf{k}} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(0) - g\frac{e^{\alpha T} - 1}{\alpha}\hat{\mathbf{k}} = \mathbf{v}_0 - g\frac{e^{\alpha T} - 1}{\alpha}\hat{\mathbf{k}}$$

Substituting in $\mathbf{u}(T) = e^{\alpha t}\frac{d\mathbf{r}}{dt}(T)$ and multiplying through by $e^{-\alpha T}$ gives

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(T) = e^{-\alpha T}\mathbf{v}_0 - g\frac{1 - e^{-\alpha T}}{\alpha}\hat{\mathbf{k}}$$

Integrating both sides of this equation from $T = 0$ to $T = t$ gives

$$\mathbf{r}(t) - \mathbf{r}(0) = \frac{e^{-\alpha t} - 1}{-\alpha}\mathbf{v}_0 - g\frac{t}{\alpha}\hat{\mathbf{k}} + g\frac{e^{-\alpha t} - 1}{-\alpha^2}\hat{\mathbf{k}}$$

so that

$$\mathbf{r}(t) = \mathbf{r}_0 - \frac{e^{-\alpha t} - 1}{\alpha}\mathbf{v}_0 + g\frac{1 - \alpha t - e^{-\alpha t}}{\alpha^2}\hat{\mathbf{k}}$$

Solutions to Exercises 1.2 — Jump to TABLE OF CONTENTS

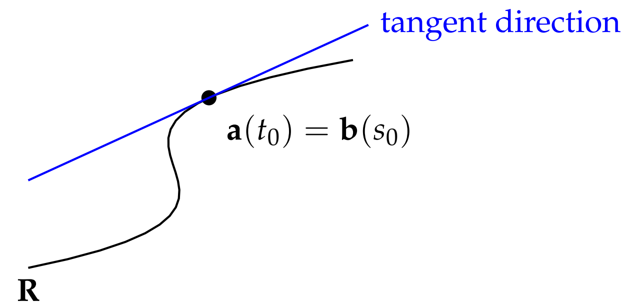
S-1: You're asked to find the arclength of the curve from $s = 1$ to $s = t$. However, arclength is given by s. So you're asked the length of the curve from the point where its arclength is one, to the point where its arclength is t. That is, $t - 1$.

S-2: The arclength from P to P will be 0, so P is the point where $s = 0$. That is, $\mathbf{r}(0)$, or $\big(\sin(1/2), \cos(1/2), \sqrt{3}/2\big)$.

S-3:

Solution 1: We consider the situation geometrically. If we plot $\mathbf{R}$ in space (of the relevant dimension), regardless of its parametrization, the derivative at a point will give a vector tangent to $\mathbf{R}$, in the direction the curve moves when the parameter is increasing. Since $\mathbf{a}(t_0)$ and $\mathbf{b}(s_0)$ describe the same spot on the curve, $\mathbf{a}'(t_0)$ and $\mathbf{b}'(s_0)$ will be parallel[1] — they're both tangent to the same piece of curve. Furthermore, as t increases, so does s, so the direction of increasing t is the same as the direction of increasing s. Therefore, A. holds.

1 Since we specified the derivatives are nonzero, there's no messiness about vectors being parallel to a zero vector.



Now we consider the magnitudes of the vectors, to rule out E. Recall $|\mathbf{a}'(t)|$ is the speed at which the curve changes relative to t; this could be any (nonnegative) number. By the same token, $|\mathbf{b}'(s)| = 1$. So, $\mathbf{b}'(s_0)$ is a unit vector, while $\mathbf{a}'(t_0)$ may or may not be. Then the two vectors are not *necessarily* equal (although they could be).

So, the best answer is A.

Solution 2: The chain rule gives us a relationship between $\mathbf{b}'(s)$ and $\mathbf{a}'(t)$.

$$\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}[\mathbf{a}(t(s))] = \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s}$$

So, the vectors $\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s}$ and $\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}$ differ only by the scalar function $\frac{\mathrm{d}t}{\mathrm{d}s}$. So, at any point along the curve, these vectors are parallel.

Furthermore, we know that t and s are positively correlated: as t increases, so does s, because we're covering more arclength. So, $\frac{\mathrm{d}t}{\mathrm{d}s}$ is nonnegative. Furthermore, since the derivatives are nonzero, $\frac{\mathrm{d}t}{\mathrm{d}s}$ is nonzero. So, $\mathbf{b}'(s_0)$ and $\mathbf{a}'(t_0)$ are positive scalar multiples of each other. That is, they are parallel, and pointing in the same direction. However, unless $\frac{\mathrm{d}t}{\mathrm{d}s} = 1$ (that is, $t(s) = s + C$ for some constant C), the vectors do not have the same magnitude, and hence are not equal.

So, A is the best solution.

S-4: (a) The velocity vector is

$$\begin{aligned}\mathbf{r}'(t) &= \big(6\sin^2(t)\cos t\,,\ -6\sin t\cos^2(t)\,,\ 3\cos^2 t - 3\sin^2 t\big)\\ &= 3\big(\sin t\sin(2t)\,,\ -\cos t\sin(2t)\,,\ \cos(2t)\big)\end{aligned}$$

In particular, since $\sin(\pi/3) = \sin(2\pi/3) = \frac{\sqrt{3}}{2}$ and $\cos(\pi/3) = -\cos(2\pi/3) = \frac{1}{2}$,

$$\mathbf{r}'(\pi/3) = 3\big(3/4\,,\ -\sqrt{3}/4\,,\ -1/2\big)$$

and the specified unit tangent vector is

$$\hat{\mathbf{T}} = \frac{\big(3/4\,,\ -\sqrt{3}/4\,,\ -1/2\big)}{\big|\big(3/4\,,\ -\sqrt{3}/4\,,\ -1/2\big)\big|} = \big(3/4\,,\ -\sqrt{3}/4\,,\ -1/2\big)$$

(b) The speed is

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}t} &= |\mathbf{r}'(t)| = 3\sqrt{\sin^2 t\sin^2(2t)+\cos^2 t\sin^2(2t)+\cos^2(2t)}\\ &= 3\sqrt{\sin^2(2t)+\cos^2(2t)}\\ &= 3\end{aligned}$$

So $s = 3t$ and the reparametrized form is

$$\mathbf{R}(s) = \big(2\sin^3(s/3), 2\cos^3(s/3), 3\sin(s/3)\cos(s/3)\big)$$

S-5: (a) We have $|\mathbf{r}(t)| = e^t \leqslant 1$ for $t \leqslant 0$. So the part of the spiral contained in the unit circle is the part of the spiral with $-\infty < t \leqslant 0$. As

$$\mathbf{r}'(t) = e^t(\cos t, \sin t) + e^t(-\sin t, \cos t) = e^t\big(\cos t - \sin t\,,\ \sin t + \cos t\big)$$

the speed

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \big|\mathbf{r}'(t)\big| = e^t\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2} = \sqrt{2}e^t$$

and the arclength from $t = -\infty$ to $\mathbf{r}(t)$ is

$$s(t) = \int_{-\infty}^{t} \frac{\mathrm{d}s}{\mathrm{d}t}(\tilde{t})\,\mathrm{d}\tilde{t} = \int_{-\infty}^{t} \sqrt{2}e^{\tilde{t}}\,\mathrm{d}\tilde{t} = \sqrt{2}e^t$$

In particular the length of the part of the spiral contained in the unit circle is $s(0) = \sqrt{2}$.

(b) The inverse function of $s(t) = \sqrt{2}e^t$ is $t(s) = \ln\left(\frac{s}{\sqrt{2}}\right)$. So the reparametrization is

$$\mathbf{R}(s) = e^t(\cos t, \sin t)\Big|_{t=\ln\left(\frac{s}{\sqrt{2}}\right)} = \frac{s}{\sqrt{2}}\left(\cos\left(\ln\left(\frac{s}{\sqrt{2}}\right)\right), \sin\left(\ln\left(\frac{s}{\sqrt{2}}\right)\right)\right)$$

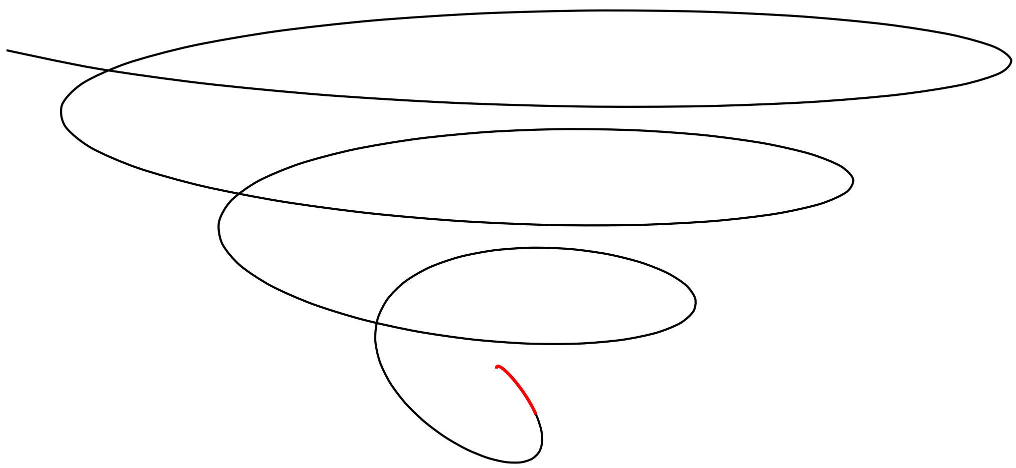
S-6: Using $\arctan t = z$, and so $t = \tan z$:

$$\begin{aligned}\mathbf{r}(t) &= \left(\frac{1}{\sqrt{1+t^2}}, \frac{\arctan t}{\sqrt{1+t^{-2}}}, \arctan t\right)\\ &= \left(\frac{1}{\sqrt{1+\tan^2 z}}, \frac{z}{\sqrt{1+\cot^2 z}}, z\right)\\ &= \left(\frac{1}{|\sec z|}, \frac{z}{|\csc z|}, z\right)\\ &= (|\cos z|,\ z|\sin z|,\ z)\end{aligned}$$

Since $0 \leqslant t$, and $\arctan t < \pi/2$ we have $0 \leqslant z < \pi/2$, so $\cos z$ and $\sin z$ are both nonnegative.

$$= (\cos z,\ z\sin z,\ z)$$

If we didn't have the restricted domain, this would make a spiral going up: z is both the height of the spiral and a radian measure. The $\hat{\boldsymbol{\imath}}$-component of the spiral stays between -1 and 1, while the $\hat{\boldsymbol{\jmath}}$-component increases. So, our spiral gets increasingly "wide," while staying the same "thickness."



Due to the restricted domain, our actual curve is only one-quarter of a "turn" of this spiral, indicated in red above.

The parameter z is a measure of height, and it is also a radian measure as the spiral turns.

S-7:

$$\begin{aligned}
\mathbf{r}(t) &= \left(\tfrac{1}{2}t^2, \tfrac{1}{3}t^3\right)\\
\mathbf{r}'(t) &= (t, t^2)\\
|\mathbf{r}'(t)| &= \sqrt{t^2+t^4} = |t|\sqrt{1+t^2}\\
s(t) &= \int_{-1}^{t} |x|\sqrt{1+x^2}\mathrm{d}x = \begin{cases} \int_{-1}^{t} -x\sqrt{1+x^2}\mathrm{d}x & \text{when } t \leqslant 0\\ \int_{-1}^{0} -x\sqrt{1+x^2}\mathrm{d}x + \int_0^t x\sqrt{1+x^2}\mathrm{d}x & \text{when } t > 0\end{cases}
\end{aligned}$$

Let $u = 1+x^2$, $\frac{1}{2}\mathrm{d}u = x\mathrm{d}x$

$$\begin{aligned}
&= \begin{cases} -\int_2^{1+t^2} \frac{1}{2}\sqrt{u}\mathrm{d}u & \text{when } t \leqslant 0\\ -\int_2^1 \frac{1}{2}\sqrt{u}\mathrm{d}u + \int_1^{1+t^2} \frac{1}{2}\sqrt{u}\mathrm{d}u & \text{when } t > 0\end{cases}\\
&= \begin{cases} -\frac{1}{3}u^{3/2}|_2^{1+t^2} & \text{when } t \leqslant 0\\ -\frac{1}{3}u^{3/2}|_2^{1} + \frac{1}{3}u^{3/2}|_1^{1+t^2} & \text{when } t > 0\end{cases}\\
&= \begin{cases} \frac{2^{3/2}}{3} - \frac{1}{3}(1+t^2)^{3/2} & \text{when } t \leqslant 0\\ -\frac{2}{3} + \frac{2^{3/2}}{3} + \frac{1}{3}(1+t^2)^{3/2} & \text{when } t > 0\end{cases}
\end{aligned}$$

Solving for t in terms of s:

$$1+t^2 = \begin{cases} (2\sqrt{2}-3s)^{2/3} & \text{when } t \leqslant 0 \\ (3s+2-2\sqrt{2})^{2/3} & \text{when } t > 0 \end{cases}$$

$$t^2 = \begin{cases} (2\sqrt{2}-3s)^{2/3}-1 & \text{when } t \leqslant 0 \\ (3s+2-2\sqrt{2})^{2/3}-1 & \text{when } t > 0 \end{cases}$$

Remembering that $\sqrt{t^2} = |t|$:

$$t = \begin{cases} -\sqrt{(2\sqrt{2}-3s)^{2/3}-1} & \text{when } t \leqslant 0 \\ \sqrt{(3s+2-2\sqrt{2})^{2/3}-1} & \text{when } t > 0 \end{cases}$$

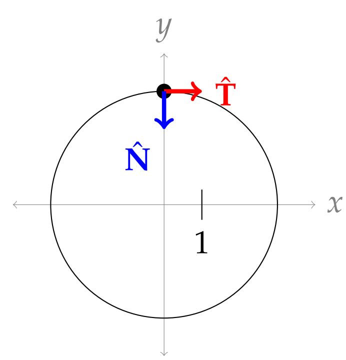
Noting that $t=0$ when $s = \frac{1}{3}(2\sqrt{2}-1)$, we find our reparametrization of $(\frac{1}{2}t^2, \frac{1}{3}t^3)$.

$$\mathbf{R}(s) = \begin{cases} \left(\frac{1}{2}\left[(2\sqrt{2}-3s)^{2/3}-1)\right], -\frac{1}{3}\left[(2\sqrt{2}-3s)^{2/3}-1\right]^{3/2}\right) & \text{when } s \leqslant \frac{1}{3}(2\sqrt{2}-1) \\ \left(\frac{1}{2}\left[(3s+2-2\sqrt{2})^{2/3}-1\right], \frac{1}{3}\left[(3s+2-2\sqrt{2})^{2/3}-1\right]^{3/2}\right) & \text{when } s > \frac{1}{3}(2\sqrt{2}-1) \end{cases}$$

Remark: after a computation with this much detail, it's nice to find a few points to check, to verify that our answer is reasonable. For instance, when $s=0$, t should be -1, and vice-versa. Also, we found that $t=0$ corresponds to $s = \frac{1}{3}(2\sqrt{2}-1)$. So, we should be able to verify that $\mathbf{r}(0) = \mathbf{R}\left(\frac{1}{3}(2\sqrt{2}-1)\right)$ and $\mathbf{r}(-1) = \mathbf{R}(0)$.

Solutions to Exercises 1.3 — Jump to TABLE OF CONTENTS

S-1: The curve is a circle of radius 3, centred at the origin. So, the "circle of best fit" is just the curve itself. $\hat{\mathbf{T}}$ is the unit vector tangent to the circle in direction of increasing t, and $\hat{\mathbf{N}}$ is the unit vector pointing towards the origin.



The radius of the (osculating) circle is 3, so $\rho = 3$ and $\kappa = \frac{1}{\rho} = \frac{1}{3}$.

S-2: The arclength of $\mathbf{r}(t)$ traced out by an interval of t of length θ is 3θ. That is, $s = 3t$. Our reparametrization of the circle in terms of arclength is
$\mathbf{R}(s) = (3\sin(s/3), 3\cos(s/3))$.

We can calculate the vectors tangent to the circle, then normalize them (i.e. make them length one) to find $\hat{\mathbf{T}}$.

$$\mathbf{v}(t) = \mathbf{r}'(t) = (3\cos t, -3\sin t) \qquad \hat{\mathbf{T}}(s) = \mathbf{R}'(s) = (\cos(s/3), -\sin(s/3))$$

$$\hat{\mathbf{T}}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{(3\cos t, -3\sin t)}{3} = (\cos t, -\sin t)$$

Note $\mathbf{R}'(s)$, because it's parametrized in terms of arclength, has derivative vectors of length one. So, we don't need to normalize them (although if we did, it wouldn't change anything).

Note also that we can check out answers using Question 1. In that question, we found $\hat{\mathbf{T}}$ was $\hat{\boldsymbol{\imath}}$ when $t = s = 0$; this fits with the vectors we just found.

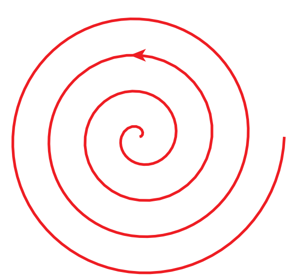
As in Question 1, $\kappa = \frac{1}{3}$. So, using Theorem 1.3.3 Part (b):

$$\begin{aligned}
\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s}(s) &= \kappa(s)\,\hat{\mathbf{N}}(s)\\
\left(-\frac{1}{3}\sin(s/3)\,,\ -\frac{1}{3}\cos(s/3)\right) &= \frac{1}{3}\hat{\mathbf{N}}(s)\\
(-\sin(s/3)\,,\ -\cos(s/3)) &= \hat{\mathbf{N}}(s)
\end{aligned}$$

Remember $s = 3t$. Using Theorem 1.3.3 Part (c):

$$\begin{aligned}
\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}t} &= \kappa\frac{\mathrm{d}s}{\mathrm{d}t}\hat{\mathbf{N}}(t)\\
(-\sin t, -\cos t) &= \frac{1}{3}(3)\hat{\mathbf{N}}(t)\\
(-\sin t, -\cos t) &= \hat{\mathbf{N}}(t)
\end{aligned}$$

S-3: As t increases, the arms of the spiral "flatten out," looking like a circle of bigger and bigger radius. So, we would expect the curvature to decrease: $\lim\limits_{t\to\infty} \kappa(t) = 0$.



S-4: $\frac{\mathrm{d}s}{\mathrm{d}t} = |\mathbf{v}(t)| = |\mathbf{r}'(t)| = \left|(e^t, 3, \cos t)\right| = \sqrt{e^{2t} + 9 + \cos^2 t}$

S-5:

$$\hat{\mathbf{T}}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

We use the chain rule to differentiate $\mathbf{r}(t)$.

$$\begin{aligned}
&= \frac{\big(e^t(\cos t - \sin t)\ ,\ e^t(\cos t + \sin t)\big)}{\sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2}} \\
&= \frac{1}{\sqrt{2}}\big(\cos t - \sin t\ ,\ \ \cos t + \sin t\big) \\
\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}t} &= \frac{1}{\sqrt{2}}\big(-\sin t - \cos t\ ,\ \ -\sin t + \cos t\big)
\end{aligned}$$

Since $\mathbf{R}(s)$ is parametrized with respect to arclength, $|\mathbf{R}'(s)| = 1$.

$$\hat{\mathbf{T}}(s) = \mathbf{R}'(s)$$

Making ample use of the chain rule, and setting $U(s) = \big(\ln\big(s/\sqrt{2}\big)\big)$, we have $U'(s) = \frac{1}{s}$:

$$\begin{aligned}
\hat{\mathbf{T}}(s) &= \frac{1}{\sqrt{2}}\left(\cos U(s) - \sin U(s)\ ,\ \ \cos U(s) + \sin U(s)\right) \\
\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s} &= \frac{1}{\sqrt{2}\,s}\left(-\sin U(s) - \cos U(s)\ ,\ \ -\sin U(s) + \cos U(s)\right)
\end{aligned}$$

S-6: The circle of radius r centred at $(0, r)$ is $x^2 + (y-r)^2 = r^2$. The bottom half of this circle is

$$y = g(x) = r - \sqrt{r^2 - x^2}$$

So

$$\begin{aligned}
g'(x) &= \frac{x}{\sqrt{r^2 - x^2}} & g'(0) &= 0 \\
g''(x) &= \frac{1}{\sqrt{r^2 - x^2}} + \frac{x^2}{[r^2 - x^2]^{3/2}} & g''(0) &= \frac{1}{r}
\end{aligned}$$

As $f(x)$ and $g(x)$ have the same second order Taylor approximation at $x = 0$, $f''(0) = g''(0) = \frac{1}{r}$.

We may parametrize the curve by $\mathbf{r}(x) = x\,\hat{\imath} + f(x)\,\hat{\jmath}$. So

$$\begin{aligned}
\mathbf{r}'(x) &= \hat{\imath} + f'(x)\,\hat{\jmath} & \mathbf{r}'(0) &= \hat{\imath} + f'(0)\,\hat{\jmath} = \hat{\imath} \\
\mathbf{r}''(x) &= f''(x)\,\hat{\jmath} & \mathbf{r}''(0) &= f''(0)\,\hat{\jmath}
\end{aligned}$$

$$\kappa(0) = \frac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3} = \frac{|f''(0)\,\hat{\imath} \times \hat{\jmath}|}{|\hat{\imath}|^3} = f''(0)$$

So $\kappa(0) = f''(0) = \frac{1}{r}$ and r is indeed the radius of curvature of $y = f(x)$ at $x = 0$.

S-7:

A. $\mathbf{v}(t) = \mathbf{r}'(t) = (e^t, 2t+1)$

B. $\mathbf{a}(t) = \mathbf{r}''(t) = (e^t, 2)$

C. $\frac{\mathrm{d}s}{\mathrm{d}t} = |\mathbf{v}(t)| = \sqrt{e^{2t} + (2t+1)^2}$

D. $\hat{\mathbf{T}}(t) = \dfrac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \dfrac{(e^t, 2t+1)}{\sqrt{e^{2t} + (2t+1)^2}} = \left(\dfrac{e^t}{\sqrt{e^{2t} + (2t+1)^2}}, \dfrac{2t+1}{\sqrt{e^{2t} + (2t+1)^2}}\right)$

E. $\kappa(t) = \dfrac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3} = \dfrac{\left|(e^t, 2t+1) \times (e^t, 2)\right|}{\sqrt{e^{2t} + (2t+1)^2}^3} = \dfrac{e^t|1-2t|}{(e^{2t} + (2t+1)^2)^{3/2}}$

S-8:

Solution 1: Note that $(\cos t + \sin t)^2 + (\sin t - \cos t)^2 = 2$ for all t. So, the points (x, y) on our curve obey $x^2 + y^2 = 2$. That is, we have a circle of radius $\sqrt{2}$. So, $\kappa = \frac{1}{\sqrt{2}}$.

Solution 2: We use the formula $\kappa = \dfrac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{\left|\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3\right|}$, remembering that $\mathbf{v}(t) = \mathbf{r}'(t)$, $\mathbf{a}(t) = \mathbf{r}''(t)$, and $\frac{\mathrm{d}s}{\mathrm{d}t} = |\mathbf{r}'(t)|$.

$$\begin{aligned}
\mathbf{v}(t) &= \mathbf{r}'(t) = (-\sin t + \cos t, \cos t + \sin t) \\
\mathbf{a}(t) &= \mathbf{r}''(t) = (-\cos t - \sin t, -\sin t + \cos t) \\
\mathbf{v}(t) \times \mathbf{a}(t) &= \left[(-\sin t + \cos t)^2 + (\cos t + \sin t)^2\right]\hat{\mathbf{k}} = 2\hat{\mathbf{k}} \\
\frac{\mathrm{d}s}{\mathrm{d}t} &= \left|\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right| = \sqrt{(-\sin t + \cos t)^2 + (\cos t + \sin t)^2} = \sqrt{2} \\
\kappa &= \left|\frac{\mathbf{v}(t) \times \mathbf{a}(t)}{\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3}\right| = \left|\frac{2\hat{\mathbf{k}}}{\sqrt{2}^3}\right| = \frac{1}{\sqrt{2}}
\end{aligned}$$

S-9: For the given ellipse

$$\begin{aligned}
\mathbf{r}(t) &= a\cos t\,\hat{\boldsymbol{\imath}} + b\sin t\,\hat{\boldsymbol{\jmath}} \\
\mathbf{v}(t) &= -a\sin t\,\hat{\boldsymbol{\imath}} + b\cos t\,\hat{\boldsymbol{\jmath}} \\
|\mathbf{v}(t)| &= \sqrt{a^2\sin^2 t + b^2\cos^2 t} \\
\mathbf{a}(t) &= -a\cos t\,\hat{\boldsymbol{\imath}} - b\sin t\,\hat{\boldsymbol{\jmath}} \\
\mathbf{v}(t) \times \mathbf{a}(t) &= \det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ -a\sin t & b\cos t & 0 \\ -a\cos t & -b\sin t & 0 \end{bmatrix} = ab\,\hat{\mathbf{k}} \\
\kappa(t) &= \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{|\mathbf{v}(t)|^3} = \frac{ab}{\left[a^2\sin^2 t + b^2\cos^2 t\right]^{3/2}}
\end{aligned}$$

Hence the maximum (minimum) curvature is achieved when the denominator is a minimum (maximum) which is the case when $\sin t = 0$ ($\cos t = 0$). So $\kappa_{\text{max}} = \frac{a}{b^2}$ and $\kappa_{\text{min}} = \frac{b}{a^2}$.

S-10: Parametrize the curve by $\mathbf{r}(t) = t\,\hat{\boldsymbol{\imath}} + e^t\,\hat{\boldsymbol{\jmath}}$. Then

$$\begin{array}{llll} \mathbf{v}(t) = \hat{\boldsymbol{\imath}} + e^t\,\hat{\boldsymbol{\jmath}} & \mathbf{v}(0) = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} & \dfrac{ds}{dt} = |\mathbf{v}(t)| = \sqrt{1+e^{2t}} & \hat{\mathbf{T}}(t) = \dfrac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \dfrac{\hat{\boldsymbol{\imath}} + e^t\hat{\boldsymbol{\jmath}}}{\sqrt{1+e^{2t}}} \\ \mathbf{a}(t) = e^t\,\hat{\boldsymbol{\jmath}} & \mathbf{a}(0) = \hat{\boldsymbol{\jmath}} & \dfrac{ds}{dt}(0) = \sqrt{2} & \hat{\mathbf{T}}(0) = \dfrac{\mathbf{v}(0)}{|\mathbf{v}(0)|} = \dfrac{\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}}{\sqrt{2}} \end{array}$$

(a) We're given y in terms of x, so let's use Part (e) of Theorem 1.3.3:

$$\begin{aligned} \kappa &= \frac{\left|\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|}{\left[1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{3/2}} = \frac{e^x}{\left[1+\left(e^x\right)^2\right]^{3/2}} \\ \kappa(0) &= \frac{1}{[1+1]^{3/2}} = 2^{-3/2} \end{aligned}$$

(b)

- The radius of the circle we want is $\rho = \frac{1}{\kappa} = 2^{3/2}$. If its centre is at (a,b), then the circle will have equation $(x-a)^2 + (y-b)^2 = 2^3$. So, we will find its centre.
- The unit vector $\hat{\mathbf{N}}$ points from our point $(0,1)$ towards the centre of the circle. Since the radius of the circle is $2^{3/2}$, the centre of the circle will be at $(0,1) + 2^{3/2}\hat{\mathbf{N}}$. So, we'll find $\hat{\mathbf{N}}$.
- Since $\hat{\mathbf{N}}$ is a unit vector perpendicular to $\hat{\mathbf{T}} = \dfrac{\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}}{\sqrt{2}}$, we know $\hat{\mathbf{N}}$ will be either $\dfrac{\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}}{\sqrt{2}}$ or $\dfrac{-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}}{\sqrt{2}}$.
- Using Part (1.3) of the proof of Theorem 1.3.3:

$$\begin{aligned} \mathbf{v}(t) \times \mathbf{a}(t) &= \kappa \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3 \hat{\mathbf{T}} \times \hat{\mathbf{N}} \\ (\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}) \times (\hat{\boldsymbol{\jmath}}) &= 2^{-3/2}\left(\sqrt{2}\right)^3 \frac{\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}}{\sqrt{2}} \times \hat{\mathbf{N}} \\ \hat{\mathbf{k}} &= \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}) \times \hat{\mathbf{N}} \\ \hat{\mathbf{N}} &= \frac{-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}}{\sqrt{2}} \end{aligned}$$

So, the centre of our circle is at point $(0,1) + \rho\hat{\mathbf{N}} = (0,1) + 2^{3/2}\frac{-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}}{2^{1/2}} = (-2,3)$. Then the equation of the circle is $(x+2)^2 + (y-3)^2 = 8$.

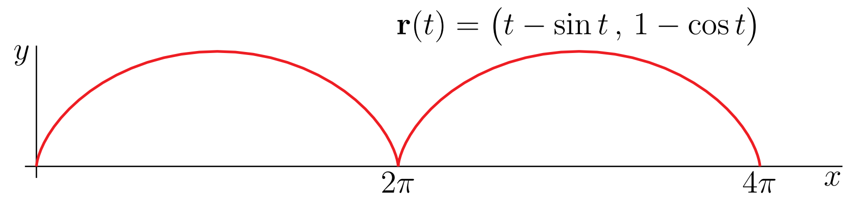
S-11: (a) Think of

$$\mathbf{r}(t) = (t, 1) - (\sin t, \cos t)$$

The $(t, 1)$ part gives the position of the centre of the wheel at time t. The other part gives the position of the thumbtack with respect to the centre of the wheel. In particular,

- at time $t = 0$, $\mathbf{r}(0) = (0, 0)$. The thumbtack is on the ground (i.e. at $y = 0$).
- At time $t = \pi$, $\mathbf{r}(\pi) = (\pi, 2)$. The thumbtack is at its highest point (i.e. at $y = 2$) and is above the centre of the wheel at $x = \pi$.
- At time $t = 2\pi$, $\mathbf{r}(2\pi) = (2\pi, 0)$. The thumbtack is back on the ground (i.e. at $y = 0$) and is below the centre of the wheel at $x = 2\pi$.
- At time $t = 3\pi$, $\mathbf{r}(3\pi) = (3\pi, 2)$. The thumbtack is again at its highest point (i.e. at $y = 2$) and is above the centre of the wheel at $x = 3\pi$.
- At time $t = 4\pi$, $\mathbf{r}(4\pi) = (4\pi, 0)$. The thumbtack is back on the ground (i.e. at $y = 0$) and is below the centre of the wheel at $x = 4\pi$.

Here is a sketch of the curve.



(b) Since

$$\begin{aligned}
\mathbf{r}(t) &= \big(t - \sin t\,,\ 1 - \cos t\big)\\
\mathbf{v}(t) = \mathbf{r}'(t) &= \big(1 - \cos t\,,\ \sin t\big)\\
\frac{\mathrm{d}s}{\mathrm{d}t}(t) = |\mathbf{v}(t)| &= \sqrt{2 - 2\cos t}\\
\mathbf{a}(t) = \mathbf{v}'(t) &= \big(\sin t\,,\ \cos t\big)\\
\mathbf{v}(t) \times \mathbf{a}(t) = \det &\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 1 - \cos t & \sin t & 0 \\ \sin t & \cos t & 0 \end{bmatrix} = \big(\cos t - 1\big)\,\hat{\mathbf{k}}
\end{aligned}$$

the curvature

$$\kappa(t) = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{|\mathbf{v}(t)|^3} = \frac{|\cos t - 1|}{(2 - 2\cos t)^{3/2}} = \frac{1}{2^{3/2}\sqrt{1 - \cos t}}$$

(c) The radius of curvature at time $t = \pi$ is

$$\rho(\pi) = \frac{1}{\kappa(\pi)} = \frac{1}{1/2^{3/2}\sqrt{2}} = 4$$

(d) At time π, the tack is at $\mathbf{r}(\pi) = (\pi, 2)$, which is at the top of its trajectory. Looking at the sketch in part (a), we see that, at that time $\hat{\mathbf{N}}(\pi) = -\hat{\jmath}$. So the osculating circle at time $t = \pi$ has center

$$\mathbf{r}(\pi) + \rho(\pi)\hat{\mathbf{N}}(\pi) = (\pi, 2) + 4(0, -1) = (\pi, -2)$$

and radius $\rho(\pi) = 4$. So the equation of the osculating circle at time π is

$$(x - \pi)^2 + (y + 2)^2 = 16$$

S-12: The velocity vector is

$$\mathbf{v}(\theta) = x'(\theta)\,\hat{\boldsymbol{\imath}} + y'(\theta)\,\hat{\boldsymbol{\jmath}} = \cos\left(\tfrac{1}{2}\pi\theta^2\right)\hat{\boldsymbol{\imath}} + \sin\left(\tfrac{1}{2}\pi\theta^2\right)\hat{\boldsymbol{\jmath}}$$

Consequently the speed

$$\frac{\mathrm{d}s}{\mathrm{d}\theta}(\theta) = |\mathbf{v}(\theta)| = 1 \implies s(\theta) = \theta + s(0)$$

Since $s(\theta)$ is zero when $\theta = 0$, we have $s(\theta) = \theta$ and hence

$$\hat{\mathbf{T}}(s) = \mathbf{v}(s) = \cos\left(\tfrac{1}{2}\pi s^2\right)\hat{\boldsymbol{\imath}} + \sin\left(\tfrac{1}{2}\pi s^2\right)\hat{\boldsymbol{\jmath}}$$

so that

$$\kappa(s) = \left|\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s}(s)\right| = \left| -\pi s\sin\left(\tfrac{1}{2}\pi s^2\right)\hat{\boldsymbol{\imath}} + \pi s\cos\left(\tfrac{1}{2}\pi s^2\right)\hat{\boldsymbol{\jmath}}\right| = \pi s$$

S-13: The curve is $y = y(x) = {}^{x^3}\!/_3$. Since $y'(x) = x^2$ and $y''(x) = 2x$, the curvature is

$$\kappa(x) = \frac{\left|\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(x)\right|}{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}(x)\right)^2\right]^{3/2}} = \frac{|2x|}{\left[1 + x^4\right]^{3/2}}$$

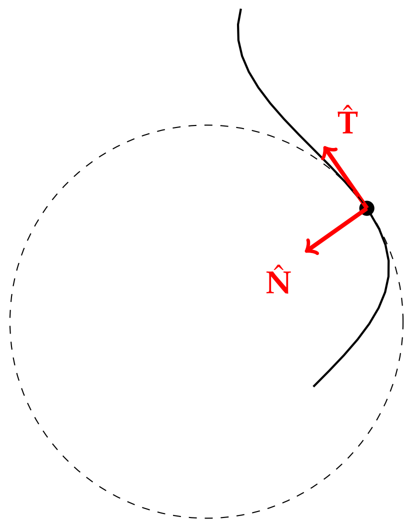
We'd like to find the critical points of $\kappa(x)$, but differentiating it looks messy. Since $\kappa(x)$ has only nonnegative values, its maxima correspond the the maxima of the function $\kappa^2(x)$. So, we find the critical points of $\kappa^2(x)$ instead, to save ourselves some computational toil.

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}x}\kappa(x)^2 = \frac{\mathrm{d}}{\mathrm{d}x}\frac{4x^2}{(1+x^4)^3} = \frac{8x}{(1+x^4)^3} - 3\frac{16x^5}{(1+x^4)^4} = \frac{8x(1+x^4) - 3\times 16x^5}{(1+x^4)^4} \\ &= \frac{8x(1-5x^4)}{(1+x^4)^4} \end{aligned}$$

Note that $\kappa(0) = 0$ and $\kappa(x) \to 0$ as $x \to \pm\infty$. So the maximum occurs when $x = \pm 1/\sqrt[4]{5}$.

Solutions to Exercises 1.4 — Jump to TABLE OF CONTENTS

S-1: $\hat{\mathbf{T}}$ is tangent to the curve, while $\hat{\mathbf{N}}$ is perpendicular to it.



Using the right-hand rule and $\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}$, $\hat{\mathbf{B}}$ points out of the page (towards the reader).

To see this, point the fingers of your right hand in the direction of $\hat{\mathbf{T}}$, and curl them inwards until they are in the direction of $\hat{\mathbf{N}}$. To do this, your thumb must be pointing towards you, not away from you. Your thumb shows the direction of $\hat{\mathbf{T}} \times \hat{\mathbf{N}}$.

S-2: In this equation, s stands for arclength.

When we take a very small interval from t to $t+h$, the change in arclength $s(t+h)-s(t)$ is approximately $|\mathbf{r}(t+h)-\mathbf{r}(t)|$, because our curve is approximated by a straight line. So, $\frac{s(t+h)-s(t)}{h} \approx \frac{|\mathbf{r}(t+h)-\mathbf{r}(t)|}{h}$, leading to $\frac{\mathrm{d}s}{\mathrm{d}t} = |\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}| = |\mathbf{v}(t)|$.

The magnitude of velocity is speed; in this text we generally call this v. That is, $v = |\mathbf{v}(t)|$. This leads to the potentially confusing (but standard) convention that s stands for arclength, while v stands for speed.

S-3: **Solution 1:**
Curves **a** and **b** are the same curve, just parametrized differently (replace t with $-t$ to convince yourself if the picture isn't enough). So, they ought to have the same torsion.

As in Example 1.4.4, we imagine that the curve is the thread on a bolt. Take a look at your right hand. If your thumb is pointing up (corresponding to the $+z$ direction), and you're looking at the tip of your thumb, your fingers curl anticlockwise. Imagine a screw has threads matching the curves **a** and **b**, and we turn it anticlockwise. The screw would move down — *not* in the same direction as our thumb. So these curves are not right-handed helices, so they have negative torsion.

The curve **c** sits entirely in a plane (the plane $x = 0$) so its torsion is zero everywhere.

Solution 2:
Here is the conventional computation for both $\mathbf{a}(t)$ and $\mathbf{b}(t)$. (The upper sign is for **a** and

the lower sign is for **b**.)

$$\begin{aligned}
\mathbf{r}(t) &= \big(\cos t\,,\, \mp 2\sin t\,,\, \pm t/2\big)\\
\mathbf{v}(t) &= \big(-\sin t\,,\, \mp 2\cos t\,,\, \pm 1/2\big)\\
\mathbf{a}(t) &= \big(-\cos t\,,\, \pm 2\sin t\,,\, 0\big)\\
\mathbf{v}(t)\times\mathbf{a}(t) &= \big(-\sin t\,,\, \mp\cos t/2\,,\, \mp 2\big)\\
\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}(t) &= \big(\sin t\,,\, \pm 2\cos t\,,\, 0\big)\\
\mathbf{v}(t)\times\mathbf{a}(t)\cdot\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}(t) &= -1\\
\tau(t) = \frac{\mathbf{v}(t)\times\mathbf{a}(t)\cdot\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}(t)}{|\mathbf{v}(t)\times\mathbf{a}(t)|^2} &= -\frac{1}{\sin^2 t+\frac{1}{4}\cos^2 t+4} < 0
\end{aligned}$$

S-4: (a) If $\kappa(s)\equiv 0$, then $\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s}=\kappa(s)\hat{\mathbf{N}}(s)\equiv 0$ so that $\hat{\mathbf{T}}$ is a constant. As a result $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}(s)=\hat{\mathbf{T}}$ and $\mathbf{r}(s)=s\hat{\mathbf{T}}+\mathbf{r}(0)$ so that the curve is the straight line with direction vector $\hat{\mathbf{T}}$ that passes through $\mathbf{r}(0)$.

(b) If $\tau(s)\equiv 0$, then $\frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}s}=-\tau(s)\hat{\mathbf{N}}(s)\equiv 0$ so that $\hat{\mathbf{B}}$ is a constant. As $\hat{\mathbf{T}}(s)\perp\hat{\mathbf{B}}$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\big(\mathbf{r}(s)-\mathbf{r}(0)\big)\cdot\hat{\mathbf{B}}=\hat{\mathbf{T}}(s)\cdot\hat{\mathbf{B}}=0$$

and $\big(\mathbf{r}(s)-\mathbf{r}(0)\big)\cdot\hat{\mathbf{B}}$ must be a constant. The constant must be zero (set $s=0$), so $\big(\mathbf{r}(s)-\mathbf{r}(0)\big)\cdot\hat{\mathbf{B}}=0$ and $\mathbf{r}(s)$ always lies in the plane through $\mathbf{r}(0)$ with normal vector $\hat{\mathbf{B}}$.

(c) Parametrize the curve by arc length. Define the "centre of curvature" at s by

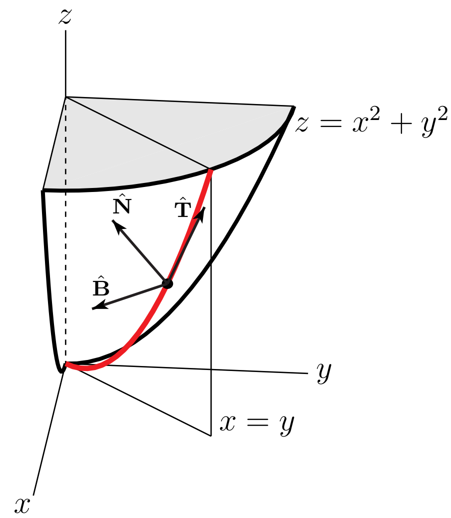
$$\mathbf{r}_c(s)=\mathbf{r}(s)+\frac{1}{\kappa(s)}\hat{\mathbf{N}}(s)$$

Since $\kappa(s)=\kappa_0$ is a constant and $\tau(s)\equiv 0$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{r}_c(s)=\hat{\mathbf{T}}(s)+\frac{1}{\kappa_0}\big[\tau(s)\hat{\mathbf{B}}-\kappa(s)\hat{\mathbf{T}}\big]=\hat{\mathbf{T}}(s)+\frac{1}{\kappa_0}\big[0\hat{\mathbf{B}}-\kappa_0\hat{\mathbf{T}}\big]=0$$

Thus $\mathbf{r}_c(s)=\mathbf{r}_c$ is a constant and $\big|\mathbf{r}(s)-\mathbf{r}_c\big|=\frac{1}{\kappa_0}$ lies on the sphere of radius $\frac{1}{\kappa_0}$ centred on $\mathbf{r}_c$. Since $\tau(s)\equiv 0$, the curve also lies on a plane, so it is a circle.

S-5: (a), (b): $\hat{\mathbf{T}}$ points in the direction of the curve; $\hat{\mathbf{N}}$ is perpendicular to it, in the same plane, pointing towards the centre of curvature. Using the right-hand rule in the picture, we see $\hat{\mathbf{B}}$ is pointing to the left.



(c) The torsion is zero, since the curve lies in a plane (the plane $x = y$).

S-6: (a) As

$$\mathbf{r}'(t) = \left(e^t + e^{-t}\right)\hat{\boldsymbol{\imath}} + \left(e^t - e^{-t}\right)\hat{\boldsymbol{\jmath}} + 2\,\hat{\mathbf{k}} \qquad \frac{\mathrm{d}s}{\mathrm{d}t}(t) = |\mathbf{r}'(t)| = \sqrt{4 + 2e^{2t} + 2e^{-2t}} = \sqrt{2}\left(e^t + e^{-t}\right)$$

$$\mathbf{r}''(t) = \left(e^t - e^{-t}\right)\hat{\boldsymbol{\imath}} + \left(e^t + e^{-t}\right)\hat{\boldsymbol{\jmath}} \qquad \mathbf{r}'(t) \times \mathbf{r}''(t) = -2\left(e^t + e^{-t}\right)\hat{\boldsymbol{\imath}} + 2\left(e^t - e^{-t}\right)\hat{\boldsymbol{\jmath}} + 4\,\hat{\mathbf{k}}$$

the curvature

$$\kappa(t) = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3} = \frac{2\sqrt{4 + 2e^{2t} + 2e^{-2t}}}{[4 + 2e^{2t} + 2e^{-2t}]^{3/2}} = \frac{1}{2 + e^{2t} + e^{-2t}}$$

(b) The length of C between $\mathbf{r}(0)$ and $\mathbf{r}(1)$ is

$$\int_0^1 \frac{\mathrm{d}s}{\mathrm{d}t}(t)\,\mathrm{d}t = \sqrt{2}\int_0^1 (e^t + e^{-t})\,\mathrm{d}t = \sqrt{2}\Big[e^t - e^{-t}\Big]_0^1 = \sqrt{2}\Big[e - \frac{1}{e}\Big]$$

S-7: The point $(2, 4, 8)$ occurs when $t = 2$.

$$\begin{aligned} \mathbf{v}(t) &= (1, 2t, 3t^2) & \mathbf{v}(2) &= (1, 4, 12) \\ \mathbf{a}(t) &= (0, 2, 6t) & \mathbf{a}(2) &= (0, 2, 12) \\ \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}(t) &= (0, 0, 6) & \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}(2) &= (0, 0, 6) \\ & & \mathbf{v}(2) \times \mathbf{a}(2) &= (24, -12, 2) \\ & & |\mathbf{v}(2) \times \mathbf{a}(2)| &= 2\sqrt{181} \end{aligned}$$

Now, we use a formula for torsion:

$$\tau(t) = \frac{(\mathbf{v}(t) \times \mathbf{a}(t)) \cdot \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}(t)}{|\mathbf{v}(t) \times \mathbf{a}(t)|^2}$$

$$\tau(2) = \frac{(24, -12, 2) \cdot (0, 0, 6)}{(2\sqrt{181})^2} = \frac{3}{181}$$

S-8: For the specified curve

$$\begin{aligned}
\mathbf{r}(t) &= t\,\hat{\boldsymbol{\imath}} + \frac{t^2}{2}\,\hat{\boldsymbol{\jmath}} + \frac{t^3}{3}\,\hat{\mathbf{k}} \\
\mathbf{v}(t) = \mathbf{r}'(t) &= \hat{\boldsymbol{\imath}} + t\,\hat{\boldsymbol{\jmath}} + t^2\,\hat{\mathbf{k}} \\
\mathbf{a}(t) = \mathbf{r}''(t) &= \hat{\boldsymbol{\jmath}} + 2t\,\hat{\mathbf{k}} \\
\mathbf{v}(t) \times \mathbf{a}(t) &= \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 1 & t & t^2 \\ 0 & 1 & 2t \end{bmatrix} \\
&= t^2\,\hat{\boldsymbol{\imath}} - 2t\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}} \\
\mathbf{a}'(t) &= 2\,\hat{\mathbf{k}}
\end{aligned}$$

From this, we read off

$$\begin{aligned}
\hat{\mathbf{T}}(t) &= \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\hat{\boldsymbol{\imath}} + t\,\hat{\boldsymbol{\jmath}} + t^2\,\hat{\mathbf{k}}}{\sqrt{1+t^2+t^4}} \\
\kappa(t) &= \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{|\mathbf{v}(t)|^3} = \frac{\sqrt{1+4t^2+t^4}}{[1+t^2+t^4]^{3/2}} \\
\hat{\mathbf{B}}(t) &= \frac{\mathbf{v}(t) \times \mathbf{a}(t)}{|\mathbf{v}(t) \times \mathbf{a}(t)|} = \frac{t^2\,\hat{\boldsymbol{\imath}} - 2t\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}}{\sqrt{1+4t^2+t^4}} \\
\hat{\mathbf{N}}(t) &= \hat{\mathbf{B}}(t) \times \hat{\mathbf{T}}(t) \\
&= \frac{1}{\sqrt{1+t^2+t^4}\sqrt{1+4t^2+t^4}} \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ t^2 & -2t & 1 \\ 1 & t & t^2 \end{bmatrix} \\
&= \frac{-(t+2t^3)\,\hat{\boldsymbol{\imath}} + (1-t^4)\,\hat{\boldsymbol{\jmath}} + (2t+t^3)\hat{\mathbf{k}}}{\sqrt{1+t^2+t^4}\sqrt{1+4t^2+t^4}} \\
\tau(t) &= \frac{\big(\mathbf{v}(t) \times \mathbf{a}(t)\big) \cdot \mathbf{a}'(t)}{|\mathbf{v}(t) \times \mathbf{a}(t)|^2} = \frac{2}{1+4t^2+t^4}
\end{aligned}$$

S-9: First, some preliminaries:

$$\begin{aligned}
\mathbf{r}(t) &= (t^3, t, e^{ct}) & \mathbf{r}(5) &= (5^3, 5, e^{5c}) \\
\mathbf{v}(t) &= (3t^2, 1, ce^{ct}) & \mathbf{v}(5) &= (3 \cdot 5^2, 1, ce^{5c}) \\
\mathbf{a}(t) &= (6t, 0, c^2e^{ct}) & \mathbf{a}(5) &= (6 \cdot 5, 0, c^2e^{5c}) \\
\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}(t) &= (6, 0, c^3e^{ct}) & \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}(5) &= (6, 0, c^3e^{5c}) \\
& & \mathbf{v}(5) \times \mathbf{a}(5) &= (c^2e^{5c}, 15ce^{5c}(2-5c), -30)
\end{aligned}$$

Second, we figure out what value of c makes $\tau(5) = 0$.

$$\begin{aligned}
0 = \tau(5) &= \frac{(\mathbf{v}(5) \times \mathbf{a}(5)) \cdot \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}(5)}{|\mathbf{v}(5) \times \mathbf{a}(5)|^2} \\
0 &= (\mathbf{v}(5) \times \mathbf{a}(5)) \cdot \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}(5) \\
&= (c^2 e^{5c}, 15ce^{5c}(2-5c), -30) \cdot (6, 0, c^3 e^{5c}) \\
&= 6c^2 e^{5c}(1-5c) \\
c = 0 \text{ or } c &= \frac{1}{5}
\end{aligned}$$

If $c = 0$, then $\mathbf{r}(t) = (t^3, t, 1)$, and so the entire curve is contained inside the plane $z = 1$. (Its torsion is zero everywhere — not just at $t = 5$.)

Consider the case $c = \frac{1}{5}$. When $t = 5$, our curve (and its osculating circle) passes through the point $\mathbf{r}(5) = (5^3, 5, e)$. The normal vector to the plane of the osculating curve is the binormal vector $\hat{\mathbf{B}}(5) = \frac{\mathbf{v}(5)\times\mathbf{a}(5)}{|\mathbf{v}(5)\times\mathbf{a}(5)|}$. Since we don't need the normal vector to the plane to be a unit vector, we can take as the normal vector to the plane simply $\mathbf{v}(5) \times \mathbf{a}(5)$, or $(e/25, 3e, -30)$. Then, an equation of the plane containing the osculating circle is $(e/25)x + (3e)y - 30z = -10e$. An equivalent equation for this plane is $(1/25)x + 3y - (30/e)z = -10$.

S-10: (a) Since $\mathbf{r}'(t) = (2t, 1, 3t^2)$, we have $\mathbf{r}'(1) = (2, 1, 3)$. So the normal plane must pass through $\mathbf{r}(1) = (1, 1, 1)$ and be perpendicular to $(2, 1, 3)$. The equation of the normal plane is then

$$2(x-1) + (y-1) + 3(z-1) = 0 \qquad \text{or} \qquad 2x + y + 3z = 6$$

(b) As

$$\begin{aligned}
\mathbf{v}(t) &= \mathbf{r}'(t) = \big(2t, 1, 3t^2\big) & \frac{\mathrm{d}s}{\mathrm{d}t} &= \sqrt{1 + 4t^2 + 9t^4} \\
\mathbf{a}(t) &= \mathbf{v}'(t) = \big(2, 0, 6t\big) & \mathbf{v}(t) \times \mathbf{a}(t) &= \big(6t, -6t^2, -2\big)
\end{aligned}$$

the curvature

$$\kappa(t) = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3} = \frac{2\sqrt{1 + 9t^2 + 9t^4}}{[1 + 4t^2 + 9t^4]^{3/2}}$$

S-11: First some preliminaries.

$$\begin{aligned}
\mathbf{v}(t) &= \mathbf{r}'(t) = -\sin t\,\hat{\boldsymbol{\imath}} + \cos t\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}} \\
\mathbf{a}(t) &= \mathbf{r}''(t) = -\cos t\,\hat{\boldsymbol{\imath}} - \sin t\,\hat{\boldsymbol{\jmath}}
\end{aligned}$$

(a), (b) From $\mathbf{v}(t)$ we read off

$$\frac{\mathrm{d}s}{\mathrm{d}t} = |\mathbf{v}(t)| = \sqrt{2}$$

From $\mathbf{a}(t) = \frac{\mathrm{d}^2s}{\mathrm{d}t^2}(t)\,\hat{\mathbf{T}}(t) + \kappa(t)\big(\frac{\mathrm{d}s}{\mathrm{d}t}(t)\big)^2\hat{\mathbf{N}}(t)$, and the fact that $\frac{\mathrm{d}^2s}{\mathrm{d}t^2} = 0$, we read off that

$$\kappa(t) = \Big(\frac{\mathrm{d}s}{\mathrm{d}t}(t)\Big)^{-2}|\mathbf{a}| = \frac{1}{2} \qquad \hat{\mathbf{N}}(t) = \frac{\mathbf{a}}{|\mathbf{a}|} = -\cos t\,\hat{\boldsymbol{\imath}} - \sin t\,\hat{\boldsymbol{\jmath}}$$

So radius of curvature is $\frac{1}{\kappa} = 2$ and the centre of curvature is

$$\begin{aligned}\Big[\mathbf{r}(t) + \frac{1}{\kappa(t)}\hat{\mathbf{N}}(t)\Big]_{t=\pi/6} &= \Big[\big(\cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}} + t\hat{\mathbf{k}}\big) + 2\big(-\cos t\,\hat{\boldsymbol{\imath}} - \sin t\,\hat{\boldsymbol{\jmath}}\big)\Big]_{t=\pi/6}\\ &= \Big[-\cos t\,\hat{\boldsymbol{\imath}} - \sin t\,\hat{\boldsymbol{\jmath}} + t\,\hat{\mathbf{k}}\Big]_{t=\pi/6}\\ &= -\frac{\sqrt{3}}{2}\,\hat{\boldsymbol{\imath}} - \frac{1}{2}\hat{\boldsymbol{\jmath}} + \frac{\pi}{6}\,\hat{\mathbf{k}}\end{aligned}$$

(c) From

$$\begin{aligned}\mathbf{v}(t)\times\mathbf{a}(t) &= \det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ -\sin t & \cos t & 1\\ -\cos t & -\sin t & 0\end{bmatrix} = \sin t\,\hat{\boldsymbol{\imath}} - \cos t\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\\ |\mathbf{v}(t)\times\mathbf{a}(t)|^2 &= 2\end{aligned}$$

we read off

$$\hat{\mathbf{B}}(t) = \frac{\mathbf{v}(t)\times\mathbf{a}(t)}{|\mathbf{v}(t)\times\mathbf{a}(t)|} = \frac{1}{\sqrt{2}}\sin t\,\hat{\boldsymbol{\imath}} - \frac{1}{\sqrt{2}}\cos t\,\hat{\boldsymbol{\jmath}} + \frac{1}{\sqrt{2}}\,\hat{\mathbf{k}}$$

so that

$$\hat{\mathbf{B}}\big(\pi/6\big) = \frac{1}{2\sqrt{2}}\,\hat{\boldsymbol{\imath}} - \frac{\sqrt{3}}{2\sqrt{2}}\,\hat{\boldsymbol{\jmath}} + \frac{1}{\sqrt{2}}\,\hat{\mathbf{k}}$$

S-12: (a) The velocity vector is

$$\mathbf{r}'(t) = (-\sin(t), \cos(t), 2t)$$

So a tangent vector at $t = \pi$ is $\mathbf{T} = (0, -1, 2\pi)$ and a parametric form for the tangent line is

$$\mathbf{R}(t) = \mathbf{r}(\pi) + t\mathbf{T} = (-1, 0, \pi^2) + t(0, -1, 2\pi)$$

(b) The speed is

$$\frac{\mathrm{d}s}{\mathrm{d}t} = |\mathbf{r}'(t)| = \sqrt{1+4t^2}$$

By Theorem 1.3.3 of the CLP-4 text, the tangential component of acceleration is

$$a_T(t) = \frac{\mathrm{d}^2s}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{1+4t^2} = \frac{4t}{\sqrt{1+4t^2}}$$

S-13: (a) The velocity vector of the particle at time t is

$$\begin{aligned}\mathbf{r}'(t) &= (\cos t - \cos t + t\sin t)\,\hat{\boldsymbol{\imath}} + (-\sin t + \sin t + t\cos t)\,\hat{\boldsymbol{\jmath}} + 2t\,\hat{\mathbf{k}} \\ &= t\sin t\,\hat{\boldsymbol{\imath}} + t\cos t\,\hat{\boldsymbol{\jmath}} + 2t\,\hat{\mathbf{k}}\end{aligned}$$

so its speed at time $1 \leqslant t < \infty$ is

$$\frac{\mathrm{d}s}{\mathrm{d}t} = |\mathbf{r}'(t)| = \sqrt{t^2\sin^2 t + t^2\cos^2 t + 4t^2} = \sqrt{5}\,t$$

(b) The unit tangent at time t is

$$\hat{\mathbf{T}}(T) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{5}}\big(\sin t\,\hat{\boldsymbol{\imath}} + \cos t\,\hat{\boldsymbol{\jmath}} + 2\,\hat{\mathbf{k}}\big)$$

So the tangential component of acceleration at time t is

$$\mathbf{a}_T(t) = \frac{\mathrm{d}^2s}{\mathrm{d}t^2}(t)\ \hat{\mathbf{T}}(t) = \sin t\,\hat{\boldsymbol{\imath}} + \cos t\,\hat{\boldsymbol{\jmath}} + 2\,\hat{\mathbf{k}}$$

(c) The (full) acceleration is

$$\mathbf{r}''(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}'(t) = \big(\sin t + t\cos t\big)\,\hat{\boldsymbol{\imath}} + \big(\cos t - t\sin t\big)\,\hat{\boldsymbol{\jmath}} + 2\,\hat{\mathbf{k}}$$

So the normal component of acceleration at time t is

$$\mathbf{a}_N(t) = \mathbf{a}(t) - \mathbf{a}_T(t) = t\cos t\,\hat{\boldsymbol{\imath}} - t\sin t\,\hat{\boldsymbol{\jmath}}$$

(d) Another formula for the normal component of acceleration is $\kappa(t)\big(\frac{\mathrm{d}s}{\mathrm{d}t}(t)\big)^2\hat{\mathbf{N}}(t)$. So the magnitude of the normal component of acceleration is $\kappa(t)\big(\frac{\mathrm{d}s}{\mathrm{d}t}(t)\big)^2$ and, by part (c),

$$\kappa(t)\left(\frac{\mathrm{d}s}{\mathrm{d}t}(t)\right)^2 = \big|t\cos t\,\hat{\boldsymbol{\imath}} - t\sin t\,\hat{\boldsymbol{\jmath}}\big| = t$$

Consequently, by part (a),

$$\kappa(t) = \frac{t}{\big(\frac{\mathrm{d}s}{\mathrm{d}t}(t)\big)^2} = \frac{1}{5t}$$

S-14: (a) If the point (x, y, z) is on the curve, it obeys both $z = x^2 + y^2$ and $z = 8 - 2x$ and hence is also obeys

$$x^2 + y^2 = 8 - 2x \qquad \text{or} \qquad (x+1)^2 + y^2 = 9$$

So the curve C is also the intersection of

$$(x+1)^2 + y^2 = 9 \qquad \text{and} \qquad z = 8 - 2x$$

$(x+1)^2 + y^2 = 9$ is the circle of radius 3 centred on $(-1, 0)$ and can be parametrized by $x(\theta) = -1 + 3\cos\theta$, $y(\theta) = 3\sin\theta$, $0 \leqslant \theta \leqslant 2\pi$. So C can be parametrized by

$$\begin{aligned}
x(\theta) &= -1 + 3\cos\theta \\
y(\theta) &= 3\sin\theta \\
z(\theta) &= 8 - 2x(\theta) = 10 - 6\cos\theta \\
\text{or } \mathbf{r}(\theta) &= [-1 + 3\cos\theta]\,\hat{\boldsymbol{\imath}} + 3\sin\theta\,\hat{\boldsymbol{\jmath}} + [10 - 6\cos\theta]\,\hat{\mathbf{k}}
\end{aligned}$$

with $0 \leqslant \theta < 2\pi$.

Remark: if we tried to parametrize the equation as $(x, y, z) = (x, \sqrt{8 - 2x - x^2}, 8 - 2x)$, then we would miss the negative y-values.

(b) Note that $\mathbf{r}(\theta)$ is $(2, 0, 4)$ when $\theta = 0$. As

$$\begin{aligned}
\mathbf{v}(\theta) &= \mathbf{r}'(\theta) = -3\sin\theta\,\hat{\boldsymbol{\imath}} + 3\cos\theta\,\hat{\boldsymbol{\jmath}} + 6\sin\theta\,\hat{\mathbf{k}} & \mathbf{v}(0) &= 3\hat{\boldsymbol{\jmath}} \\
\mathbf{a}(\theta) &= \mathbf{v}'(\theta) = -3\cos\theta\,\hat{\boldsymbol{\imath}} - 3\sin\theta\,\hat{\boldsymbol{\jmath}} + 6\cos\theta\,\hat{\mathbf{k}} & \mathbf{a}(0) &= -3\hat{\boldsymbol{\imath}} + 6\hat{\mathbf{k}}
\end{aligned}$$

the unit tangent vector at $(2, 0, 4)$ is

$$\hat{\mathbf{T}}(0) = \frac{\mathbf{v}(0)}{|\mathbf{v}(0)|} = \hat{\boldsymbol{\jmath}}$$

and, since $\mathbf{v}(0) \times \mathbf{a}(0) = 9\hat{\mathbf{k}} + 18\hat{\boldsymbol{\imath}}$, the unit binormal vector and curvature at $(2, 0, 4)$ are

$$\hat{\mathbf{B}}(0) = \frac{\mathbf{v}(0) \times \mathbf{a}(0)}{|\mathbf{v}(0) \times \mathbf{a}(0)|} = \frac{2\hat{\boldsymbol{\imath}} + \hat{\mathbf{k}}}{\sqrt{5}} \qquad \kappa(0) = \frac{|\mathbf{v}(0) \times \mathbf{a}(0)|}{|\mathbf{v}(0)|^3} = \frac{9\sqrt{5}}{3^3} = \frac{\sqrt{5}}{3}$$

and the unit normal vector $\hat{\mathbf{N}}$ at $(2, 0, 4)$

$$\hat{\mathbf{N}}(0) = \hat{\mathbf{B}}(0) \times \hat{\mathbf{T}}(0) = \frac{1}{\sqrt{5}}(2\hat{\boldsymbol{\imath}} + \hat{\mathbf{k}}) \times \hat{\boldsymbol{\jmath}} = \frac{1}{\sqrt{5}}(2\hat{\mathbf{k}} - \hat{\boldsymbol{\imath}})$$

S-15: We have

$$\begin{aligned}
\mathbf{v}(t) &= \mathbf{r}'(t) = t^2\,\hat{\boldsymbol{\imath}} + \sqrt{2}\,t\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}} & |\mathbf{v}(t)| &= \sqrt{t^4 + 2t^2 + 1} = t^2 + 1 \\
\mathbf{a}(t) &= \mathbf{v}'(t) = 2t\,\hat{\boldsymbol{\imath}} + \sqrt{2}\,\hat{\boldsymbol{\jmath}}
\end{aligned}$$

(a) The unit tangent vector is

$$\hat{\mathbf{T}}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{t^2\,\hat{\boldsymbol{\imath}} + \sqrt{2}\,t\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}}{t^2 + 1}$$

(b) Since

$$\mathbf{v}(t) \times \mathbf{a}(t) = \det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ t^2 & \sqrt{2}\,t & 1 \\ 2t & \sqrt{2} & 0 \end{bmatrix} = -\sqrt{2}\,\hat{\boldsymbol{\imath}} + 2t\,\hat{\boldsymbol{\jmath}} - \sqrt{2}\,t^2\,\hat{\mathbf{k}}$$

The curvature is

$$\kappa(t) = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{|\mathbf{v}(t)|^3} = \frac{\sqrt{2 + 4t^2 + 2t^4}}{\left(t^2+1\right)^3} = \frac{\sqrt{2}}{\left(t^2+1\right)^2}$$

(c) Note that $\mathbf{r}(2)$ is $\left(\frac{8}{3}, 2\sqrt{2}, 2\right)$.

Solution 1: Since

$$\begin{aligned}
\hat{\mathbf{T}}'(t) &= \frac{2t\,\hat{\imath} + \sqrt{2}\,\hat{\jmath}}{t^2+1} - 2t\frac{t^2\,\hat{\imath} + \sqrt{2}\,t\,\hat{\jmath} + \hat{\mathbf{k}}}{\left(t^2+1\right)^2} \\
\hat{\mathbf{T}}'(2) &= \frac{4\,\hat{\imath} + \sqrt{2}\,\hat{\jmath}}{5} - 4\frac{4\,\hat{\imath} + 2\sqrt{2}\,\hat{\jmath} + \hat{\mathbf{k}}}{25} \\
&= \frac{4\,\hat{\imath} - 3\sqrt{2}\,\hat{\jmath} - 4\hat{\mathbf{k}}}{25} \\
|\mathbf{T}'(2)| &= \frac{5\sqrt{2}}{25}
\end{aligned}$$

the principal normal vector $\hat{\mathbf{N}}$ at $\left(\frac{8}{3}, 2\sqrt{2}, 2\right)$ is

$$\hat{\mathbf{N}}(2) = \frac{\hat{\mathbf{T}}'(2)}{|\hat{\mathbf{T}}'(2)|} = \frac{4\,\hat{\imath} - 3\sqrt{2}\,\hat{\jmath} - 4\hat{\mathbf{k}}}{5\sqrt{2}}$$

Solution 2: Perhaps we'd rather not differentiate $\hat{\mathbf{T}}(t)$.

$$\hat{\mathbf{B}} = \frac{\mathbf{v}(t) \times \mathbf{a}(t)}{|\mathbf{v}(t) \times \mathbf{a}(t)|} \quad \text{and} \quad \hat{\mathbf{N}} = \hat{\mathbf{B}} \times \hat{\mathbf{T}}$$

Using our previous work:

$$\begin{aligned}
\hat{\mathbf{B}}(2) &= \frac{\mathbf{v}(2) \times \mathbf{a}(2)}{|\mathbf{v}(2) \times \mathbf{a}(2)|} = \frac{-\sqrt{2}\hat{\imath} + 4\hat{\jmath} - 4\sqrt{2}\hat{\mathbf{k}}}{\sqrt{2+16+32}} = \frac{1}{5}\left(-\hat{\imath} + 2\sqrt{2}\hat{\jmath} - 4\hat{\mathbf{k}}\right) \\
\hat{\mathbf{T}}(2) &= \frac{1}{5}\left(4\hat{\imath} + 2\sqrt{2}\hat{\jmath} + \hat{\mathbf{k}}\right) \\
\hat{\mathbf{N}}(2) &= \hat{\mathbf{B}}(2) \times \hat{\mathbf{T}}(2) = \frac{1}{5}\left(-\hat{\imath} + 2\sqrt{2}\hat{\jmath} - 4\hat{\mathbf{k}}\right) \times \frac{1}{5}\left(4\hat{\imath} + 2\sqrt{2}\hat{\jmath} + \hat{\mathbf{k}}\right) \\
&= \left(\frac{2\sqrt{2}}{5}\right)\hat{\imath} + \left(-\frac{3}{5}\right)\hat{\jmath} + \left(-\frac{2\sqrt{2}}{5}\right)\hat{\mathbf{k}}
\end{aligned}$$

S-16: (a) The curve $x^2 + y^2 = 1$ is a circle of radius 1. So we can parametrize it by $x(\theta) = \cos\theta, y(\theta) = \sin\theta, 0 \leqslant \theta < 2\pi$. The z-coordinate of any point on the intersection is determined by $z = 1 - x - y$. So we can use

$$\mathbf{r}(\theta) = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} + (1 - \cos\theta - \sin\theta)\,\hat{\mathbf{k}} \qquad 0 \leqslant \theta < 2\pi$$

(b) As

$$\begin{aligned}\mathbf{v}(\theta) &= \mathbf{r}'(\theta) = \big(-\sin\theta, \cos\theta, \sin\theta - \cos\theta\big)\\ \mathbf{a}(\theta) &= \mathbf{v}'(\theta) = \big(-\cos\theta, -\sin\theta, \cos\theta + \sin\theta\big)\end{aligned}$$

we have

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}\theta} &= |\mathbf{r}'(\theta)| = \sqrt{\sin^2\theta + \cos^2\theta + (\sin\theta - \cos\theta)^2}\\ &= \sqrt{2 - 2\sin\theta\cos\theta}\\ &= \sqrt{2 - \sin(2\theta)}\\ \mathbf{v}(\theta) \times \mathbf{a}(\theta) &= \big(1, 1, 1\big)\end{aligned}$$

and the curvature

$$\kappa(\theta) = \frac{|\mathbf{v}(\theta) \times \mathbf{a}(\theta)|}{\big(\frac{\mathrm{d}s}{\mathrm{d}\theta}\big)^3} = \frac{\sqrt{3}}{[2 - \sin(2\theta)]^{3/2}}$$

(c) The curvature is

- a maximum (minimum) when $2 - \sin(2\theta)$ is a minimum (maximum),
- which is the case when $\sin(2\theta) = 1$ ($\sin(2\theta) = -1$),
- which in turn is the case when $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$ ($\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$).

So

$$\begin{aligned}\text{maximum curvature } &= \frac{\sqrt{3}}{[2-1]^{3/2}} = \sqrt{3} &&\text{at} \quad \frac{\hat{\imath}}{\sqrt{2}} + \frac{\hat{\jmath}}{\sqrt{2}} + (1 - \sqrt{2})\,\hat{\mathbf{k}}\\ & &&\text{and} \quad -\frac{\hat{\imath}}{\sqrt{2}} - \frac{\hat{\jmath}}{\sqrt{2}} + (1 + \sqrt{2})\,\hat{\mathbf{k}}\\ \text{minimum curvature } &= \frac{\sqrt{3}}{[2-(-1)]^{3/2}} = \frac{1}{3} &&\text{at} \quad -\frac{\hat{\imath}}{\sqrt{2}} + \frac{\hat{\jmath}}{\sqrt{2}} + \hat{\mathbf{k}}\\ & &&\text{and} \quad \frac{\hat{\imath}}{\sqrt{2}} - \frac{\hat{\jmath}}{\sqrt{2}} + \hat{\mathbf{k}}\end{aligned}$$

S-17: For $\mathbf{r}(t)$ to be well-defined, we need $t > 0$ (because of the $\ln t$.)

$$\mathbf{v}(t) = \mathbf{r}'(t) = 2t\,\hat{\imath} + 2\,\hat{\jmath} + \frac{1}{t}\,\hat{\mathbf{k}} \qquad\qquad \frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{4t^2 + 4 + 1/t^2} = 2t + \frac{1}{t}$$

The unit tangent vector is

$$\hat{\mathbf{T}}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{2t\,\hat{\imath} + 2\,\hat{\jmath} + \frac{1}{t}\,\hat{\mathbf{k}}}{2t + \frac{1}{t}} = \frac{2t^2\,\hat{\imath} + 2t\,\hat{\jmath} + \hat{\mathbf{k}}}{2t^2 + 1}$$

so, from §1.5 of the CLP-4 text,

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}t}(t)\,\kappa(t)\,\hat{\mathbf{N}}(t) = \hat{\mathbf{T}}'(t) &= \frac{4t\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}}}{2t^2+1} - 4t\frac{2t^2\,\hat{\boldsymbol{\imath}} + 2t\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}}{(2t^2+1)^2} = \frac{4t\,\hat{\boldsymbol{\imath}} + (-4t^2+2)\,\hat{\boldsymbol{\jmath}} - 4t\,\hat{\mathbf{k}}}{(2t^2+1)^2}\\ &= 2\frac{2t\,\hat{\boldsymbol{\imath}} - (2t^2-1)\,\hat{\boldsymbol{\jmath}} - 2t\,\hat{\mathbf{k}}}{(2t^2+1)^2}\end{aligned}$$

Since the length of $2t\,\hat{\boldsymbol{\imath}} - (2t^2-1)\,\hat{\boldsymbol{\jmath}} - 2t\,\hat{\mathbf{k}}$ is

$$\begin{aligned}\sqrt{4t^2+(2t^2-1)^2+4t^2} = \sqrt{8t^2+4t^4-4t^2+1} &= \sqrt{4t^4+4t^2+1}\\ &= \sqrt{(2t^2+1)^2} = 2t^2+1\end{aligned}$$

we have

$$\hat{\mathbf{N}}(t) = \frac{2t\,\hat{\boldsymbol{\imath}} - (2t^2-1)\,\hat{\boldsymbol{\jmath}} - 2t\,\hat{\mathbf{k}}}{2t^2+1}$$

and

$$\kappa(t) = \frac{|\hat{\mathbf{T}}'(t)|}{\frac{\mathrm{d}s}{\mathrm{d}t}(t)} = \frac{\frac{2}{2t^2+1}}{2t+\frac{1}{t}} = \frac{2t}{(2t^2+1)^2}$$

S-18: (a) Since

$$\begin{aligned}\mathbf{r}(t) &= \hat{\boldsymbol{\imath}} + \frac{t^2}{2}\,\hat{\boldsymbol{\jmath}} + \frac{t^3}{3}\,\hat{\mathbf{k}}\\ \mathbf{r}'(t) &= t\,\hat{\boldsymbol{\jmath}} + t^2\,\hat{\mathbf{k}}\\ \frac{\mathrm{d}s}{\mathrm{d}t}(t) &= |\mathbf{r}'(t)| = \sqrt{t^2+t^4} = t\sqrt{1+t^2}\end{aligned}$$

the length of the curve is

$$\int_0^1 \frac{\mathrm{d}s}{\mathrm{d}t}(t)\,\mathrm{d}t = \int_0^1 t\sqrt{1+t^2}\,\mathrm{d}t = \frac{1}{3}\big[1+t^2\big]^{3/2}\Big|_0^1 = \frac{1}{3}\big[2^{3/2}-1\big]$$

(b) For the specified curve

$$\begin{aligned}\mathbf{r}(t) &= \cos(t)\,\hat{\boldsymbol{\imath}} + \sin(t)\,\hat{\boldsymbol{\jmath}} + t\,\hat{\mathbf{k}}\\ \mathbf{r}'(t) &= -\sin(t)\,\hat{\boldsymbol{\imath}} + \cos(t)\,\hat{\boldsymbol{\jmath}} + 1\,\hat{\mathbf{k}}\\ \hat{\mathbf{T}}(t) &= \frac{-\sin(t)\,\hat{\boldsymbol{\imath}} + \cos(t)\,\hat{\boldsymbol{\jmath}} + 1\,\hat{\mathbf{k}}}{\sqrt{2}}\\ \hat{\mathbf{T}}'(t) &= \frac{-\cos(t)\,\hat{\boldsymbol{\imath}} - \sin(t)\,\hat{\boldsymbol{\jmath}}}{\sqrt{2}}\\ \hat{\mathbf{T}}'\big(\pi/4\big) &= \frac{-\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}}{2}\\ \hat{\mathbf{N}}\big(\pi/4\big) &= \frac{\hat{\mathbf{T}}'\big(\pi/4\big)}{|\hat{\mathbf{T}}'\big(\pi/4\big)|} = \frac{-\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}}{\sqrt{2}}\end{aligned}$$

(c) Recalling, from §1.5 in the CLP-4 text, that

$$\hat{\mathbf{T}}'(t) = \kappa(t)\,\frac{\mathrm{d}s}{\mathrm{d}t}(t)\,\hat{\mathbf{N}}(t)$$

we have, by part (d),

$$\kappa\big(\pi/4\big) = \frac{|\hat{\mathbf{T}}'\big(\pi/4\big)|}{|\mathbf{r}'\big(\pi/4\big)|} = \frac{1/\sqrt{2}}{\sqrt{2}} = \frac{1}{2}$$

S-19: (a), (b), (c) We have

$$\begin{aligned}
\mathbf{r}(t) &= \big(t+2\,,\,1-t\,,\,t^2/2\big)\\
\mathbf{v}(t) = \mathbf{r}'(t) &= \big(1\,,\,-1\,,\,t\big)\\
\frac{\mathrm{d}s}{\mathrm{d}t}(t) = |\mathbf{v}(t)| &= \sqrt{2+t^2}\\
\mathbf{a}(t) = \mathbf{v}'(t) &= \big(0\,,\,0\,,\,1\big)
\end{aligned}$$

(d) By §1.5 of the CLP-4 text, the curvature

$$\kappa(t) = \frac{|\mathbf{v}(t)\times\mathbf{a}(t)|}{(\frac{\mathrm{d}s}{\mathrm{d}t}(t))^3} = \frac{|(-1,-1,0)|}{[2+t^2]^{3/2}} = \frac{\sqrt{2}}{[2+t^2]^{3/2}}$$

(e) Since $\frac{\mathrm{d}s}{\mathrm{d}t}(t) = \sqrt{2+t^2}$, we have $\frac{\mathrm{d}^2s}{\mathrm{d}t^2}(t) = \frac{t}{\sqrt{2+t^2}}$ and

$$\begin{aligned}
\big(0\,,\,0\,,\,1\big) = \mathbf{a}(t) &= \frac{\mathrm{d}^2s}{\mathrm{d}t^2}\hat{\mathbf{T}}(t) + \kappa(t)\Big(\frac{\mathrm{d}s}{\mathrm{d}t}\Big)^2\hat{\mathbf{N}}(t)\\
&= \frac{t}{\sqrt{2+t^2}}\,\frac{(1\,,\,-1\,,\,t)}{\sqrt{2+t^2}} + \frac{\sqrt{2}}{[2+t^2]^{3/2}}\big(\sqrt{2+t^2}\big)^2\hat{\mathbf{N}}(t)
\end{aligned}$$

or

$$\frac{\sqrt{2}}{\sqrt{2+t^2}}\hat{\mathbf{N}}(t) = \big(0\,,\,0\,,\,1\big) - \frac{(t\,,\,-t\,,\,t^2)}{2+t^2} = \frac{(-t\,,\,t\,,\,2)}{2+t^2}$$

which implies

$$\hat{\mathbf{N}}(t) = \frac{(-t\,,\,t\,,\,2)}{\sqrt{2(2+t^2)}}$$

(f) At $t=0$

$$\begin{aligned}
\mathbf{r}(0) &= (2,1,0)\\
\hat{\mathbf{T}}(0) &= \frac{(1,-1,0)}{\sqrt{2}}\\
\hat{\mathbf{N}}(0) &= (0,0,1)\\
\hat{\mathbf{B}}(0) &= \hat{\mathbf{T}}(0)\times\hat{\mathbf{N}}(0) = \frac{1}{\sqrt{2}}(1,-1,0)\times(0,0,1) = \frac{1}{\sqrt{2}}(-1,-1,0)
\end{aligned}$$

The osculating plane is the plane through $\mathbf{r}(0)$ which is perpendicular to $\hat{\mathbf{B}}(0)$, which is

$$\frac{1}{\sqrt{2}}(-1,-1,0)\cdot\big\{(x,y,z)-(2,1,0)\big\}=0 \qquad \text{or} \qquad x+y=3$$

(g) The osculating circle has centre

$$\mathbf{r}(0)+\frac{1}{\kappa(0)}\hat{\mathbf{N}}(0)=(2,1,0)+\frac{1}{1/2}(0,0,1)=(2,1,2)$$

S-20: First some preliminary computations.

$$\begin{aligned}
\mathbf{r}(t)&=\frac{t^3}{3}\,\hat{\boldsymbol{\imath}}+\frac{t^2}{\sqrt{2}}\,\hat{\boldsymbol{\jmath}}+t\,\hat{\mathbf{k}}\\
\mathbf{r}'(t)&=t^2\,\hat{\boldsymbol{\imath}}+\sqrt{2}t\,\hat{\boldsymbol{\jmath}}+\hat{\mathbf{k}} \qquad\qquad |\mathbf{r}'(t)|=\sqrt{t^4+2t^2+1}=t^2+1\\
\mathbf{r}''(t)&=2t\,\hat{\boldsymbol{\imath}}+\sqrt{2}\,\hat{\boldsymbol{\jmath}}\\
\mathbf{r}'(t)\times\mathbf{r}''(t)&=\det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ t^2 & \sqrt{2}t & 1\\ 2t & \sqrt{2} & 0\end{bmatrix}=-\sqrt{2}\,\hat{\boldsymbol{\imath}}+2t\,\hat{\boldsymbol{\jmath}}-\sqrt{2}t^2\,\hat{\mathbf{k}}
\end{aligned}$$

(a) The unit tangent vector is

$$\hat{\mathbf{T}}(t)=\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}=\frac{t^2\,\hat{\boldsymbol{\imath}}+\sqrt{2}t\,\hat{\boldsymbol{\jmath}}+\hat{\mathbf{k}}}{t^2+1}$$

(b) The curvature is (see §1.5 of the CLP-4 text)

$$\begin{aligned}
\kappa(t)&=\frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}=\frac{|-\sqrt{2}\,\hat{\boldsymbol{\imath}}+2t\,\hat{\boldsymbol{\jmath}}-\sqrt{2}t^2\,\hat{\mathbf{k}}|}{(t^2+1)^3}=\frac{\sqrt{2+4t^2+2t^4}}{(t^2+1)^3}\\
&=\frac{\sqrt{2}}{(t^2+1)^2}
\end{aligned}$$

(c) At $t=0$

$$\kappa(0)=\sqrt{2}$$

For ease of computation, we'll find $\hat{\mathbf{B}}$ first, then use it to find $\hat{\mathbf{N}}$.

(e) At $t=0$, the binormal vector is (see §1.5 of the CLP-4 text)

$$\hat{\mathbf{B}}(0)=\frac{\mathbf{r}'(0)\times\mathbf{r}''(0)}{|\mathbf{r}'(0)\times\mathbf{r}''(0)|}=\frac{-\sqrt{2}\,\hat{\boldsymbol{\imath}}}{\sqrt{2}}=-\hat{\boldsymbol{\imath}}$$

(d) At $t=0$, the principal normal vector is (see §1.5 of the CLP-4 text)

$$\hat{\mathbf{N}}(0)=\hat{\mathbf{B}}(0)\times\hat{\mathbf{T}}(0)=-\hat{\boldsymbol{\imath}}\times\hat{\mathbf{k}}=\hat{\boldsymbol{\jmath}}$$

S-21: The curve has

$$\begin{aligned} \mathbf{r}(t) &= (t^2\,,\,t\,,\,t^3) \\ \mathbf{v}(t) = \mathbf{r}'(t) &= (2t\,,\,1\,,\,3t^2) \\ \mathbf{a}(t) = \mathbf{r}''(t) &= (2\,,\,0\,,\,6t) \end{aligned}$$

(a) In particular, a (non unit) tangent vector at $\mathbf{r}(-1) = (1,-1,-1)$ is $\mathbf{r}'(-1) = (-2,1,3)$. So the tangent line to the curve at $(1,-1,-1)$ is

$$(x,y,z) - (1,-1,-1) = t(-2,1,3)$$

or

$$\begin{aligned} x &= 1 - 2t \\ y &= -1 + t \\ z &= -1 + 3t \end{aligned}$$

(b) At $\mathbf{r}(1) = (1,1,1)$,

$$\begin{aligned} \mathbf{v}(1) = \mathbf{r}'(1) &= (2,1,3) \\ \mathbf{a}(1) = \mathbf{r}''(1) &= (2,0,6) \\ \mathbf{v}(1) \times \mathbf{a}(1) &= (6,-6,-2) \end{aligned}$$

So the unit binormal vector is

$$\hat{\mathbf{B}}(1) = \frac{\mathbf{v}(1) \times \mathbf{a}(1)}{|\mathbf{v}(1) \times \mathbf{a}(1)|} = \frac{(3,-3,-1)}{|(3,-3,-1)|} = \frac{1}{\sqrt{19}}(3,-3,-1)$$

An equation for the osculating plane is

$$(3,-3,-1) \cdot (x-1\,,\,y-1\,,\,z-1) = 0 \qquad \text{or} \qquad 3x - 3y - z = -1$$

S-22: (a) For this curve

$$\begin{aligned} \mathbf{r}'(t) &= t\sin t\,\hat{\boldsymbol{\imath}} + t\cos t\,\hat{\boldsymbol{\jmath}} + 2t\,\hat{\mathbf{k}} \\ \frac{\mathrm{d}s}{\mathrm{d}t}(t) = |\mathbf{r}'(t)| &= \sqrt{5}\,t \end{aligned}$$

so the length of the curve from $t = 0$ to $t = \pi$ is

$$\int_0^\pi \frac{\mathrm{d}s}{\mathrm{d}t}(t)\,\mathrm{d}t = \sqrt{5}\int_0^\pi t\,\mathrm{d}t = \frac{\sqrt{5}\,\pi^2}{2}$$

(b) The unit tangent vector is

$$\hat{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{5}}\big(\sin t\,\hat{\boldsymbol{\imath}} + \cos t\,\hat{\boldsymbol{\jmath}} + 2\,\hat{\mathbf{k}}\big)$$

so that

$$\kappa(t)\,\frac{\mathrm{d}s}{\mathrm{d}t}(t)\,\hat{\mathbf{N}}(t) = \frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}t}(t) = \frac{1}{\sqrt{5}}\big(\cos t\,\hat{\boldsymbol{\imath}} - \sin t\,\hat{\boldsymbol{\jmath}}\big)$$

which implies that

$$\kappa(t)\,\overbrace{\sqrt{5}\,t}^{\frac{\mathrm{d}s}{\mathrm{d}t}(t)} = \frac{1}{\sqrt{5}}\big|\big(\cos t\,\hat{\boldsymbol{\imath}} - \sin t\,\hat{\boldsymbol{\jmath}}\big)\big| = \frac{1}{\sqrt{5}} \implies \kappa(t) = \frac{1}{5t}$$

S-23: (a) For the specified curve

$$\begin{aligned}
\mathbf{r}(t) &= \left(\frac{4\sqrt{2}}{3}t^{3/2}, \frac{4\sqrt{2}}{3}t^{3/2}, t(2-t)\right) \\
\mathbf{v}(t) &= \big(2\sqrt{2}t^{1/2}, 2\sqrt{2}t^{1/2}, 2-2t\big) \\
|\mathbf{v}| &= \sqrt{8t+8t+4-8t+4t^2} = \sqrt{4(1+2t+t^2)} = 2(1+t)
\end{aligned}$$

The rocket is at $z = 0$ when $t = 0$ and when $t = 2$. So the distance travelled is

$$\int_0^2 |\mathbf{v}(t)|\,dt = \int_0^2 2(1+t)\,dt = 2\left[t + \frac{t^2}{2}\right]_0^2 = 8$$

(b) The rocket is at its maximum height when $\frac{\mathrm{d}z}{\mathrm{d}t} = 2 - 2t = 0$. That is, when $t = 1$. Its velocity then is $(2\sqrt{2}, 2\sqrt{2}, 0)$. A unit vector in this direction is $\hat{\mathbf{T}}(1) = \frac{1}{\sqrt{2}}(1,1,0)$. That is the unit tangent vector.

At general t, the unit tangent is

$$\hat{\mathbf{T}}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\big(\sqrt{2}t^{1/2}, \sqrt{2}t^{1/2}, 1-t\big)}{1+t}$$

So

$$\begin{aligned}
\hat{\mathbf{T}}'(t) &= \frac{\big(\sqrt{2}t^{-1/2}/2, \sqrt{2}t^{-1/2}/2, -1\big)}{1+t} - \frac{\big(\sqrt{2}t^{1/2}, \sqrt{2}t^{1/2}, 1-t\big)}{(1+t)^2} \\
\hat{\mathbf{T}}'(1) &= \frac{\big(\sqrt{2}/2, \sqrt{2}/2, -1\big)}{2} - \frac{\big(\sqrt{2}, \sqrt{2}, 0\big)}{4} \\
&= \big(0, 0, -1/2\big)
\end{aligned}$$

So the principal unit normal vector is

$$\hat{\mathbf{N}}(1) = \frac{\hat{\mathbf{T}}'(1)}{|\hat{\mathbf{T}}'(1)|} = (0,0,-1)$$

(c) As

$$\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}t}(1) = \big(0, 0, -1/2\big) \qquad \frac{\mathrm{d}s}{\mathrm{d}t}(1) = |\mathbf{v}(1)| = 4$$

the curvature

$$\kappa(1) = \frac{|\hat{\mathbf{T}}'(1)|}{|\mathbf{v}(1)|} = \frac{1}{8}$$

S-24: (a) For the specified curve

$$\begin{aligned}
\mathbf{r}(t) &= \big(\cos^3 t, \sin^3 t, 2\sin^2 t\big) \\
\mathbf{v}(t) &= \big(-3\cos^2 t \sin t, 3\sin^2 t\cos t, 4\sin t\cos t\big) = \sin t\cos t(-3\cos t, 3\sin t, 4) \\
|\mathbf{v}(t)| &= \sin t\cos t\sqrt{9\cos^2 t + 9\sin^2 t + 16} = 5\sin t\cos t
\end{aligned}$$

So the distance travelled is

$$\int_0^{\pi/2} |\mathbf{v}(t)|\,\mathrm{d}t = \int_0^{\pi/2} 5\sin t\cos t\,\mathrm{d}t = \frac{5}{2}\sin^2 t\Big|_0^{\pi/2} = \frac{5}{2}$$

(b) Since

$$\mathbf{v}(t) = \sin t\cos t(-3\cos t, 3\sin t, 4) \qquad |\mathbf{v}(t)| = 5\sin t\cos t$$

we have

$$\begin{aligned}
\hat{\mathbf{T}}(t) &= \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{1}{5}(-3\cos t, 3\sin t, 4) & \hat{\mathbf{T}}\big(\pi/6\big) &= \frac{1}{5}\Big(-\frac{3}{2}, \frac{3\sqrt{3}}{2}, 4\Big) \\
\hat{\mathbf{T}}'(t) &= \frac{1}{5}(3\sin t, 3\cos t, 0) & \hat{\mathbf{T}}'\big(\pi/6\big) &= \frac{1}{5}\Big(\frac{3\sqrt{3}}{2}, \frac{3}{2}, 0\Big) = \frac{3}{10}\Big(\sqrt{3}, 1, 0\Big) \\
\hat{\mathbf{N}}\big(\pi/6\big) &= \frac{\hat{\mathbf{T}}'\big(\pi/6\big)}{|\hat{\mathbf{T}}'\big(\pi/6\big)|} = \frac{1}{2}\big(\sqrt{3}, 1, 0\big) & \hat{\mathbf{B}}\big(\pi/6\big) &= \hat{\mathbf{T}}\big(\pi/6\big) \times \hat{\mathbf{N}}\big(\pi/6\big) \\
& & &= \frac{1}{10}\big(-4, 4\sqrt{3}, -6\big) \\
& & &= \frac{1}{5}\big(-2, 2\sqrt{3}, -3\big)
\end{aligned}$$

S-25: (a) The curve $x^2 + y^2 = 1$ is a circle of radius 1. So we can parametrize it by $x(\theta) = \cos\theta$, $y(\theta) = \sin\theta$, $0 \leqslant \theta < 2\pi$. The z-coordinate of any point on the intersection is determined by $z = x^2 - y^2$. So we can use the parametrization

$$\begin{aligned}
\mathbf{r}(\theta) &= \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} + [\cos^2\theta - \sin^2\theta]\,\hat{\mathbf{k}} \\
&= \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} + \cos(2\theta)\,\hat{\mathbf{k}} \qquad 0 \leqslant \theta < 2\pi
\end{aligned}$$

(b) Note that $\mathbf{r}(\theta) = \big(1/\sqrt{2}, 1/\sqrt{2}, 0\big)$ when $\theta = \frac{\pi}{4}$. For general θ, the velocity and acceleration are

$$\begin{aligned}
\mathbf{v}(\theta) = \mathbf{r}'(\theta) &= -\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}} - 2\sin(2\theta)\,\hat{\mathbf{k}} \\
\mathbf{a}(\theta) = \mathbf{v}'(\theta) &= -\cos\theta\,\hat{\boldsymbol{\imath}} - \sin\theta\,\hat{\boldsymbol{\jmath}} - 4\cos(2\theta)\,\hat{\mathbf{k}}
\end{aligned}$$

In particular,

$$\begin{aligned}
\mathbf{v}(\pi/4) &= -\frac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}} + \frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}} - 2\,\hat{\mathbf{k}} \\
\mathbf{a}(\pi/4) &= -\frac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}} - \frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}} \\
\frac{\mathrm{d}s}{\mathrm{d}\theta}(\pi/4) &= |\mathbf{v}(\pi/4)| = \sqrt{5} \\
\mathbf{v}(\pi/4) \times \mathbf{a}(\pi/4) &= -\sqrt{2}\,\hat{\boldsymbol{\imath}} + \sqrt{2}\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}} \\
|\mathbf{v}(\pi/4) \times \mathbf{a}(\pi/4)| &= \sqrt{5}
\end{aligned}$$

So the curvature

$$\kappa(\pi/4) = \frac{|\mathbf{v}(\pi/4) \times \mathbf{a}(\pi/4)|}{|\mathbf{v}(\pi/4)|^3} = \frac{1}{5}$$

(c) The binormal to C at $\big(1/\sqrt{2}, 1/\sqrt{2}, 0\big)$ is

$$\hat{\mathbf{B}}(\pi/4) = \frac{\mathbf{v}(\pi/4) \times \mathbf{a}(\pi/4)}{|\mathbf{v}(\pi/4) \times \mathbf{a}(\pi/4)|} = \frac{-\sqrt{2}\,\hat{\boldsymbol{\imath}} + \sqrt{2}\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}}{\sqrt{5}}$$

So the osculating plane to C at $\big(1/\sqrt{2}, 1/\sqrt{2}, 0\big)$ is

$$\begin{aligned}
&\big(-\sqrt{2}, \sqrt{2}, 1\big) \cdot \big(x - 1/\sqrt{2}, y - 1/\sqrt{2}, z - 0\big) = 0 \qquad \text{or} \\
&z = \sqrt{2}\,x - \sqrt{2}\,y
\end{aligned}$$

(d) From the computations in parts (b) and (c), we have

$$\begin{aligned}
\hat{\mathbf{T}}(\pi/4) &= \frac{\mathbf{v}(\pi/4)}{|\mathbf{v}(\pi/4)|} = \frac{-1/\sqrt{2}\,\hat{\boldsymbol{\imath}} + 1/\sqrt{2}\,\hat{\boldsymbol{\jmath}} - 2\,\hat{\mathbf{k}}}{\sqrt{5}} \\
\hat{\mathbf{B}}(\pi/4) &= \frac{\mathbf{v}(\pi/4) \times \mathbf{a}(\pi/4)}{|\mathbf{v}(\pi/4) \times \mathbf{a}(\pi/4)|} = \frac{-\sqrt{2}\,\hat{\boldsymbol{\imath}} + \sqrt{2}\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}}{\sqrt{5}} \\
\hat{\mathbf{N}}(\pi/4) &= \hat{\mathbf{B}}(\pi/4) \times \hat{\mathbf{T}}(\pi/4) = \frac{-\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}}{\sqrt{2}}
\end{aligned}$$

So the osculating circle has radius $1/\kappa(\pi/4) = 5$ and centre

$$\begin{aligned}
\mathbf{r}_c(\pi/4) = \mathbf{r}(\pi/4) + \frac{\hat{\mathbf{N}}(\pi/4)}{\kappa(\pi/4)} &= \big(1/\sqrt{2}, 1/\sqrt{2}, 0\big) - 5\big(1/\sqrt{2}, 1/\sqrt{2}, 0\big) \\
&= \big(-2\sqrt{2}, -2\sqrt{2}, 0\big)
\end{aligned}$$

S-26: We'll solve this problem twice, using two different strategies. (The second strategy will be much more efficient than the first one.) Both strategies use that $\mathbf{F} = m\mathbf{a}$. Since we are told that $m = 2$, we just have to find the acceleration $\mathbf{a}$ at $(1, 1, 1)$.

Strategy 1: In the first strategy, we'll find the position $\mathbf{r}(t)$, as a function of time t and then differentiate twice to get the acceleration $\mathbf{a}(t)$.

- First we'll find any old parametrization. We are told that, on the path, $z = x$ and $z = y^2$. So let's use y as the parameter. Then $x = z = y^2$. So the parametrization is $\mathbf{R}(y) = y^2\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + y^2\,\hat{\mathbf{k}}$. (We'll save the notation "$\mathbf{r}(t)$" for the parametrization with respect to time.)
- Next we'll reparametrize to get the time t as the parameter. Since

$$\begin{aligned}\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}y} &= 2y\,\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + 2y\,\hat{\mathbf{k}}\\ \implies \frac{\mathrm{d}s}{\mathrm{d}y} &= \big|2y\,\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + 2y\,\hat{\mathbf{k}}\big| = \sqrt{1+8y^2}\end{aligned}$$

  We are told that the speed $\frac{\mathrm{d}s}{\mathrm{d}t} = 3$ for all t. So, choosing our zero point for time to coincide with our zero point for s, we have $s = 3t$, or $t = s/3$ so that

$$\frac{\mathrm{d}t}{\mathrm{d}y} = \frac{1}{3}\sqrt{1+8y^2}$$

  We could now integrate to get t as a function of y. But that looks quite messy. Fortunately we only need the acceleration at one point, namely $(1, 1, 1)$. We'll now see that that saves quite a bit of work. Pretend that we have integrated to get t as a function of y and call the answer $t(y)$. Call the inverse function, which gives y as a function of t, $y(t)$.
- We now have $\mathbf{r}(t) = \mathbf{R}\big(y(t)\big)$. So, by the chain rule,

$$\begin{aligned}\mathbf{r}'(t) &= \mathbf{R}'\big(y(t)\big)\; y'(t)\\ \mathbf{r}''(t) &= \mathbf{R}'\big(y(t)\big)\; y''(t) + \mathbf{R}''\big(y(t)\big)\; y'(t)^2\end{aligned}$$

  We're only interest in the time, call it t_0, at which $y(t_0) = 1$. The acceleration at time t_0 is

$$\begin{aligned}\mathbf{r}''(t_0) &= \mathbf{R}'\big(y(t_0)\big)\; y''(t_0) + \mathbf{R}''\big(y(t_0)\big)\; y'(t_0)^2\\ &= \mathbf{R}'(1)\; y''(t_0) + \mathbf{R}''(1)\; y'(t_0)^2\\ &= \big[2\,\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + 2\,\hat{\mathbf{k}}\big]\; y''(t_0) + \big[2\,\hat{\boldsymbol{\imath}} + 2\,\hat{\mathbf{k}}\big]\; y'(t_0)^2\end{aligned}$$

  so we just have to find $y'(t_0)$ and $y''(t_0)$.
- We know that $\frac{\mathrm{d}t}{\mathrm{d}y} = \frac{1}{3}\sqrt{1+8y^2}$. So by the inverse function theorem

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t}(t) &= \frac{3}{\sqrt{1+8y(t)^2}}\\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2}(t) &= -\frac{1}{2}\,\frac{3\big(16y(t)y'(t)\big)}{[1+8y(t)^2]^{3/2}}\end{aligned}$$

In particular

$$y'(t_0) = \frac{3}{\sqrt{1+8y(t_0)^2}} = \frac{3}{\sqrt{1+8}} = 1$$

$$y''(t_0) = -\frac{24\,y(t_0)y'(t_0)}{[1+8y(t_0)^2]^{3/2}} = -\frac{24\times 1\times 1}{(1+8)^{3/2}} = -\frac{8}{9}$$

◦ *Finally*, the force is

$$\begin{aligned} 2\mathbf{r}''(t_0) &= 2\big[2\,\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + 2\,\hat{\mathbf{k}}\big]\; y''(t_0) + 2\big[2\,\hat{\boldsymbol{\imath}} + 2\,\hat{\mathbf{k}}\big]\; y'(t_0)^2 \\ &= -\frac{16}{9}\big[2\,\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + 2\,\hat{\mathbf{k}}\big] + 2\big[2\,\hat{\boldsymbol{\imath}} + 2\,\hat{\mathbf{k}}\big] \\ &= \frac{4}{9}\,\hat{\boldsymbol{\imath}} - \frac{16}{9}\,\hat{\boldsymbol{\jmath}} + \frac{4}{9}\,\hat{\mathbf{k}} \end{aligned}$$

Strategy 2: The second strategy will be based on (see §1.5 in the CLP-4 text)

$$\mathbf{a} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\,\hat{\mathbf{T}} + \kappa\Big(\frac{\mathrm{d}s}{\mathrm{d}t}\Big)^2\hat{\mathbf{N}}$$

In this problem, we are told that $\frac{\mathrm{d}s}{\mathrm{d}t} = 3$ for all t, so that $\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = 0$ and

$$\mathbf{a} = 9\kappa\hat{\mathbf{N}}$$

So we just have to find the curvature, κ, and unit normal, $\hat{\mathbf{N}}$, at $(1,1,1)$. We have already found one parametrization of the path in strategy 1, namely

$$\mathbf{R}(y) = y^2\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + y^2\,\hat{\mathbf{k}}$$

Note that $\mathbf{R}(1) = (1,1,1)$. Since

$$\begin{aligned} \mathbf{R}'(y) &= 2y\,\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + 2y\,\hat{\mathbf{k}} \\ \hat{\mathbf{T}}(y) &= \frac{\mathbf{R}'(y)}{|\mathbf{R}'(y)|} = \frac{2y\,\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + 2y\,\hat{\mathbf{k}}}{\sqrt{1+8y^2}} \\ \hat{\mathbf{T}}'(y) &= \frac{2\,\hat{\boldsymbol{\imath}} + 2\,\hat{\mathbf{k}}}{\sqrt{1+8y^2}} - \frac{16y}{2}\,\frac{2y\,\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + 2y\,\hat{\mathbf{k}}}{[1+8y^2]^{3/2}} \\ \hat{\mathbf{T}}'(1) &= \frac{2\,\hat{\boldsymbol{\imath}} + 2\,\hat{\mathbf{k}}}{3} - 8\frac{2\,\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + 2\,\hat{\mathbf{k}}}{27} = \frac{2\,\hat{\boldsymbol{\imath}} - 8\,\hat{\boldsymbol{\jmath}} + 2\,\hat{\mathbf{k}}}{27} \end{aligned}$$

we have (again see §1.5 in the CLP-4 text)

$$\begin{aligned} \kappa(1) &= \frac{|\hat{\mathbf{T}}'(1)|}{|\mathbf{R}'(1)|} \\ \hat{\mathbf{N}}(1) &= \frac{\hat{\mathbf{T}}'(1)}{|\hat{\mathbf{T}}'(1)|} \end{aligned}$$

$$\begin{aligned} \mathbf{F} &= m\mathbf{a} = 2\times 9\kappa(1)\hat{\mathbf{N}}(1) = 18\frac{\hat{\mathbf{T}}'(1)}{|\mathbf{R}'(1)|} = 18\frac{2\,\hat{\boldsymbol{\imath}} - 8\,\hat{\boldsymbol{\jmath}} + 2\,\hat{\mathbf{k}}}{27\sqrt{1+8\times 1^2}} \\ &= \frac{4}{9}(\hat{\boldsymbol{\imath}} - 4\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}) \end{aligned}$$

S-27: (a) As

$$\begin{aligned}
\mathbf{r}(t) &= 2t\hat{\boldsymbol{\imath}} + t^2\hat{\boldsymbol{\jmath}} + \sqrt{3}t^2\hat{\mathbf{k}} \\
\mathbf{r}'(t) &= 2\hat{\boldsymbol{\imath}} + 2t\hat{\boldsymbol{\jmath}} + 2\sqrt{3}t\hat{\mathbf{k}}
\end{aligned}$$

the unit tangent vector is

$$\hat{\mathbf{T}}(t) = \frac{\hat{\boldsymbol{\imath}} + t\hat{\boldsymbol{\jmath}} + \sqrt{3}t\hat{\mathbf{k}}}{|\hat{\boldsymbol{\imath}} + t\hat{\boldsymbol{\jmath}} + \sqrt{3}t\hat{\mathbf{k}}|} = \frac{\hat{\boldsymbol{\imath}} + t\hat{\boldsymbol{\jmath}} + \sqrt{3}t\hat{\mathbf{k}}}{\sqrt{1+4t^2}}$$

(b) Since

$$\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}t}(t) = \frac{\hat{\boldsymbol{\jmath}} + \sqrt{3}\hat{\mathbf{k}}}{\sqrt{1+4t^2}} - 4t\frac{\hat{\boldsymbol{\imath}} + t\hat{\boldsymbol{\jmath}} + \sqrt{3}t\hat{\mathbf{k}}}{(1+4t^2)^{3/2}} = \frac{-4t\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \sqrt{3}\hat{\mathbf{k}}}{(1+4t^2)^{3/2}}$$

the unit normal is

$$\hat{\mathbf{N}}(t) = \frac{-4t\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \sqrt{3}\hat{\mathbf{k}}}{|-4t\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \sqrt{3}\hat{\mathbf{k}}|} = \frac{-4t\,\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \sqrt{3}\hat{\mathbf{k}}}{2\sqrt{1+4t^2}}$$

(c) The unit binormal is

$$\begin{aligned}
\hat{\mathbf{B}}(t) &= \hat{\mathbf{T}}(t) \times \hat{\mathbf{N}}(t) \\
&= \frac{1}{2(1+4t^2)} \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 1 & t & \sqrt{3}t \\ -4t & 1 & \sqrt{3} \end{bmatrix} \\
&= \frac{-\sqrt{3}(1+4t^2)\hat{\boldsymbol{\jmath}} + (1+4t^2)\hat{\mathbf{k}}}{2(1+4t^2)} \\
&= -\frac{\sqrt{3}}{2}\hat{\boldsymbol{\jmath}} + \frac{1}{2}\hat{\mathbf{k}}
\end{aligned}$$

which is (3).

(d) The plane contains the point $\mathbf{r}(0) = \mathbf{0}$ and is perpendicular to the vector $-\frac{\sqrt{3}}{2}\hat{\boldsymbol{\jmath}} + \frac{1}{2}\hat{\mathbf{k}}$ and so is

$$-\sqrt{3}y + z = 0$$

(e) The curvature is

$$\begin{aligned}
\kappa(t) &= \left|\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}t}(t)\right| / \left|\frac{\mathrm{d}s}{\mathrm{d}t}\right| = \frac{|-4t\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \sqrt{3}\hat{\mathbf{k}}|}{(1+4t^2)^{3/2}} \frac{1}{|2\hat{\boldsymbol{\imath}} + 2t\hat{\boldsymbol{\jmath}} + 2\sqrt{3}t\hat{\mathbf{k}}|} \\
&= \frac{\sqrt{4+16t^2}}{(1+4t^2)^{3/2}} \frac{1}{2\sqrt{1+4t^2}} = \frac{1}{(1+4t^2)^{3/2}}
\end{aligned}$$

(f), (g) The denominator $(1+4t^2)^{3/2}$ of $\kappa(t)$ is a minimum at $t=0$ and grows without bound as $|t|$ increases. So the denominator never achieves a maximum. Consquently, the curvature $\kappa(t)$ achieves its maximum value when $t = 0$ and so at $\mathbf{r}(0) = (0, 0, 0)$. The curvature never achieves a minimum.

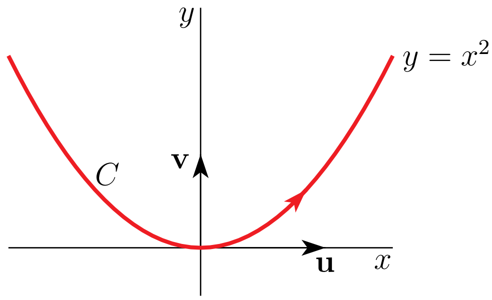
(h) Since $\sqrt{3}\,\mathbf{v} + \mathbf{w} = 4\,\hat{\mathbf{k}}$ and $\mathbf{v} - \sqrt{3}\,\mathbf{w} = 4\,\hat{\boldsymbol{\jmath}}$,

$$\hat{\boldsymbol{\imath}} = \frac{\mathbf{u}}{2} \qquad \hat{\boldsymbol{\jmath}} = \frac{\mathbf{v} - \sqrt{3}\,\mathbf{w}}{4} \qquad \hat{\mathbf{k}} = \frac{\sqrt{3}\,\mathbf{v} + \mathbf{w}}{4}$$

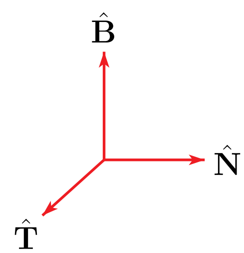
Since $\mathbf{u} = 2\,\hat{\boldsymbol{\imath}}$ and $\mathbf{v} = \hat{\boldsymbol{\jmath}} + \sqrt{3}\,\hat{\mathbf{k}}$,

$$\mathbf{r}(t) = t\,\mathbf{u} + t^2\,\mathbf{v} = a(t)\mathbf{u} + b(t)\mathbf{v} + c(t)\mathbf{w} \qquad \text{with } a(t) = t,\ b(t) = t^2,\ c(t) = 0$$

The curve $\big(a(t), b(t)\big) = (t, t^2)$ is the curve $y = x^2$. It is "curviest" at the origin, which is consistent with part (f). It becomes flatter and flatter as $|t|$ increases, but never achieves "perfect flatness", which is consistent with (g).



S-28: The three unit vectors $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$ and $\hat{\mathbf{B}}$ are mutually perpendicular and form a right handed triple.



So

$$\hat{\mathbf{N}} = \hat{\mathbf{B}} \times \hat{\mathbf{T}} \qquad \hat{\mathbf{N}} \times \hat{\mathbf{T}} = -\hat{\mathbf{B}} \qquad \hat{\mathbf{B}} \times \hat{\mathbf{N}} = -\hat{\mathbf{T}}$$

and

$$\frac{\mathrm{d}\hat{\mathbf{N}}}{\mathrm{d}s} = \frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}s} \times \hat{\mathbf{T}} + \hat{\mathbf{B}} \times \frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s} = -\tau\,\hat{\mathbf{N}} \times \hat{\mathbf{T}} + \hat{\mathbf{B}} \times \big(\kappa\hat{\mathbf{N}}\big) = \tau\,\hat{\mathbf{B}} - \kappa\hat{\mathbf{T}}$$

S-29: (a) Parametrizing the curve by θ gives

$$\begin{aligned} \mathbf{r}(\theta) &= \big(\sin(2\theta), 1 - \cos(2\theta), 2\cos\theta\big), \\ \mathbf{v} = \mathbf{r}'(\theta) &= \big(2\cos(2\theta), 2\sin(2\theta), -2\sin\theta\big), \\ \mathbf{a} = \mathbf{r}''(\theta) &= \big(-4\sin(2\theta), 4\cos(2\theta), -2\cos\theta\big). \end{aligned}$$

At the point P, we have $\theta = \pi/4$, giving instantaneous values

$$\mathbf{r} = (1, 1, \sqrt{2}), \quad \mathbf{v} = \big(0, 2, -\sqrt{2}\big), \quad v = |\mathbf{v}| = \sqrt{6}, \quad \mathbf{a} = \big(-4, 0, -\sqrt{2}\big).$$

Hence $\hat{\mathbf{T}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{6}}\big(0, 2, -\sqrt{2}\big)$.

Now $\hat{\mathbf{B}} = \frac{\mathbf{v}\times\mathbf{a}}{|\mathbf{v}\times\mathbf{a}|} = \frac{1}{\sqrt{26}}\big(-\sqrt{2}, 2\sqrt{2}, 4\big) = \frac{1}{\sqrt{13}}\big(-1, 2, 2\sqrt{2}\big)$, since

$$\mathbf{v}\times\mathbf{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 0 & 2 & -\sqrt{2} \\ -4 & 0 & -\sqrt{2} \end{vmatrix} = \big(-2\sqrt{2}, 4\sqrt{2}, 8\big), \qquad |\mathbf{v}\times\mathbf{a}| = \sqrt{104} = 2\sqrt{26}.$$

This leads to

$$\hat{\mathbf{N}} = \hat{\mathbf{B}} \times \hat{\mathbf{T}} = \frac{1}{\sqrt{78}} \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ -1 & 2 & 2\sqrt{2} \\ 0 & 2 & -\sqrt{2} \end{vmatrix} = -\frac{1}{\sqrt{78}}\big(6\sqrt{2}, \sqrt{2}, 2\big) = -\frac{1}{\sqrt{39}}\big(6, 1, \sqrt{2}\big).$$

Finally,

$$\kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{v^3} = \frac{2\sqrt{26}}{\big(\sqrt{6}\big)^3} = \frac{2\sqrt{2}\sqrt{13}}{6\sqrt{2}\sqrt{3}} = \frac{\sqrt{13}}{3\sqrt{3}} = \frac{\sqrt{39}}{9}.$$

(b) Now parametrize the curve by time, t, and write $\mathbf{v} = \mathbf{r}'(t)$, $v = |\mathbf{r}'(t)|$ and $\mathbf{a} = \mathbf{r}''(t)$. Note that in part (a) we used $\mathbf{v}$, v and $\mathbf{a}$ with different meanings. We use the dot product to extract the tangential and normal components of $\mathbf{a} = \frac{\mathrm{d}v}{\mathrm{d}t}\hat{\mathbf{T}} + v^2\kappa\hat{\mathbf{N}}$:

$$\begin{aligned} \mathbf{a}\cdot\hat{\mathbf{T}} &= \left(\frac{\mathrm{d}v}{\mathrm{d}t}\hat{\mathbf{T}} + v^2\kappa\hat{\mathbf{N}}\right)\cdot\hat{\mathbf{T}} \\ &= \frac{\mathrm{d}v}{\mathrm{d}t}\hat{\mathbf{T}}\cdot\hat{\mathbf{T}} + (v^2\kappa)\hat{\mathbf{N}}\cdot\hat{\mathbf{T}} \end{aligned}$$

Since $\hat{\mathbf{T}}$ is a unit vector, $\hat{\mathbf{T}}\cdot\hat{\mathbf{T}} = \|\hat{\mathbf{T}}\|^2 = 1$; since $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ are perpendicular, $\hat{\mathbf{T}}\cdot\hat{\mathbf{N}} = 0$.

$$= \frac{\mathrm{d}v}{\mathrm{d}t}$$

This gives us a nice way to compute $\frac{\mathrm{d}v}{\mathrm{d}t}$, the rate of change of speed.

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}t} &= \mathbf{a}\cdot\hat{\mathbf{T}} = (-2, 3, -2\sqrt{2})\cdot\frac{1}{\sqrt{6}}\big(0, 2, -\sqrt{2}\big) \\ &= \frac{1}{\sqrt{6}}[0 + 6 + 4] = \frac{10}{\sqrt{6}} = \frac{5}{3}\sqrt{6}. \end{aligned}$$

Similarly, $\mathbf{a}\cdot\hat{\mathbf{N}} = v^2\kappa$, so

$$v^2 = \frac{1}{\kappa}\mathbf{a}\cdot\hat{\mathbf{N}} = \frac{9}{\sqrt{39}}\frac{-1}{\sqrt{39}}(-2, 3, -2\sqrt{2})\cdot\big(6, 1, \sqrt{2}\big) = \frac{9\times 13}{39} = 3.$$

Hence $|v| = \sqrt{3}$; since $v = |\mathbf{v}|$, $v = \sqrt{3}$. Then
$\mathbf{v} = |\mathbf{v}|\hat{\mathbf{T}} = v\hat{\mathbf{T}} = \frac{\sqrt{3}}{\sqrt{6}}\big(0, 2, -\sqrt{2}\big) = (0, \sqrt{2}, -1)$.

S-30: (a) The position, velocity and acceleration are

$$\begin{aligned}
\mathbf{r}(t) &= \big(\cos t\,,\ \sin t\,,\ c\sin t\big)\\
\mathbf{v}(t) = \mathbf{r}'(t) &= \big(-\sin t\,,\ \cos t\,,\ c\cos t\big)\\
\mathbf{a}(t) = \mathbf{r}''(t) &= \big(-\cos t\,,\ -\sin t\,,\ -c\sin t\big)
\end{aligned}$$

(b) The speed is

$$v(t) = |\mathbf{v}(t)| = \sqrt{1+c^2\cos^2 t}$$

(c) By Theorem 1.3.3.c in the CLP-4 text, the tangential component of the acceleration is

$$\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{1+c^2\cos^2 t} = \frac{-c^2\sin t\cos t}{\sqrt{1+c^2\cos^2 t}}$$

(d) $y(t) = \sin t$ and $z(t) = c\sin t$ obey $z(t) = cy(t)$ for all t. So the curve lies on the plane $z = cy$.

S-31: (a) For the specified curve $\mathbf{r}(\pi) = \big(-4, 0, \frac{1}{4}\big)$ and

$$\begin{aligned}
\mathbf{r}(\theta) &= \Big(4\cos\theta\,,\ 2\sin\theta\,,\ \frac{1}{4}\cos(2\theta)\Big)\\
\mathbf{v}(\theta) = \mathbf{r}'(\theta) &= \Big(-4\sin\theta\,,\ 2\cos\theta\,,\ -\frac{1}{2}\sin(2\theta)\Big)\\
\mathbf{a}(\theta) = \mathbf{r}''(\theta) &= \Big(-4\cos\theta\,,\ -2\sin\theta\,,\ -\cos(2\theta)\Big)\\
\mathbf{v}(\pi) &= \big(0\,,\ -2\,,\ 0\big)\\
\mathbf{a}(\pi) &= \big(4\,,\ 0\,,\ -1\big)\\
\mathbf{v}(\pi)\times\mathbf{a}(\pi) &= \big(2\,,\ 0\,,\ 8\big)
\end{aligned}$$

So the curvature at $\theta = \pi$ is

$$\kappa(\pi) = \frac{|\mathbf{v}(\pi)\times\mathbf{a}(\pi)|}{|\mathbf{v}(\pi)|^3} = \frac{|(2,0,8)|}{|(0,-2,0)|^3} = \frac{\sqrt{17}}{4}$$

(b) The radius is

$$\frac{1}{\kappa(\pi)} = \frac{4}{\sqrt{17}}$$

(c) Set $\mathbf{R}(t) = \mathbf{r}(t^2)$. Then

$$\begin{aligned}
\mathbf{R}'(t) &= 2t\,\mathbf{r}'(t^2)\\
\mathbf{R}''(t) &= 2\,\mathbf{r}'(t^2) + 4t^2\,\mathbf{r}''(t^2)
\end{aligned}$$

In particular,

$$\begin{aligned}
\mathbf{R}(\sqrt{\pi}) &= \left(-4\,,\,0\,,\,\frac{1}{4}\right)\\
\mathbf{R}'(\sqrt{\pi}) &= 2\sqrt{\pi}\,\mathbf{v}(\pi) = \left(0\,,\,-4\sqrt{\pi}\,,\,0\right)\\
\text{speed} &= \left|\mathbf{R}'(\sqrt{\pi})\right| = 4\sqrt{\pi}\\
\text{acceleration} &= \mathbf{R}''(\sqrt{\pi}) = 2\,\mathbf{v}(\pi) + 4\pi\,\mathbf{a}(\pi) = \left(16\pi\,,\,-4\,,\,-4\pi\right)
\end{aligned}$$

The normal component of the acceleration has magnitude

$$\kappa\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = \frac{\sqrt{17}}{4}\left(4\sqrt{\pi}\right)^2 = 4\sqrt{17}\,\pi$$

Solutions to Exercises 1.6 — Jump to TABLE OF CONTENTS

S-1: We want to add up all the tiny pieces of arclength $\mathrm{d}s$ along a curve C. So, the integral would simply be $\int_C \mathrm{d}s$.

To see this another way, if we define $\mathbf{r} = (x(t), y(t), z(t))$ for $a \leqslant t \leqslant b$ to be the equation of C, we could calculate the arclength as:

$$\int_a^b |\mathbf{r}'(t)|\mathrm{d}t = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\mathrm{d}t$$

This fits the form of Definition 1.6.1 with $f(x,y,z) = 1$, so we write it as a line integral as $\int_C 1\mathrm{d}s$, which is equivalent to $\int_C \mathrm{d}s$.

S-2: (a) The curve is $\mathbf{r}(\theta) = x(\theta)\,\hat{\boldsymbol{\imath}} + y(\theta)\,\hat{\boldsymbol{\jmath}}$ with $x(\theta) = r(\theta)\cos\theta$, $y(\theta) = r(\theta)\sin\theta$ and $\theta_1 \leqslant \theta \leqslant \theta_2$. On this curve

$$\begin{aligned}
\mathbf{v}(\theta) &= \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\theta}(\theta) = x'(\theta)\hat{\boldsymbol{\imath}} + y'(\theta)\hat{\boldsymbol{\jmath}} = \left[r'(\theta)\cos\theta - r(\theta)\sin\theta\right]\hat{\boldsymbol{\imath}} + \left[r'(\theta)\sin\theta + r(\theta)\cos\theta\right]\hat{\boldsymbol{\jmath}}\\
\implies \frac{\mathrm{d}s}{\mathrm{d}\theta}(\theta) &= \sqrt{\left[r'(\theta)\cos\theta - r(\theta)\sin\theta\right]^2 + \left[r'(\theta)\sin\theta + r(\theta)\cos\theta\right]^2}\\
&= \sqrt{r'(\theta)^2 + r(\theta)^2}
\end{aligned}$$

Hence

$$\begin{aligned}
\int_C f(x,y)\,ds &= \int_{\theta_1}^{\theta_2} f\big(x(\theta), y(\theta)\big)\frac{\mathrm{d}s}{\mathrm{d}\theta}\,\mathrm{d}\theta\\
&= \int_{\theta_1}^{\theta_2} f\big(r(\theta)\cos\theta, r(\theta)\sin\theta\big)\sqrt{r(\theta)^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}(\theta)\right)^2}\,\mathrm{d}\theta
\end{aligned}$$

(b) In this case $f(x,y) = 1$, $r(\theta) = 1 + \cos\theta$, $\theta_1 = 0$ and $\theta_2 = 2\pi$,

$$\begin{aligned}
\int_C ds &= \int_0^{2\pi} \sqrt{[1+\cos\theta]^2 + [-\sin\theta]^2}\,\mathrm{d}\theta = \int_0^{2\pi}\sqrt{2(1+\cos\theta)}\,\mathrm{d}\theta\\
&= \int_0^{2\pi}\sqrt{4\cos^2\frac{\theta}{2}}\,\mathrm{d}\theta = 2\int_0^{2\pi}\left|\cos\frac{\theta}{2}\right|\,\mathrm{d}\theta = 4\int_0^{\pi}\cos\frac{\theta}{2}\,\mathrm{d}\theta = 8\sin\frac{\theta}{2}\Big|_0^{\pi} = 8
\end{aligned}$$

S-3: Following Definition 1.6.1:

$$\int_C \left(\frac{xy}{z}\right) \mathrm{d}s = \int_1^2 \left(\frac{\frac{2}{3}t^3 \cdot \sqrt{3}t^2}{3t}\right) \sqrt{(2t^2)^2 + (2\sqrt{3}t)^2 + (3)^2}\, \mathrm{d}t$$

$$= \int_1^2 \left(\frac{2}{3\sqrt{3}}\, t^4\right) (2t^2 + 3)\, \mathrm{d}t = \frac{4}{21\sqrt{3}}(2^7 - 1) + \frac{2}{5\sqrt{3}}(2^5 - 1)$$

S-4: We parametrize the unit circle as $(\cos t, \sin t)$, $0 \leqslant t \leqslant 2\pi$.

A tiny slice of the hoop with length $\mathrm{d}s$ has mass $(x^2 \text{ kg/m})(\mathrm{d}s \text{ m}) = x^2 \mathrm{d}s$ kg. So, the entire hoop has mass:

$$\int_C x^2\, \mathrm{d}s = \int_0^{2\pi} \cos^2 t \sqrt{(-\sin t)^2 + (\cos t)^2}\, \mathrm{d}t = \int_0^{2\pi} \cos^2 t\, \mathrm{d}t$$

$$= \int_0^{2\pi} \frac{1 + \cos(2t)}{2}\, \mathrm{d}t = \left[\frac{t}{2} + \frac{\sin(2t)}{4}\right]_0^{2\pi} = \pi \text{ kg}$$

For an efficient, sneaky, way to evaluate $\int_0^{2\pi} \cos^2 t\, \mathrm{d}t$, see Example 2.4.4 in the CLP-4 text.

S-5: To parametrize C, we note the vector between the two points is $(2-1, 4-2, 5-3) = (1, 2, 2)$. So, the line is $(1, 2, 3) + t(1, 2, 2)$ for $0 \leqslant t \leqslant 1$. That is, $x(t) = 1 + t$, $y(t) = 2 + 2t$, and $z(t) = 3 + 2t$.

$$\int_C (xy + z)\mathrm{d}s = \int_0^1 \left((1+t)(2+2t) + (3+2t)\right) \sqrt{1^1 + 2^2 + 2^2}\mathrm{d}t$$

$$= \int_0^1 3\big(5 + 6t + 2t^2\big)\, \mathrm{d}t = 26$$

S-6: (a) In this case $\mathbf{r}(t) = t\hat{\boldsymbol{\imath}} + t^2\hat{\boldsymbol{\jmath}}$, so that $\mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t) = \hat{\boldsymbol{\imath}} + 2t\hat{\boldsymbol{\jmath}}$ and $\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{1 + 4t^2}$. Hence

$$\int_C f(x, y, z)\, \mathrm{d}s = \int_0^1 x(t) \cos z(t) \frac{\mathrm{d}s}{\mathrm{d}t}\, \mathrm{d}t = \int_0^1 t(\cos 0)\sqrt{1 + 4t^2}\, \mathrm{d}t = \frac{1}{8} \frac{(1 + 4t^2)^{3/2}}{3/2}\bigg|_0^1$$

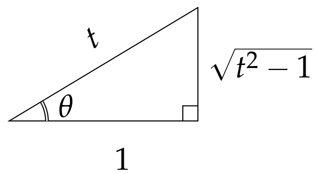
$$= \frac{5^{3/2} - 1}{12}$$

(b) In this case $\mathbf{r}(t) = \big(t, \frac{2}{3}t^{3/2}, t\big)$, so that $\mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t) = \big(1, t^{1/2}, 1\big)$ and $\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{2 + t}$. Hence

$$\int_C f(x, y, z)\, \mathrm{d}s = \int_1^2 \frac{x(t) + y(t)}{y(t) + z(t)} \frac{\mathrm{d}s}{\mathrm{d}t}\, \mathrm{d}t = \int_1^2 \frac{t + \frac{2}{3}t^{3/2}}{\frac{2}{3}t^{3/2} + t} \sqrt{2 + t}\, \mathrm{d}t = \frac{(2 + t)^{3/2}}{3/2}\bigg|_1^2$$

$$= \frac{8 - 3^{3/2}}{3/2}$$

S-7: In the figure below, we construct a triangle with $\theta = \operatorname{arcsec} t$; the hypotenuse has length t, while the side adjacent to θ has length 1. By the Pythagorean Theorem, the remaining side has length $\sqrt{t^2-1}$, so $\sin\theta = \sin(\operatorname{arcsec} t) = \frac{\sqrt{t^2-1}}{t}$.



Remember $\frac{\mathrm{d}}{\mathrm{d}t}\{\ln t\} = \frac{1}{t}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\{\operatorname{arcsec} t\} = \frac{1}{|t|\sqrt{t^2-1}}$. In our range, $1 \leqslant t \leqslant \sqrt{2}$, we have $|t| = t$.

$$\begin{aligned}\int_C \sin x\,\mathrm{d}s &= \int_1^{\sqrt{2}} \sin\left(\operatorname{arcsec} t\right)\sqrt{\left(\frac{1}{t\sqrt{t^2-1}}\right)^2 + \left(\frac{1}{t}\right)^2}\,\mathrm{d}t\\ &= \int_1^{\sqrt{2}} \frac{\sqrt{t^2-1}}{t}\sqrt{\frac{1}{t^2(t^2-1)} + \frac{1}{t^2}}\,\mathrm{d}t\\ &= \int_1^{\sqrt{2}} \frac{1}{t}\,\mathrm{d}t = \frac{1}{2}\ln 2\end{aligned}$$

S-8: (a) Since the particle has mass $m = 1$, Newton's law of motion $m\mathbf{a} = \mathbf{F}$ simplifies to

$$\mathbf{r}''(t) = \hat{\boldsymbol{\jmath}} - \sin t\,\hat{\mathbf{k}}$$

Integrating once gives

$$\mathbf{r}'(t) = t\,\hat{\boldsymbol{\jmath}} + \cos t\,\hat{\mathbf{k}} + \mathbf{C}$$

for some constant vector $\mathbf{C}$. To satisfy the initial condition that $\mathbf{r}'(0) = \mathbf{v}_0 = \hat{\boldsymbol{\imath}} + \hat{\mathbf{k}}$, we need

$$\hat{\boldsymbol{\imath}} + \hat{\mathbf{k}} = \mathbf{r}'(0) = \hat{\mathbf{k}} + \mathbf{C} \implies \mathbf{C} = \hat{\boldsymbol{\imath}}$$

So

$$\mathbf{r}'(t) = \hat{\boldsymbol{\imath}} + t\,\hat{\boldsymbol{\jmath}} + \cos t\,\hat{\mathbf{k}}$$

Integrating a second time, and imposing the initial condition that $\mathbf{r}(0) = \hat{\boldsymbol{\jmath}}$, gives

$$\mathbf{r}(t) = t\,\hat{\boldsymbol{\imath}} + \frac{t^2}{2}\hat{\boldsymbol{\jmath}} + \sin t\,\hat{\mathbf{k}} + \hat{\boldsymbol{\jmath}} = t\,\hat{\boldsymbol{\imath}} + \left(1 + \frac{t^2}{2}\right)\hat{\boldsymbol{\jmath}} + \sin t\,\hat{\mathbf{k}}$$

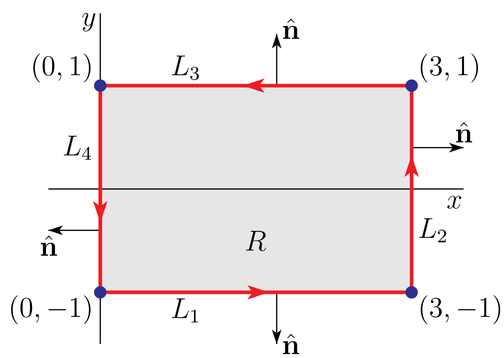
(b) The particle has $x(t) = \pi/2$ when $t = \pi/2$. So

$$\mathbf{r}(\pi/2) = \frac{\pi}{2}\hat{\boldsymbol{\imath}} + \left(1 + \frac{\pi^2}{8}\right)\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}$$

(c) The work done is

$$\begin{aligned}
\text{Work} &= \int_0^{\pi/2} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, \mathrm{d}t \\
&= \int_0^{\pi/2} \big(\hat{\boldsymbol{\jmath}} - \sin t\, \hat{\mathbf{k}}\big) \cdot \big(\hat{\boldsymbol{\imath}} + t\, \hat{\boldsymbol{\jmath}} + \cos t\, \hat{\mathbf{k}}\big)\, \mathrm{d}t \\
&= \int_0^{\pi/2} \big(t - \sin t \cos t\big)\, \mathrm{d}t \\
&= \left[\frac{t^2}{2} + \frac{1}{2}\cos^2 t\right]_0^{\pi/2} \\
&= \frac{\pi^2}{8} - \frac{1}{2}
\end{aligned}$$

S-9: Here is a sketch of the rectangle R.



Its boundary consists of four line segments.

- L_1 from $(0, -1)$ to $(3, -1)$, with $\hat{\mathbf{n}} = -\hat{\boldsymbol{\jmath}}$
- L_2 from $(3, -1)$ to $(3, 1)$, with $\hat{\mathbf{n}} = \hat{\boldsymbol{\imath}}$
- L_3 from $(3, 1)$ to $(0, 1)$, with $\hat{\mathbf{n}} = \hat{\boldsymbol{\jmath}}$
- L_4 from $(0, 1)$ to $(0, -1)$, with $\hat{\mathbf{n}} = -\hat{\boldsymbol{\imath}}$

So

$$\begin{aligned}
\int_C \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}s &= \int_{L_1} \mathbf{F} \cdot (-\hat{\boldsymbol{\jmath}})\, \mathrm{d}s + \int_{L_2} \mathbf{F} \cdot \hat{\boldsymbol{\imath}}\, \mathrm{d}s + \int_{L_3} \mathbf{F} \cdot (\hat{\boldsymbol{\jmath}})\, \mathrm{d}s + \int_{L_4} \mathbf{F} \cdot (-\hat{\boldsymbol{\imath}})\, \mathrm{d}s \\
&= \int_0^3 - \overbrace{(-1)}^{y}\, e^x\, \mathrm{d}x + \int_{-1}^1 \overbrace{(3)}^{x}\, y^2\, \mathrm{d}y + \int_3^0 \overbrace{(1)}^{y}\, e^x\, \overbrace{(-\mathrm{d}x)}^{\mathrm{d}s} + \int_1^{-1} \overbrace{(0)}^{x}\, y^2\, \overbrace{(-\mathrm{d}y)}^{\mathrm{d}s} \\
&= \big[e^3 - 1\big] + \big[1^3 - (-1)^3\big] + \big[e^3 - 1\big] + 0 \\
&= 2e^3
\end{aligned}$$

The trickiest part of this computation is getting $\mathrm{d}s$ correct on L_3 and L_4 (remembering that $\mathrm{d}s$ is the arc length traveled and so is positive, while $\mathrm{d}x < 0$ on L_3 and $\mathrm{d}y < 0$ on L_4). To make a more detailed computation of $\int_{L_3} \mathbf{F} \cdot (\hat{\boldsymbol{\jmath}})\, \mathrm{d}s$, parametrize L_3 by

$$\mathbf{r}(t) = (3, 1) + t\big\{(0, 1) - (3, 1)\big\} = \big(3 - 3t, 1\big) \qquad 0 \leqslant t \leqslant 1$$

so that $\mathbf{r}(0) = (3,1)$ is the initial point of L_3 and $\mathbf{r}(1) = (0,1)$ is the final point of L_3. Then

$$\mathbf{r}'(t) = (-3,0) \qquad \frac{\mathrm{d}s}{\mathrm{d}t}(t) = |\mathbf{r}'(t)| = 3$$

and

$$\int_{L_3} \mathbf{F}\cdot\hat{\boldsymbol{\jmath}}\,\mathrm{d}s = \int_0^1 \mathbf{F}\big(\mathbf{r}(t)\big)\cdot\hat{\boldsymbol{\jmath}}\,\frac{\mathrm{d}s}{\mathrm{d}t}(t)\,\mathrm{d}t = \int_0^1 \overbrace{e^{3-3t}}^{y(t)e^{x(t)}}\ \overbrace{3}^{\frac{\mathrm{d}s}{\mathrm{d}t}(t)}\,\mathrm{d}t = -e^{3-3t}\Big|_0^1 = e^3-1$$

S-10: (a) Since $\mathbf{r}(t) = t\cos t\,\hat{\boldsymbol{\imath}} + t\sin t\,\hat{\boldsymbol{\jmath}} + t^2\,\hat{\mathbf{k}}$

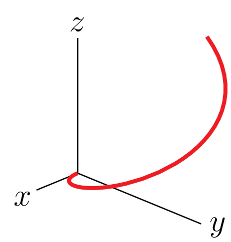
$$\begin{aligned}
\mathbf{r}'(t) &= \big(\cos t - t\sin t\big)\hat{\boldsymbol{\imath}} + \big(\sin t + t\cos t\big)\,\hat{\boldsymbol{\jmath}} + 2t\,\hat{\mathbf{k}}\\
\frac{\mathrm{d}s}{\mathrm{d}t} &= |\mathbf{r}'(t)| = \sqrt{\big(\cos t - t\sin t\big)^2 + \big(\sin t + t\cos t\big)^2 + (2t)^2}\\
&= \sqrt{1+5t^2}\\
\mathbf{r}'(\pi) &= -\hat{\boldsymbol{\imath}} - \pi\,\hat{\boldsymbol{\jmath}} + 2\pi\,\hat{\mathbf{k}}\\
\hat{\mathbf{T}}(\pi) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{1+5\pi^2}}\big(-\hat{\boldsymbol{\imath}} - \pi\,\hat{\boldsymbol{\jmath}} + 2\pi\,\hat{\mathbf{k}}\big)
\end{aligned}$$

(b)

$$\begin{aligned}
\int_C \sqrt{x^2+y^2}\,\mathrm{d}s &= \int_0^\pi \sqrt{x^2(t)+y^2(t)}\,\frac{\mathrm{d}s}{\mathrm{d}t}\,\mathrm{d}t = \int_0^\pi t\,\sqrt{1+5t^2}\,\mathrm{d}t = \left[\frac{1}{15}(1+5t^2)^{3/2}\right]_0^\pi\\
&= \frac{1}{15}\big[(1+5\pi^2)^{3/2}-1\big]
\end{aligned}$$

(c) For every t, the coordinates $x(t) = t\cos t$, $y(t) = t\sin t$, $z(t) = t^2$ obey $x(t)^2 + y(t)^2 = t^2 = z(t)$ and so the curve lies on $z = x^2+y^2$.

(d) First concentrate on $\big(x(t)\,,\,y(t)\big)$. As t runs from 0 to π, the curve $\big(r\cos t\,,\,r\sin t\big)$ sweeps out half of a circle of radius r. Our $\big(x(t)\,,\,y(t)\big)$ does something similar, but the radius $r = t$ increases from 0 to π. Thus our $\big(x(t)\,,\,y(t)\big)$ sweeps out the beginning of a spiral. At the same time $z(t)$ increases from 0 to π^2. So the curve $\mathcal{C}$ looks like



S-11: We use the centre of mass formulae $\bar{x} = \dfrac{\int_C x\rho\,\mathrm{d}s}{\int_C \rho\,\mathrm{d}s}$, etc. To make the working clearer, we'll break these calculations into several steps.

$$\begin{aligned} x(t) &= t + \frac{1}{2}t^2 & x'(t) &= 1+t \\ y(t) &= t - \frac{1}{2}t^2 & y'(t) &= 1-t \\ z(t) &= \frac{4}{3}t^{3/2} & z'(t) &= 2\sqrt{t} \end{aligned}$$

$$\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = \sqrt{1+2t+t^2+1-2t+t^2+4t} = \sqrt{2(t^2+2t+1)} = \sqrt{2}(t+1)$$

$$\rho(x(t),y(t),z(t)) = \frac{x(t)+y(t)}{2} = \frac{(t+t^2/2)+(t-t^2/2)}{2} = t$$

$$\int_C \rho\,\mathrm{d}s = \int_0^4 t\,\sqrt{2}(t+1)\,\mathrm{d}t = \frac{2^3\cdot 11\sqrt{2}}{3}$$

$$\int_C x\rho\,\mathrm{d}s = \int_0^4 \left(t+\frac{1}{2}t^2\right) t\,\sqrt{2}(t+1)\,\mathrm{d}t = \sqrt{2}\int_0^{2^2}\left(\frac{t^4}{2}+\frac{3}{2}t^3+t^2\right)\mathrm{d}t$$

$$= \sqrt{2}\left(\frac{2^9}{5}+3(2^5)+\frac{2^6}{3}\right) = \frac{2^5\cdot 103\sqrt{2}}{15}$$

$$\int_C y\rho\,\mathrm{d}s = \int_0^4 \left(t-\frac{1}{2}t^2\right) t\,\sqrt{2}(t+1)\,\mathrm{d}t = \sqrt{2}\int_0^{2^2}\left(-\frac{t^4}{2}+\frac{t^3}{2}+t^2\right)\mathrm{d}t$$

$$= \sqrt{2}\left(-\frac{2^9}{5}+2^5+\frac{2^6}{3}\right) = -\frac{2^5\cdot 23\sqrt{2}}{15}$$

$$\int_C z\rho\,\mathrm{d}s = \int_0^4 \left(\frac{4}{3}t^{3/2}\right) t\,\sqrt{2}(t+1)\,\mathrm{d}t = \frac{4\sqrt{2}}{3}\int_0^{2^2}\left(t^{7/2}+t^{5/2}\right)\mathrm{d}t$$

$$= \frac{4\sqrt{2}}{3}\left(\frac{2^{10}}{9}+\frac{2^8}{7}\right) = \frac{2^{10}\cdot 37\sqrt{2}}{7\cdot 3^3}$$

$$\overline{x} = \frac{\int x\rho\,\mathrm{d}s}{\int \rho\,\mathrm{d}s} = \frac{\frac{2^5\cdot 103\sqrt{2}}{15}}{\frac{2^3\cdot 11\sqrt{2}}{3}} = \frac{412}{55} \approx 7.5$$

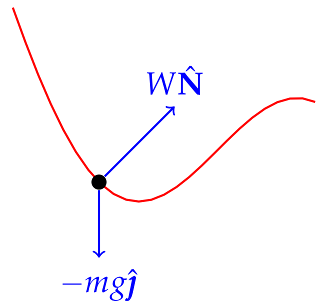
$$\overline{y} = \frac{\int y\rho\,\mathrm{d}s}{\int \rho\,\mathrm{d}s} = \frac{-\frac{2^5\cdot 23\sqrt{2}}{15}}{\frac{2^3\cdot 11\sqrt{2}}{3}} = -\frac{92}{55} \approx -1.7$$

$$\overline{z} = \frac{\int z\rho\,\mathrm{d}s}{\int \rho\,\mathrm{d}s} = \frac{\frac{2^{10}\cdot 37\sqrt{2}}{7\cdot 3^3}}{\frac{2^3\cdot 11\sqrt{2}}{3}} = \frac{4736}{693} \approx 6.8$$

After these long calculations, it's nice to do a sanity check. Using $0 \leqslant t \leqslant 4$, we see our wire takes up space in the following intervals: $0 \leqslant x \leqslant 12$, $-4 \leqslant y \leqslant 1/2$, and $0 \leqslant z \leqslant 32/3$. The coordinates of our centre of mass all fall in these intervals, which doesn't guarantee our answer is correct, but it is a nice sign. If, say $\overline{x}$ had been negative, or $\overline{z}$ were greater than 11, we would have known there was something wrong.

Solutions to Exercises 1.7 — Jump to TABLE OF CONTENTS

S-1: We don't have enough information to gauge the size of the vectors, but we can figure out their direction. Gravity pulls straight down, so the vector $-mg\hat{\boldsymbol{\jmath}}$ points straight down. The normal force will be normal to the curve.



S-2: This equation stems from $\mathbf{F} = m\mathbf{a}$. In that equation, $\mathbf{a}$ is acceleration — the second derivative of position with respect to *time*. So, $\mathbf{v}$ is the derivative of position with respect to time.

We previously used $\mathbf{v}$ as the derivative of position with respect to the parameter we use to define our position — which was often called t, but was not the necessarily time. So this is a good point to keep straight.

S-3: **Solution 1:**
For large, negative values of x, the wire is closer and closer to a vertical line. If the bead were sliding down a vertical wire, it could do so without even touching the wire, so the force exerted on the bead would be zero. As x approaches 0 from the left, the wire approximates a horizontal line. If the bead were sitting on a horizontal line, the wire would be pushing up to counter gravity. So, we imagine the magnitude of the force exerted by the wire might increase as x increases. That is, $\frac{\mathrm{d}W}{\mathrm{d}x} > 0$.

Solution 2:
The net force exerted on the bead is

$$F = m\mathbf{a} = W\hat{\mathbf{N}} - mg\hat{\boldsymbol{\jmath}}$$

We dot both sides with $\hat{\mathbf{N}}$.

$$W\hat{\mathbf{N}} \cdot \hat{\mathbf{N}} - mg\hat{\boldsymbol{\jmath}} \cdot \hat{\mathbf{N}} = m\mathbf{a} \cdot \hat{\mathbf{N}}$$

Using the equation $\mathbf{a}(t) = \frac{d^2s}{dt^2}\hat{\mathbf{T}} + \kappa\left(\frac{ds}{dt}\right)^2\hat{\mathbf{N}}$,

$$W - mg\hat{\boldsymbol{\jmath}} \cdot \hat{\mathbf{N}} = m\kappa\left(\frac{ds}{dt}\right)^2$$
$$W = mg\hat{\mathbf{n}} \cdot \hat{\mathbf{N}} + m\kappa\left(\frac{ds}{dt}\right)^2$$
$$= mg\cos\theta + m\kappa\left(\frac{ds}{dt}\right)^2$$

where θ is the angle between $\hat{\boldsymbol{\jmath}}$ and $\hat{\mathbf{N}}$.

As x moves from a highly negative number to zero, θ moves from nearly $\pi/2$ to nearly 0. Therefore $\cos\theta$ increases from nearly zero to nearly one. Then $mg\cos\theta$ is increasing.

Furthermore, as x increases, we see from the picture that the curvature κ increases, and speed $\frac{ds}{dt}$ increases as well (kinetic energy is increasing as potential energy decreases).

So, $\frac{dW}{dx} > 0$.

S-4: Equation 1.7.1 defines $E = \frac{1}{2}m|\mathbf{v}|^2 + mgy$. The skater reaches their highest point when $|\mathbf{v}| = 0$, so when $y = \frac{E}{mg}$. This is the same equation as a sufficiently large circular culvert: it's the height where all the kinetic energy has been converted into potential energy. That's why we never even used the equation $y = x^2$!

S-5: The skateboarder starts going back down at $y_S = \frac{E}{mg}$, so we solve $3\text{ m} = \frac{E}{100\text{kg}\cdot 9.8\ \frac{\text{m}}{\text{s}^2}}$ to find $E = 2940\ \frac{\text{kg}\cdot\text{m}^2}{\text{s}^2} = 2940J$

Remark: we needed the diameter to be greater than 3m for the skateboarder to not be going all the way around the culvert, but choosing $r = 5$ leads to an answer no different from, say, $r = 50$.

S-6: From the text, the skateboarder will make it all the way around when $\frac{5}{2}(5) \leqslant \frac{E}{mg}$. Energy E is given by $E = \frac{1}{2}m|\mathbf{v}|^2 + mgy$, the sum of the kinetic and potential energy of the system. At $y = 0$, all the energy is kinetic, so $E = \frac{1}{2}m|\mathbf{v}|^2$, where $|\mathbf{v}|$ is the skater's velocity at the bottom of the culvert.

So, we solve:

$$\frac{25}{2} \leqslant \frac{E}{mg} = \frac{\frac{1}{2}m|\mathbf{v}|^2}{m \cdot 9.8}$$
$$|\mathbf{v}| \geqslant 5\sqrt{9.8}$$

So, a speed of $5\sqrt{9.8}$ m/s or higher is needed. (That's about 56 kph.)

S-7: Equation 1.7.2 tells us the normal force exerted by the track is $W\hat{\mathbf{N}}$, where $W = m\kappa|\mathbf{v}|^2 + mg\hat{\mathbf{k}} \cdot \hat{\mathbf{N}}$. (Note in our problem, the vertical direction is $\hat{\mathbf{k}}$, not $\hat{\boldsymbol{\jmath}}$ as in the text.) So, we ought to find κ and $\mathbf{N}$.

$$\begin{aligned}
\mathbf{r}(\theta) &= (3\cos\theta, 5\sin\theta, 4+4\cos\theta)\\
\mathbf{v}(\theta) &= (-3\sin\theta, 5\cos\theta, -4\sin\theta)\\
|\mathbf{v}(\theta)| = \frac{\mathrm{d}s}{\mathrm{d}\theta} &= \sqrt{9\sin^2\theta + 25\cos^2\theta + 16\sin^2\theta} = 5\\
\mathbf{a}(\theta) &= (-3\cos\theta, -5\sin\theta, -4\cos\theta)\\
\mathbf{v}\times\mathbf{a} &= 5(-4, 0, 3)\\
\kappa(\theta) &= \frac{|\mathbf{v}\times\mathbf{a}|}{\left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^3} = \frac{25}{5^3} = \frac{1}{5}
\end{aligned}$$

Since $\frac{\mathrm{d}^2s}{\mathrm{d}\theta^2} = 0$, we use the following theorem to find $\hat{\mathbf{N}}$:

$$\begin{aligned}
\mathbf{a}(\theta) &= \frac{\mathrm{d}^2s}{\mathrm{d}\theta^2}\hat{\mathbf{T}} + \kappa\left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^2\hat{\mathbf{N}}\\
(-3\cos\theta, -5\sin\theta, -4\cos\theta) &= 0 + \frac{25}{5}\hat{\mathbf{N}}\\
\hat{\mathbf{N}}(\theta) &= \left(-\frac{3}{5}\cos\theta, -\sin\theta, -\frac{4}{5}\cos\theta\right)
\end{aligned}$$

Using the given quantity $|\mathbf{v}(t)| = 5$ at the specified point,

$$\begin{aligned}
W\Big|_{\theta=\pi/4} &= m\kappa|\mathbf{v}|^2 + mg\hat{\mathbf{k}}\cdot\hat{\mathbf{N}}\\
&= (1)\frac{1}{5}5^2 + 1(9.8)\left(-\frac{4}{5}\cos(\pi/4)\right) = 5 - \frac{39.2}{5\sqrt{2}}\\
W\hat{\mathbf{N}}\Big|_{\theta=\pi/4} &= \left(5 - \frac{39.2}{5\sqrt{2}}\right)\left(-\frac{3}{5}\cos(\pi/4), -\sin(\pi/4), -\frac{4}{5}\cos(\pi/4)\right)\\
&= \left(5 - \frac{39.2}{5\sqrt{2}}\right)\left(-\frac{3}{5\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{4}{5\sqrt{2}}\right)\\
&= \left(-\frac{3}{\sqrt{2}} + 2.352\,,\ -\frac{5}{\sqrt{2}} + 3.92\,,\ -2\sqrt{2} + 3.136\right)
\end{aligned}$$

S-8: Equation 1.7.2 tells us the normal force exerted by the track is $W\hat{\mathbf{N}}$, where $W = m\kappa|\mathbf{v}|^2 + mg\hat{\boldsymbol{\jmath}}\cdot\hat{\mathbf{N}}$. So, we need to find κ and $\hat{\boldsymbol{\jmath}}\cdot\hat{\mathbf{N}}$ at the point $\theta = \frac{13\pi}{3}$.

Note that θ is the parameter used to describe the track, but it is *not* time. So $|\mathbf{v}(\theta)| = \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\theta}\right|$

is not the same as $|\mathbf{v}|$, the *speed* of the bead.

$$\begin{aligned}
\mathbf{r}(\theta) &= (\sin\theta, \sin\theta - \theta) & \mathbf{v}(\theta) &= (\cos\theta, \cos\theta - 1)\\
|\mathbf{v}(\theta)| &= \frac{\mathrm{d}s}{\mathrm{d}\theta} = \sqrt{2\cos^2\theta - 2\cos\theta + 1} & \mathbf{a}(\theta) &= (-\sin\theta, -\sin\theta)\\
|\mathbf{v}\times\mathbf{a}| &= |\sin\theta| & \kappa(\theta) &= \frac{|\mathbf{v}\times\mathbf{a}|}{\left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^3} = \frac{|\sin\theta|}{(2\cos^2\theta - 2\cos\theta + 1)^{3/2}}\\
\frac{\mathrm{d}^2s}{\mathrm{d}\theta^2} &= \frac{\sin\theta(1-2\cos\theta)}{\sqrt{2\cos^2\theta - 2\cos\theta + 1}}
\end{aligned}$$

Equation 1.3.3 part (c) gives us the relation $\mathbf{a}(\theta) = \frac{\mathrm{d}^2s}{\mathrm{d}\theta^2}\hat{\mathbf{T}} + \kappa\left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^2\hat{\mathbf{N}}$. We use this to find $\hat{\boldsymbol{\jmath}}\cdot\hat{\mathbf{N}}$ at $\theta = 13\pi/3$ without differentiating (actually, without even finding) $\hat{\mathbf{T}}$.

$$\begin{aligned}
\mathbf{a}(13\pi/3) &= \left(-\sqrt{3}/2, -\sqrt{3}/2\right)\\
\frac{\mathrm{d}^2s}{\mathrm{d}\theta^2}(13\pi/3) &= 0\\
\kappa(13\pi/3) &= \sqrt{6}\\
\frac{\mathrm{d}s}{\mathrm{d}\theta}(13\pi/3) &= 1/\sqrt{2}\\
\mathbf{a}(\theta)\cdot\hat{\boldsymbol{\jmath}} &= \left(\frac{\mathrm{d}^2s}{\mathrm{d}\theta^2}\hat{\mathbf{T}} + \kappa\left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^2\hat{\mathbf{N}}\right)\cdot\hat{\boldsymbol{\jmath}}\\
-\frac{\sqrt{3}}{2} &= 0 + \sqrt{6}(1/2)\hat{\mathbf{N}}\cdot\hat{\boldsymbol{\jmath}}\\
\hat{\mathbf{N}}\cdot\hat{\boldsymbol{\jmath}} &= -\frac{1}{\sqrt{2}}
\end{aligned}$$

Now we can find the speed $|\mathbf{v}|$ of the bead when $|W| = 100$ and it breaks off the track.

$$\begin{aligned}
W &= m\kappa|\mathbf{v}|^2 + mg\hat{\boldsymbol{\jmath}}\cdot\hat{\mathbf{N}}\\
\pm 100 &= \left(\frac{1}{9.8}\right)\sqrt{6}|\mathbf{v}|^2 + \frac{9.8}{9.8}\left(-\frac{1}{\sqrt{2}}\right)\\
|\mathbf{v}| &= \sqrt{\frac{9.8}{\sqrt{6}}\left(100 + \frac{1}{\sqrt{2}}\right)} \approx 20\ \mathrm{m/s}\ \approx 72\ \mathrm{kph}
\end{aligned}$$

(Because $|\mathbf{v}| > 0$, the equation above has no solution for $W = -100$.)

Quite fast! 100 N is a lot of force for such a light object.

S-9: According to the equation in the text, the skier will become airborne when:

$$|\mathbf{v}| > \sqrt{\frac{g}{\kappa}|\hat{\boldsymbol{\jmath}}\cdot\hat{\mathbf{N}}|}$$

We'll use the equation of the curve to find κ and $\hat{\mathbf{N}}$.

Note that g is given in metres per second, while the other quantities are in kilometres and hours. Converting, $9.8\ \mathrm{m/s^2}$ is the same as
$\left(\frac{9.8\ \mathrm{m}}{1\ \mathrm{s}^2}\right)\left(\frac{1\ \mathrm{km}}{1000\ \mathrm{m}}\right)\left(\frac{3600\ \mathrm{s}}{1\ \mathrm{hr}}\right)^2 = 98\cdot 6^4\ \frac{\mathrm{km}}{\mathrm{h}^2} = 2^5\cdot 3^4\cdot 7^2\ \frac{\mathrm{km}}{\mathrm{h}^2}$.

$$\begin{aligned}
\mathbf{r}(t) &= (\ln t, 1-t)\\
\mathbf{r}'(t) &= \mathbf{v}(t) = (t^{-1}, -1) \qquad\qquad \frac{\mathrm{d}s}{\mathrm{d}t} = |\mathbf{v}(t)| = \sqrt{1+t^{-2}}\\
\mathbf{r}''(t) &= \mathbf{a}(t) = (-t^{-2}, 0)\\
\kappa(t) &= \frac{|\mathbf{v}\times\mathbf{a}|}{\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3} = \frac{t^{-2}}{\sqrt{1+t^{-2}}^3} = \frac{|t|}{(1+t^2)^{3/2}} = \frac{t}{(1+t^2)^{3/2}}
\end{aligned}$$

Note t is positive in the interval in question.

$$\begin{aligned}
\hat{\mathbf{T}}(t) &= \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{1}{\sqrt{1+t^{-2}}}(t^{-1}, -1) = \left(\frac{1}{\sqrt{1+t^2}}, \frac{-t}{\sqrt{1+t^2}}\right)\\
\hat{\mathbf{T}}'(t) &= \left(\frac{-t}{(1+t^2)^{3/2}}, \frac{-1}{(1+t^2)^{3/2}}\right) \qquad\qquad |\hat{\mathbf{T}}'(t)| = \frac{1}{t^2+1}\\
\hat{\mathbf{N}}(t) &= \frac{\hat{\mathbf{T}}'(t)}{|\hat{\mathbf{T}}'(t)|} = \left(\frac{-t}{\sqrt{1+t^2}}, \frac{-1}{\sqrt{1+t^2}}\right)\\
|\hat{\mathbf{N}}\cdot\hat{\boldsymbol{\jmath}}| &= \frac{1}{\sqrt{1+t^2}}
\end{aligned}$$

Now, we have all the pieces we need to find the "escape velocity" of the ground.

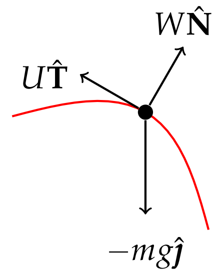
$$|\mathbf{v}| = \sqrt{\frac{g}{\kappa}|\hat{\mathbf{N}}\cdot\hat{\boldsymbol{\jmath}}|} = \sqrt{\frac{g\cdot(1+t^2)^{3/2}}{t(1+t^2)^{1/2}}} = \sqrt{\frac{g(1+t^2)}{t}}$$

Since the skier can take off anywhere on the hill, we just need their velocity to be larger than the *smallest* value of $\sqrt{\frac{g(1+t^2)}{t}}$ when $1/e \leqslant t \leqslant e$. To find that minimum, we find the location of the minimum of the simpler function $g(t) = \frac{1+t^2}{t}$. Using first-semester calculus, we find it to occur when $t=1$. So, the minimum value of $\sqrt{\frac{g(1+t^2)}{t}}$ (that is, smallest speed to achieve lift-off) occurs at $t=1$. We therefore need a minimum speed greater than:

$$\left.\sqrt{\frac{g(1+t^2)}{|t|}}\right|_{t=1} = \sqrt{2g} = \sqrt{2^6\cdot 3^4\cdot 7^2} = 2^3\cdot 3^2\cdot 7 = 504\ \text{kph}$$

(It seems unlikely that one could reach this speed on skis. The skier is probably earth-bound until they find a curvier hill.)

S-10: We now have three forces acting on the bead, rather than the two in the text. The wire still exerts a normal force $W\hat{\mathbf{N}}$ on the bead to keep it on the wire; gravity still exerts a force $-mg\hat{\jmath}$ straight down. Now our jet-pack force also exerts a force parallel to the direction of the bead's motion, i.e. parallel to $\hat{\mathbf{T}}$. This force is $U\hat{\mathbf{T}}$.



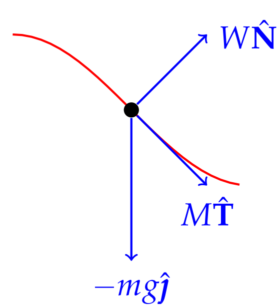
The net force acting on the bead is the sum of these three forces:

$$F = m\mathbf{a} = U\hat{\mathbf{T}} + W\hat{\mathbf{N}} - mg\hat{\jmath}$$

To focus on the force in the direction of $\hat{\mathbf{T}}$, we dot both sides of the equation with $\hat{\mathbf{T}}(s) = \left(\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s}\right)$. (Recall $\mathbf{r}(s)$ was parametrized with respect to arclength, so $\hat{\mathbf{T}}(s) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}$ everywhere.) Since the speed of the bead is constant, the tangential component of its acceleration, $\mathbf{a} \cdot \hat{\mathbf{T}}$, is 0 (see Theorem 1.3.3.c).

$$\begin{aligned}
0 &= (U\hat{\mathbf{T}} + W\hat{\mathbf{N}} - mg\hat{\jmath}) \cdot \hat{\mathbf{T}} \\
&= (U\hat{\mathbf{T}} \cdot \hat{\mathbf{T}}) + (W\hat{\mathbf{N}} \cdot \hat{\mathbf{T}}) - mg\hat{\jmath} \cdot \hat{\mathbf{T}} \\
&= U + 0 - mg\frac{\mathrm{d}y}{\mathrm{d}s} \\
U &= mg\frac{\mathrm{d}y}{\mathrm{d}s}
\end{aligned}$$

S-11: (a) There are three forces acting on the snowmachine. If it's not accelerating, then $F = m\mathbf{a} = 0$: that is, the forces all cancel out.



So, we have the equation

$$m\mathbf{a} = W\hat{\mathbf{N}} + M\hat{\mathbf{T}} - mg\hat{\jmath}$$

To isolate M, we dot both sides of the equation with $\hat{\mathbf{T}}$. Remember $\hat{\mathbf{T}}$ is a unit vector, and it is perpendicular to $\hat{\mathbf{N}}$.

$$\begin{aligned} m\mathbf{a}\cdot\hat{\mathbf{T}} &= W\hat{\mathbf{N}}\cdot\hat{\mathbf{T}} + M\hat{\mathbf{T}}\cdot\hat{\mathbf{T}} - mg\hat{\boldsymbol{\jmath}}\cdot\hat{\mathbf{T}} \\ &= 0 + M - mg\hat{\boldsymbol{\jmath}}\cdot\hat{\mathbf{T}} \end{aligned}$$

Since the speed of the snowmachine is constant, the tangential component of its acceleration, $\mathbf{a}\cdot\hat{\mathbf{T}}$, is 0 (see Theorem 1.3.3.c).

$$\begin{aligned} 0 &= M - mg\hat{\boldsymbol{\jmath}}\cdot\hat{\mathbf{T}} \\ M &= mg\hat{\boldsymbol{\jmath}}\cdot\hat{\mathbf{T}} \end{aligned}$$

(b) We would expect, from looking at the situation, that the engine would have to provide a "backwards" force to slow the acceleration due to gravity. So, we would expect $M < 0$. Indeed, if $\hat{\mathbf{T}}$ points downhill, then the y-component of $\hat{\mathbf{T}}$ is negative, so $M = mg\hat{\boldsymbol{\jmath}}\cdot\hat{\mathbf{T}}$ is negative.

(This is the purpose of driving downhill in a low gear: the friction inside the motor provides a force *opposing* the direction of motion, slowing the vehicle.)

(c) To use the equation $M = mg\hat{\boldsymbol{\jmath}}\cdot\hat{\mathbf{T}}$, we'll need to find $\hat{\boldsymbol{\jmath}}\cdot\hat{\mathbf{T}}$.

$$\begin{aligned} \mathbf{r}(x) &= (x, 1+\cos x) & \mathbf{r}'(x) &= (1, -\sin x) \\ |\mathbf{r}'(x)| &= \sqrt{1+\sin^2 x} & \hat{\mathbf{T}}(x) &= \frac{1}{\sqrt{1+\sin^2 x}}(1, -\sin x) \\ \hat{\mathbf{T}}(3\pi/4) &= \left(\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}}\right) \end{aligned}$$

So,

$$M = (200\text{ kg})(9.8\text{ m/s}^2)\left(-\frac{1}{\sqrt{3}}\text{ m}\right) = -\frac{1960}{\sqrt{3}}\text{ N} \approx -1131.6\text{ N}$$

S-12: We begin with the usual computations.

$$\begin{aligned} \mathbf{r}(\theta) &= (4\cos\theta, 3(1+\sin\theta)) \\ \mathbf{v}(\theta) = \mathbf{r}'(\theta) &= (-4\sin\theta, 3\cos\theta) & |\mathbf{v}(\theta)| &= \frac{\mathrm{d}s}{\mathrm{d}\theta} = \sqrt{16\sin^2\theta + 9\cos^2\theta} = \sqrt{9+7\sin^2\theta} \\ \mathbf{a}(\theta) &= (-4\cos\theta, -3\sin\theta) \\ |\mathbf{v}(\theta)\times\mathbf{a}(\theta)| &= 12 & \kappa(\theta) &= \frac{|\mathbf{v}(\theta)\times\mathbf{a}(\theta)|}{\left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^3} = \frac{12}{(9+7\sin^2\theta)^{3/2}} \\ \hat{\mathbf{T}}(\theta) &= \frac{(-4\sin\theta, 3\cos\theta)}{\sqrt{9+7\sin^2\theta}} & \hat{\mathbf{T}}'(\theta) &= \frac{(36\cos\theta, 48\sin\theta)}{-(9\cos^2\theta+16\sin^2\theta)^{3/2}} \\ |\hat{\mathbf{T}}'(\theta)| &= \frac{12}{9\cos^2\theta+16\sin^2\theta} & \hat{\mathbf{N}}(\theta) &= \frac{(3\cos\theta, 4\sin\theta)}{-\sqrt{9\cos^2\theta+16\sin^2\theta}} \end{aligned}$$

We want to find the height y_S where $|\mathbf{v}| = 0$, and the height y_A where $W = 0$. Remember that $\mathbf{v}$ in these equations is the derivative of position with respect to *time*, and is not the same as $\mathbf{v}(\theta)$.

$$\begin{aligned}
\text{Equation 1.7.1:} \quad & E = \frac{1}{2}m|\mathbf{v}|^2 + mgy \\
\text{If } |\mathbf{v}| = 0: \quad & E = mgy_S \implies y_S = \frac{E}{mg}
\end{aligned}$$

This answers part a.

$$\begin{aligned}
\text{Equation 1.7.2:} \quad & W = 2\kappa(E - mgy) + mg\hat{\boldsymbol{\jmath}} \cdot \hat{\mathbf{N}} \\
\text{If } W = 0: \quad & 0 = 2\kappa(E - mgy_A) + mg\hat{\boldsymbol{\jmath}} \cdot \hat{\mathbf{N}} \\
& = \frac{24\,(E - mgy_A)}{(9 + 7\sin^2\theta)^{3/2}} - mg\left(4\frac{\sin\theta}{\sqrt{9 + 7\sin^2\theta}}\right)
\end{aligned}$$

Using $y = 3 + 3\sin\theta$:

$$= \frac{24\,(E - mgy_A)}{\left(9 + 7\left(\frac{y_A - 3}{3}\right)^2\right)^{3/2}} - 4mg\left(\frac{\frac{y_A-3}{3}}{\sqrt{9 + 7\left(\frac{y_A-3}{3}\right)^2}}\right)$$

So, for part b., we can write (say)

$$\frac{24\,(E - mgy_A)}{\left(9 + 7\left(\frac{y_A - 3}{3}\right)^2\right)^{3/2}} = 4mg\left(\frac{\frac{y_A-3}{3}}{\sqrt{9 + 7\left(\frac{y_A-3}{3}\right)^2}}\right)$$

Now, suppose the skater's speed at the bottom of the culvert ($y = 0$) is 11 m/s. Then their energy is $E = \frac{1}{2}m(11^2) + 0$, or $\frac{121m}{2}$ joules, where m is their mass. Then $y_S = \frac{E}{mg} = \frac{121}{2 \cdot 9.8} \approx 6.2$. Since the half-way height of the culvert is at height $y = 3$, the skater makes it onto the ceiling of the culvert. Now the question is: did they make it all around, or fall off the ceiling?

For this, we need to find y_A. If they go airborne on the ceiling, they fall; but if $y_A > 6$, then they never lose contact with the culvert, and they go all the way around.

$$\frac{24\left(E-mgy_A\right)}{\left(9+7\left(\frac{y_A-3}{3}\right)^2\right)^{3/2}}=4mg\left(\frac{\frac{y_A-3}{3}}{\sqrt{9+7\left(\frac{y_A-3}{3}\right)^2}}\right)$$

$$\Leftrightarrow\qquad \frac{6\left(\frac{E}{mg}-y_A\right)}{\left(9+7\left(\frac{y_A-3}{3}\right)^2\right)^{3/2}}=\frac{\frac{y_A-3}{3}}{\sqrt{9+7\left(\frac{y_A-3}{3}\right)^2}}$$

$$\Leftrightarrow\qquad \frac{6\left(\frac{11^2}{2\cdot 9.8}-y_A\right)}{\left(9+7\left(\frac{y_A-3}{3}\right)^2\right)^{3/2}}=\frac{\frac{y_A-3}{3}}{\sqrt{9+7\left(\frac{y_A-3}{3}\right)^2}}$$

To simplify to a more standard form, we multiply both sides by $\left(9+7\left(\frac{y_A-3}{3}\right)^2\right)^{3/2}$:

$$6\left(\frac{11^2}{2\cdot 9.8}-y_A\right)=\left(\frac{y_A-3}{3}\right)\left(9+7\left(\frac{y_A-3}{3}\right)^2\right)$$

Now, we simplify to

$$0=\frac{7}{9}y_A^3-7y_A^2+48y_A-\frac{7797}{49}$$

Now, solving for y_A involves solving a cubic function, which is no small task. We could ask a computer, but we can also get an idea of its root(s) by plugging in numbers and using the intermediate value theorem. In particular, we need to know whether y_A is greater than 6 (the skater makes it!) or between 3 and 6 (they fall off the ceiling).

Let $f(y)=\frac{7}{9}y^3-7y^2+48y-\frac{7797}{49}$. Note $f(4)=-\frac{12941}{441}$, which is negative, and $f(5)=\frac{1367}{441}$, which is positive. So, by the intermediate value theorem, there is a root of $f(y)$ between $y=4$ and $y=5$. That is, y_A is between 4 and 5, so the skater falls off the ceiling somewhere between these heights, rather than making it all the way around.

S-13: (a) By Newton's law of motion

$$\begin{aligned}E'(t)&=\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}m|\mathbf{v}(t)|^2+mg\mathbf{r}(t)\cdot\hat{\mathbf{k}}\right]=m\mathbf{v}(t)\cdot\mathbf{v}'(t)+mg\mathbf{v}(t)\cdot\hat{\mathbf{k}}\\&=\mathbf{v}(t)\cdot\big[\mathbf{N}\big(\mathbf{r}(t)\big)-mg\hat{\mathbf{k}}\big]+mg\mathbf{v}(t)\cdot\hat{\mathbf{k}}\\&=0\end{aligned}$$

since $\mathbf{v}(t)\cdot\mathbf{N}\big(\mathbf{r}(t)\big)=0$. So $E(t)$ is a constant, independent of t.

(b) By part (a),

$$E(t)=E(0)\implies \tfrac{1}{2}m|\mathbf{v}(t)|^2+mgb\theta(t)=mg(2\pi b)\implies |\mathbf{v}(t)|^2=2gb\big(2\pi-\theta(t)\big)$$

(c) We wish to determine the time it takes to go from $\theta = 2\pi$ to $\theta = 0$. We'll first determine $\frac{d\theta}{dt}$.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{d\theta}\frac{d\theta}{dt} = \big(-a\sin\theta, a\cos\theta, b\big)\frac{d\theta}{dt}$$
$$\implies |\mathbf{v}|^2 = \left[a^2+b^2\right]\left(\frac{d\theta}{dt}\right)^2$$
$$\implies \frac{d\theta}{dt} = -\left[\frac{|\mathbf{v}|^2}{a^2+b^2}\right]^{1/2} = -\left[\frac{2gb(2\pi-\theta)}{a^2+b^2}\right]^{1/2}$$

We have chosen the negative sign because θ must decrease from 2π to 0. The time required to do so is

$$\int dt = \int_{2\pi}^{0}\frac{dt}{d\theta}d\theta = -\left[\frac{a^2+b^2}{2gb}\right]^{1/2}\int_{2\pi}^{0}\frac{1}{(2\pi-\theta)^{1/2}}d\theta$$
$$= \left[\frac{a^2+b^2}{2gb}\right]^{1/2}\int_{0}^{2\pi}\frac{1}{(2\pi-\theta)^{1/2}}d\theta$$
$$= \left[\frac{a^2+b^2}{2gb}\right]^{1/2}\left[-2(2\pi-\theta)^{1/2}\right]_0^{2\pi} = 2\left[\frac{a^2+b^2}{gb}\pi\right]^{1/2}$$

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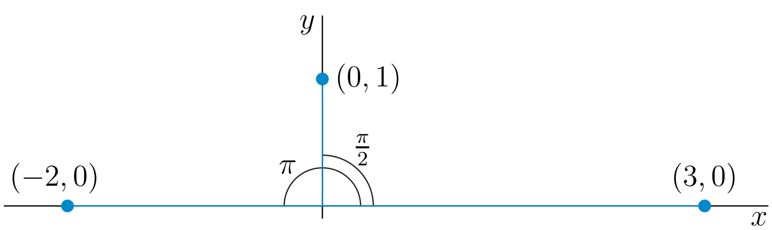
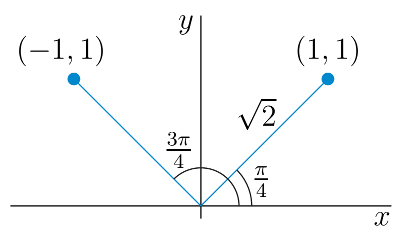
S-1: The left hand sketch below contains the points, (x_1, y_1), (x_3, y_3), (x_5, y_5), that are on the axes. The right hand sketch below contains the points, (x_2, y_2), (x_4, y_4), that are not on the axes.





Recall that the polar coordinates r, θ are related to the cartesian coordinates x, y, by $x = r\cos\theta$, $y = r\sin\theta$. So $r = \sqrt{x^2+y^2}$ and $\tan\theta = \frac{y}{x}$ (assuming that $x \neq 0$) and

$$(x_1,y_1) = (3,0) \implies r_1 = 3,\ \tan\theta_1 = 0 \implies \theta_1 = 0 \text{ as } (x_1,y_1) \text{ is on the positive } x\text{-axis}$$
$$(x_2,y_2) = (1,1) \implies r_2 = \sqrt{2},\ \tan\theta_2 = 1 \implies \theta_2 = \frac{\pi}{4} \text{ as } (x_2,y_2) \text{ is in the first octant}$$
$$(x_3,y_3) = (0,1) \implies r_3 = 1,\ \cos\theta_3 = 0 \implies \theta_3 = \frac{\pi}{2} \text{ as } (x_3,y_3) \text{ is on the positive } y\text{-axis}$$
$$(x_4,y_4) = (-1,1) \implies r_4 = \sqrt{2},\ \tan\theta_4 = -1 \implies \theta_4 = \frac{3\pi}{4} \text{ as } (x_4,y_4) \text{ is in the third octant}$$
$$(x_5,y_5) = (-2,0) \implies r_5 = 2,\ \tan\theta_5 = 0 \implies \theta_5 = \pi \text{ as } (x_5,y_5) \text{ is on the negative } x\text{-axis}$$

S-2: Note that the distance from the point $\big(r\cos\theta\,,\,r\sin\theta\big)$ to the origin is

$$\sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = \sqrt{r^2} = |r|$$

Thus r can be either the distance to the origin or minus the distance to the origin.

(a) The distance from $(-2,0)$ to the origin is 2. So either $r = 2$ or $r = -2$.

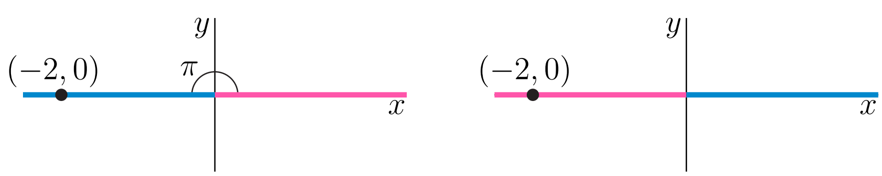
- If $r = 2$, then θ must obey

$$\begin{aligned}(-2,0) = \big(2\cos\theta\,,\,2\sin\theta\big) &\iff \sin\theta = 0,\ \cos\theta = -1\\ &\iff \theta = n\pi,\ n \text{ integer},\ \cos\theta = -1\\ &\iff \theta = n\pi,\ n \text{ odd integer}\end{aligned}$$

- If $r = -2$, then θ must obey

$$\begin{aligned}(-2,0) = \big(-2\cos\theta\,,\,-2\sin\theta\big) &\iff \sin\theta = 0,\ \cos\theta = 1\\ &\iff \theta = n\pi,\ n \text{ integer},\ \cos\theta = 1\\ &\iff \theta = n\pi,\ n \text{ even integer}\end{aligned}$$

In the figure on the left below, the blue half-line is the set of all points with polar coordinates $\theta = \pi$, $r > 0$ and the pink half-line is the set of all points with polar coordinates $\theta = \pi$, $r < 0$. In the figure on the right below, the blue half-line is the set of all points with polar coordinates $\theta = 0$, $r > 0$ and the pink half-line is the set of all points with polar coordinates $\theta = 0$, $r < 0$.



(b) The distance from $(1,1)$ to the origin is $\sqrt{2}$. So either $r = \sqrt{2}$ or $r = -\sqrt{2}$.

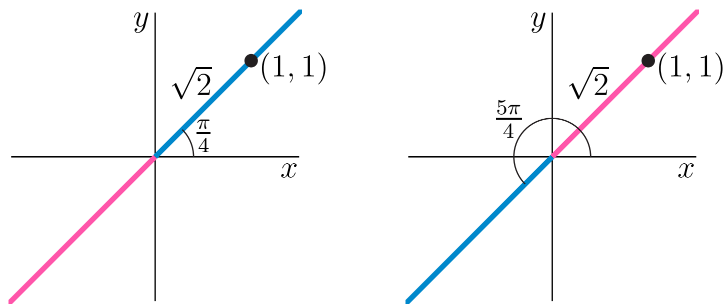
- If $r = \sqrt{2}$, then θ must obey

$$\begin{aligned}(1,1) = \big(\sqrt{2}\,\cos\theta\,,\,\sqrt{2}\,\sin\theta\big) &\iff \sin\theta = \cos\theta = 1/\sqrt{2}\\ &\iff \theta = \pi/4 + 2n\pi,\ n \text{ integer}\end{aligned}$$

- If $r = -\sqrt{2}$, then θ must obey

$$\begin{aligned}(1,1) = \big(-\sqrt{2}\,\cos\theta\,,\,-\sqrt{2}\,\sin\theta\big) &\iff \sin\theta = \cos\theta = -1/\sqrt{2}\\ &\iff \theta = 5\pi/4 + 2n\pi,\ n \text{ integer}\end{aligned}$$

In the figure on the left below, the blue half-line is the set of all points with polar coordinates $\theta = \frac{\pi}{4}$, $r > 0$ and the pink half-line is the set of all points with polar coordinates $\theta = \frac{\pi}{4}$, $r < 0$. In the figure on the right below, the blue half-line is the set of all points with polar coordinates $\theta = \frac{5\pi}{4}$, $r > 0$ and the pink half-line is the set of all points with polar coordinates $\theta = \frac{5\pi}{4}$, $r < 0$.



(c) The distance from $(-1,-1)$ to the origin is $\sqrt{2}$. So either $r = \sqrt{2}$ or $r = -\sqrt{2}$.

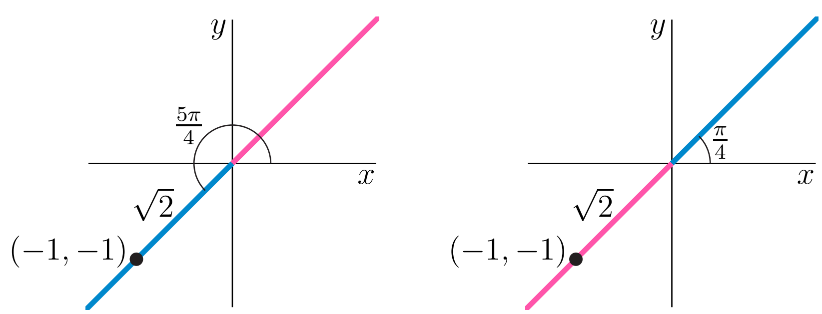
- If $r = \sqrt{2}$, then θ must obey

$$(-1,-1) = \big(\sqrt{2}\cos\theta\,,\ \sqrt{2}\sin\theta\big) \iff \sin\theta = \cos\theta = -1/\sqrt{2}$$
$$\iff \theta = 5\pi/4 + 2n\pi,\ n \text{ integer}$$

- If $r = -\sqrt{2}$, then θ must obey

$$(-1,-1) = \big(-\sqrt{2}\cos\theta\,,\ -\sqrt{2}\sin\theta\big) \iff \sin\theta = \cos\theta = 1/\sqrt{2}$$
$$\iff \theta = \pi/4 + 2n\pi,\ n \text{ integer}$$

In the figure on the left below, the blue half-line is the set of all points with polar coordinates $\theta = \frac{5\pi}{4}$, $r > 0$ and the pink half-line is the set of all points with polar coordinates $\theta = \frac{5\pi}{4}$, $r < 0$. In the figure on the right below, the blue half-line is the set of all points with polar coordinates $\theta = \frac{\pi}{4}$, $r > 0$ and the pink half-line is the set of all points with polar coordinates $\theta = \frac{\pi}{4}$, $r < 0$.



S-3: (a) The lengths are

$$|\hat{\mathbf{e}}_r(\theta)| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$
$$|\hat{\mathbf{e}}_\theta(\theta)| = \sqrt{(-\sin\theta)^2 + \cos^2\theta} = 1$$

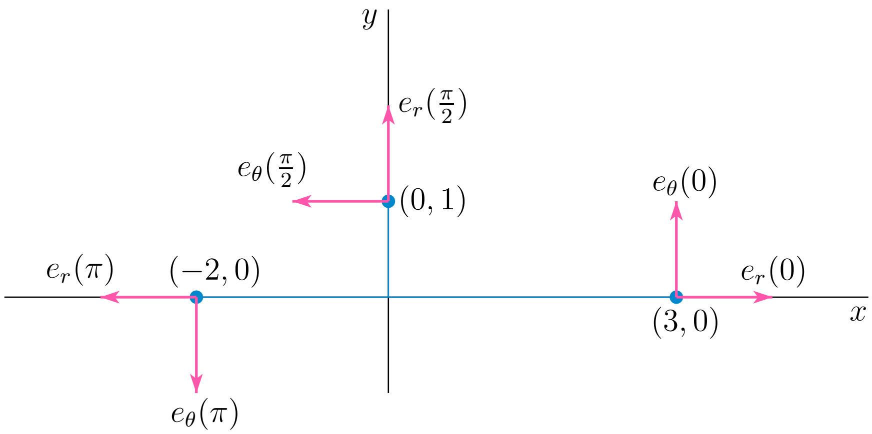
As
$$\hat{\mathbf{e}}_r(\theta)\cdot\hat{\mathbf{e}}_\theta(\theta) = (\cos\theta)(-\sin\theta) + (\sin\theta)(\cos\theta) = 0$$
the two vectors are perpendicular and the angle between them is $\frac{\pi}{2}$. The cross product is

$$\hat{\mathbf{e}}_r(\theta)\times\hat{\mathbf{e}}_\theta(\theta) = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{bmatrix} = \hat{\mathbf{k}}$$

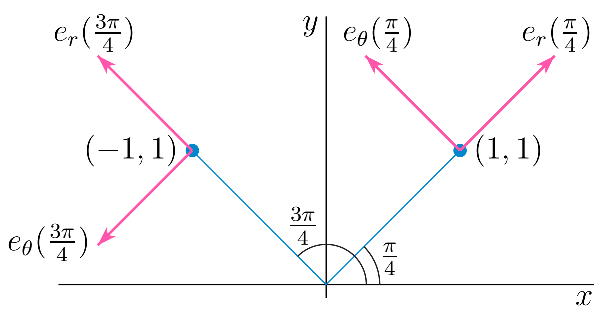
(b) Note that for θ determined by $x = r\cos\theta$, $y = r\sin\theta$,

- the vector $\hat{\mathbf{e}}_r(\theta)$ is a unit vector in the same direction as the vector from $(0, 0)$ to (x, y) and
- the vector $\hat{\mathbf{e}}_\theta(\theta)$ is a unit vector that is perpendicular to $\hat{\mathbf{e}}_r(\theta)$.
- The y-component of $\hat{\mathbf{e}}_\theta(\theta)$ has the same sign as the x-component of $\hat{\mathbf{e}}_r(\theta)$. The x-component of $\hat{\mathbf{e}}_\theta(\theta)$ has opposite sign to that of the y-component of $\hat{\mathbf{e}}_r(\theta)$.

Here is a sketch of (x_i, y_i), $\hat{\mathbf{e}}_r(\theta_i)$, $\hat{\mathbf{e}}_\theta(\theta_i)$ for $i = 1, 3, 5$ (the points on the axes)



and here is a sketch (to a different scale) of (x_i, y_i), $\hat{\mathbf{e}}_r(\theta_i)$, $\hat{\mathbf{e}}_\theta(\theta_i)$ for $i = 2, 4$ (the points off the axes).



S-4: (a) Since $-1 \leqslant \sin(4\theta) \leqslant 1$, the coordinate $r = 2 + \sin(4\theta)$ oscillates between $r = 1$ and $r = 3$ as θ runs from 0 to 2π. The maximum value $r = 3$ is achieved when $\sin(4\theta) = 1$, i.e when $4\theta = \frac{\pi}{2} + 2n\pi$, i.e. when $\theta = \frac{\pi}{8} + \frac{n\pi}{2}$. That matches figure (E).

(b) Since $-1 \leqslant \sin(4\theta) \leqslant 1$, the coordinate $r = 1 + 2\sin(4\theta)$ takes its maximum value $r = 3$ when $\sin(4\theta) = 1$, i.e. when $\theta = \frac{\pi}{8} + \frac{n\pi}{2}$, just as the case with (a). But now r can also take the value 0. That matches figure (B).

(c) $r = 1$ is completely indepedent of θ. All points on the curve $r = 1$ are a distance 1 from the origin. That is, $r = 1$ is the circle of radius 1 centred on the origin. That's figure (F).

(d) In this case, θ is subject to the restriction $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$, like figure (C). Figure (C) looks like a circle. We can verify that $r = 2\cos(\theta)$ is indeed a circle by converting to Cartesian coordinates. We can convert the right hand side to exactly $2x = 2r\cos(\theta)$ by multiplying the whole equation by r.

$$\begin{aligned} r = 2\cos(\theta) &\iff r^2 = 2r\cos(\theta) \iff x^2 + y^2 = 2x \\ &\iff (x-1)^2 + y^2 = 1 \end{aligned}$$

So $r = 2\cos(\theta)$ is the circle of radius 1 centred on $x = 1$, $y = 0$, which indeed matches figure (C).

(e) When $\theta = 0$, $r = e^{\theta/10} + e^{-\theta/10} = 2$. As

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\big(e^{\theta/10} + e^{-\theta/10}\big) = \frac{1}{10}\big(e^{\theta/10} - e^{-\theta/10}\big) > 0 \qquad \text{for all } \theta > 0$$

$r = e^{\theta/10} + e^{-\theta/10}$ increases as θ increases for all $\theta \geqslant 0$. Furthermore the rate of increase gets bigger and bigger as θ gets bigger and bigger. So r starts at $r = 2$ when $\theta = 0$ and increases faster and faster as θ increases. That matches figure (A).

(f) When $\theta = 0$, $r = \theta = 0$. As

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\theta = 1 \qquad \text{for all } \theta$$

$r = \theta$ increases as θ increases for all $\theta \geqslant 0$. Furthermore the rate of increase is independent of θ. So r starts at $r = 0$ when $\theta = 0$ and increases at a constant rate as θ increases. That matches figure (D).

S-5: Think of θ as a time parameter and recall that $\kappa(\theta) = \frac{|\mathbf{v}(\theta)\times\mathbf{a}(\theta)|}{|\mathbf{v}(\theta)|^3}$. The given curve has

$$\begin{aligned} x(\theta) &= f(\theta)\cos\theta \\ y(\theta) &= f(\theta)\sin\theta \\ \mathbf{r}(\theta) &= f(\theta)\big[\cos\theta\,\hat{\pmb{\imath}} + \sin\theta\,\hat{\pmb{\jmath}}\big] \\ \mathbf{v}(\theta) = \mathbf{r}'(\theta) &= f'(\theta)\big[\cos\theta\,\hat{\pmb{\imath}} + \sin\theta\,\hat{\pmb{\jmath}}\big] + f(\theta)\big[-\sin\theta\,\hat{\pmb{\imath}} + \cos\theta\,\hat{\pmb{\jmath}}\big] \\ \mathbf{a}(\theta) = \mathbf{r}''(\theta) &= \big\{f''(\theta) - f(\theta)\big\}\big[\cos\theta\,\hat{\pmb{\imath}} + \sin\theta\,\hat{\pmb{\jmath}}\big] + 2f'(\theta)\big[-\sin\theta\,\hat{\pmb{\imath}} + \cos\theta\,\hat{\pmb{\jmath}}\big] \end{aligned}$$

The efficient way to compute $|\mathbf{v}(\theta)|$ and the cross product $\mathbf{v}(\theta)\times\mathbf{a}(\theta)$ is to observe that

$$\begin{aligned} \mathbf{v}(\theta) &= f'(\theta)\,\hat{\mathbf{e}}_r(\theta) + f(\theta)\,\hat{\mathbf{e}}_\theta(\theta) \\ \mathbf{a}(\theta) &= \big\{f''(\theta) - f(\theta)\big\}\,\hat{\mathbf{e}}_r(\theta) + 2f'(\theta)\,\hat{\mathbf{e}}_\theta(\theta) \end{aligned}$$

where $\hat{\mathbf{e}}_r(\theta)$ and $\hat{\mathbf{e}}_\theta(\theta)$ are the vectors of Q[3]. As $\hat{\mathbf{e}}_r(\theta)$ and $\hat{\mathbf{e}}_\theta(\theta)$ are mutually perpendicular unit vectors obeying $\hat{\mathbf{e}}_r(\theta)\times\hat{\mathbf{e}}_\theta(\theta) = \hat{\mathbf{k}}$ and $\hat{\mathbf{e}}_r(\theta)\times\hat{\mathbf{e}}_r(\theta) = \hat{\mathbf{e}}_\theta(\theta)\times\hat{\mathbf{e}}_\theta(\theta) = 0$,

$$\begin{aligned} |\mathbf{v}(\theta)|^2 = \mathbf{v}(\theta)\cdot\mathbf{v}(\theta) &= \big[f'(\theta)\,\hat{\mathbf{e}}_r(\theta) + f(\theta)\,\hat{\mathbf{e}}_\theta(\theta)\big]\cdot\big[f'(\theta)\,\hat{\mathbf{e}}_r(\theta) + f(\theta)\,\hat{\mathbf{e}}_\theta(\theta)\big] \\ &= f'(\theta)^2\,\hat{\mathbf{e}}_r(\theta)\cdot\hat{\mathbf{e}}_r(\theta) + f(\theta)^2\,\hat{\mathbf{e}}_\theta(\theta)\cdot\hat{\mathbf{e}}_\theta(\theta) + 2f'(\theta)\,f(\theta)\,\hat{\mathbf{e}}_r(\theta)\cdot\hat{\mathbf{e}}_\theta(\theta) \\ &= f'(\theta)^2 + f(\theta)^2 \\ |\mathbf{v}(\theta)| &= \sqrt{f'(\theta)^2 + f(\theta)^2} \\ \mathbf{v}(\theta)\times\mathbf{a}(\theta) &= \big[f'(\theta)\,\hat{\mathbf{e}}_r(\theta) + f(\theta)\,\hat{\mathbf{e}}_\theta(\theta)\big]\times\big[\big\{f''(\theta) - f(\theta)\big\}\,\hat{\mathbf{e}}_r(\theta) + 2f'(\theta)\,\hat{\mathbf{e}}_\theta(\theta)\big] \\ &= 2f'(\theta)^2\,\hat{\mathbf{e}}_r(\theta)\times\hat{\mathbf{e}}_\theta(\theta) + f(\theta)[f''(\theta) - f(\theta)]\,\hat{\mathbf{e}}_\theta(\theta)\times\hat{\mathbf{e}}_r(\theta) \\ &= \big\{2f'(\theta)^2 - f(\theta)[f''(\theta) - f(\theta)]\big\}\,\hat{\mathbf{k}} \end{aligned}$$

So

$$\kappa(\theta) = \frac{|\mathbf{v}(\theta) \times \mathbf{a}(\theta)|}{|\mathbf{v}(\theta)|^3} = \frac{\left|f(\theta)^2 + 2f'(\theta)^2 - f(\theta)f''(\theta)\right|}{\left[f(\theta)^2 + f'(\theta)^2\right]^{3/2}}$$

S-6: By the Q[5] with

$$f(\theta) = a(1 - \cos\theta) \qquad f'(\theta) = a\sin\theta \qquad f''(\theta) = a\cos\theta$$

we have

$$\begin{aligned}
\kappa(\theta) &= \frac{\left|f(\theta)^2 + 2f'(\theta)^2 - f(\theta)f''(\theta)\right|}{[f(\theta)^2 + f'(\theta)^2]^{3/2}} \\
&= \frac{\left|a^2 - 2a^2\cos\theta + a^2\cos^2\theta + 2a^2\sin^2\theta - a^2\cos\theta + a^2\cos^2\theta\right|}{[a^2 - 2a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta]^{3/2}} \\
&= \frac{3a^2 - 3a^2\cos\theta}{[2a^2 - 2a^2\cos\theta]^{3/2}} = \frac{3}{2^{3/2}a\sqrt{1-\cos\theta}} = \frac{3}{2\sqrt{2ar(\theta)}}
\end{aligned}$$

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S-1: The vectors are pointing to the right when $x > 0$, to the left when $x < 0$, and are vertical when $x = 0$. So, at least for (x, y) shown in the sketch,

$$\mathbf{v}(x,y) \cdot \hat{\boldsymbol{\imath}} \begin{cases} > 0 & \text{when } \boxed{x > 0} \\ = 0 & \text{when } \boxed{x = 0} \\ < 0 & \text{when } \boxed{x < 0} \end{cases}$$

The behaviour of the y-values is more complicated. Vectors in one vertical line seem to be all pointing up, or all pointing down. So, the sign of $\mathbf{v} \cdot \hat{\boldsymbol{\jmath}}$ depends only on x, not on y (although the *magnitude* of $\mathbf{v} \cdot \hat{\boldsymbol{\jmath}}$ depends on both). Roughly, the vectors are pointing

- Down when $x < -2$;
- horizontally when $x = -2$ (remember the vector is positioned with the *base* of $\mathbf{v}(x,y)$ at (x, y);
- up when $-2 < x < 2$;
- horizontally when $x = 2$;
- up when $2 < x$.

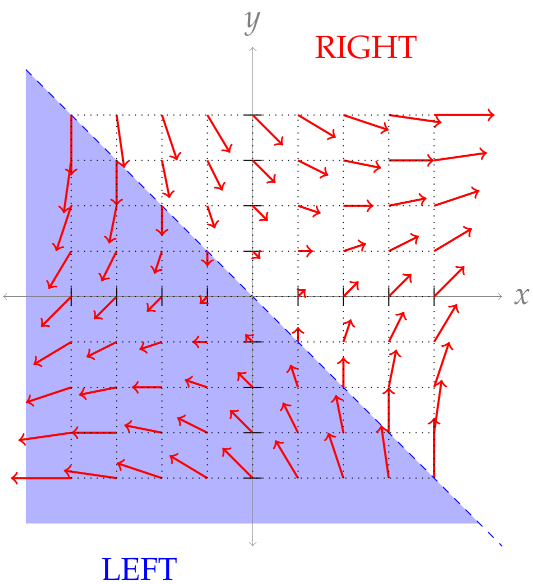
Since we're assuming there's nothing surprising happening between the samples pictured, at least for (x, y) shown in the sketch,

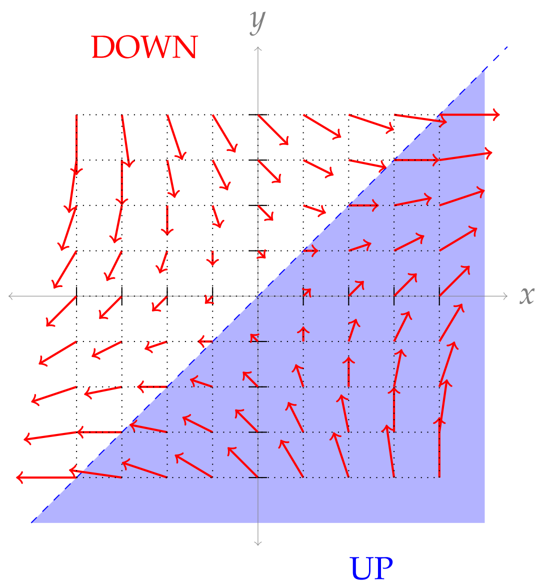
$$\mathbf{v}(x,y) \cdot \hat{\boldsymbol{\jmath}} \begin{cases} > 0 & \text{when } \boxed{-2 < x < 2} \\ = 0 & \text{when } \boxed{x \in \{-2, 2\}} \\ < 0 & \text{when } \boxed{x < -2 \text{ or } x > 2} \end{cases}$$

S-2: To start out, we find the places where $\mathbf{v}(x,y)\cdot\hat{\imath}=0$ (vertical vectors) or $\mathbf{v}(x,y)\cdot\hat{\jmath}=0$ (horizontal vectors). Remember the vector $\mathbf{v}(x,y)$ has its *tail* at (x,y).

We see the vertical vectors (those with $\mathbf{v}(x,y)\cdot\hat{\imath}=0$) occur at every point along the line $y=-x$, while horizontal vectors (those with $\mathbf{v}(x,y)\cdot\hat{\jmath}=0$) occur at every point along the line $y=x$.

Indeed, below the line $y=-x$, vectors point to the left, while above the line $y=-x$ they point to the right. Similarly, vectors point down when they're above the line $y=x$, and the point up when they're below the line $y=x$.





So, at least for (x,y) shown in the sketch,

$$\mathbf{v}(x,y)\cdot\hat{\imath}\begin{cases}>0 & \text{when } \boxed{y>-x}\\ =0 & \text{when } \boxed{y=-x}\\ <0 & \text{when } \boxed{y<-x}\end{cases}\quad\text{and}\quad \mathbf{v}(x,y)\cdot\hat{\jmath}\begin{cases}>0 & \text{when } \boxed{y<x}\\ =0 & \text{when } \boxed{y=x}\\ <0 & \text{when } \boxed{y>x}\end{cases}$$

S-3:

Since all conveyors point towards the origin, the direction of motion of an object at location (x,y) is $\frac{(-x,-y)}{\sqrt{x^2+y^2}}$. Its magnitude is y, so $\mathbf{v}(x,y)=\frac{-y}{\sqrt{x^2+y^2}}(x,y)$.

S-4: The arrows near the point A are pointing to the right, indicating that $P>0$, and upward, indicating that $Q>0$. Moving from left to right near A, the vertical component of the arrows is decreasing, indicating that $\frac{\partial Q}{\partial x}<0$. Moving vertically upwards near A, the vertical component of the arrows is increasing, indicating that $\frac{\partial Q}{\partial y}>0$.

S-5: (a) At time 0 the velocity of the twig is $\mathbf{v}(1,1) = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$. So at time $t = 0.1$, the position of the twig is approximately

$$(1,1) + 0.01(1,1) = (1.01\,,\,1.01)$$

(b) At time 0 the velocity of the twig is $\mathbf{v}(0,0) = \mathbf{0}$. So at time $t = 0.1$, the position of the twig is

$$(0,0) + 0.1(0,0) = (0\,,\,0)$$

(c) At time 0 the velocity of the twig is $\mathbf{v}(0,0) = \mathbf{0}$. So it is stationary and its velocity remains zero for all time. The position of the twig at time 10, and in fact at all times, is $(0\,,\,0)$.

S-6: The velocity of the fluid at all points of the y-axis is $-\hat{\boldsymbol{\jmath}}$. So the twig will remain on the y-axis and will consequently have velocity $-\hat{\boldsymbol{\jmath}}$ for all time. The position of the twig at time 10 will be

$$(0,0) + 10(0,-1) = (0\,,\,-10)$$

S-7:

Since all conveyors point towards the origin, the direction of motion of an object at location (x,y) is $\frac{(-x,-y)}{\sqrt{x^2+y^2}}$. Its magnitude is y, so $\mathbf{v}(x,y) = \frac{-y}{\sqrt{x^2+y^2}}(x,y)$.

S-8: Set your face to be at the origin of our coordinate system, $(0,0,0)$. A bee at position (x,y,z) is a distance of $\sqrt{x^2+y^2+z^2}$ from your face, heading in the direction $(-x,-y,-z)$. So, the unit vector indicating the direction of one friendly bee is $\frac{-1}{\sqrt{x^2+y^2+z^2}}(x,y,z)$. Now all we need to find is the length of this vector, i.e. the speed of the friendly bee.

The speed of the friendly bee is inversely proportional to $\sqrt{x^2+y^2+z^2}$, its distance from your face. (Bees that are farther away are buzzing towards you more excitedly.) So, speed is given by $\frac{\alpha}{\sqrt{x^2+y^2+z^2}}$ for some constant α.

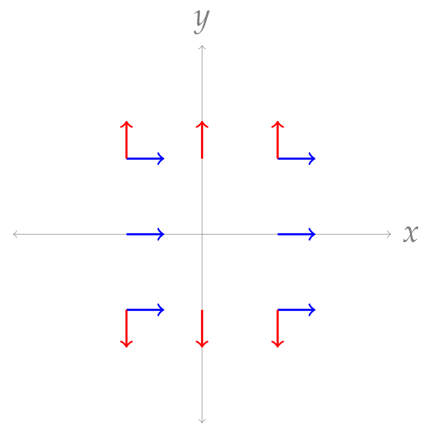
The bee velocity has the direction of the unit vector $\frac{-1}{\sqrt{x^2+y^2+z^2}}(x,y,z)$ with length $\frac{\alpha}{\sqrt{x^2+y^2+z^2}}$ for some positive constant α. That is,

$$\mathbf{v}(x,y,z) = -\frac{\alpha}{x^2+y^2+z^2}(x,y,z)$$

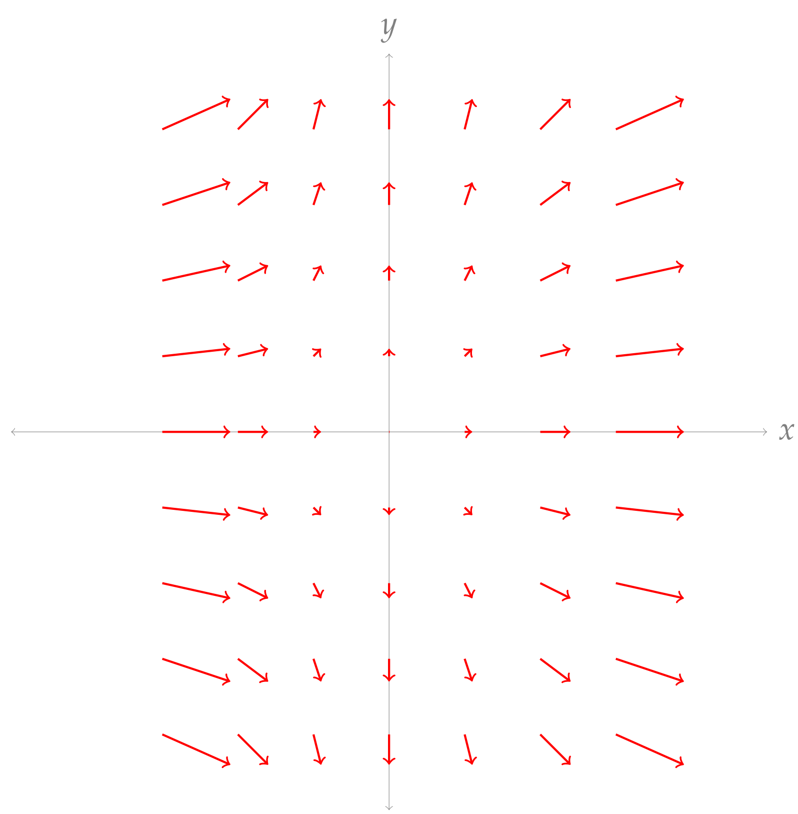
S-9: Beginning as in the text, we note

$$\mathbf{v}(x,y)\cdot\hat{\boldsymbol{\imath}} = x^2\begin{cases} >0 & x \ne 0 \\ =0 & x = 0\end{cases} \qquad \text{and} \qquad \mathbf{v}(x,y)\cdot\hat{\boldsymbol{\jmath}} = y\begin{cases} >0 & y>0 \\ =0 & y=0 \\ <0 & y<0\end{cases}.$$

That leads to the following picture:



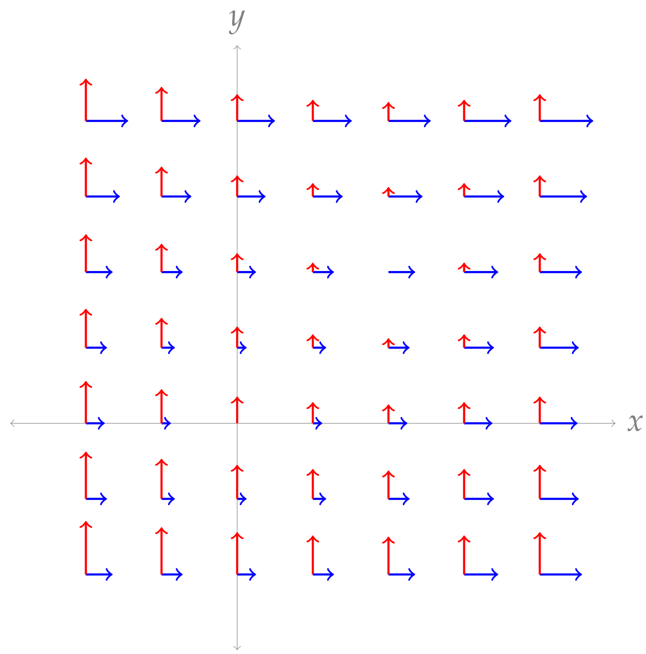
This gives us a general idea to start with. Refining, we notice that when $x^2 > |y|$, then the vector $\mathbf{v}(x, y)$ will be more horizontal than vertical. As we move away from the y-axis in a horizontal line, the difference between x^2 and $|y|$ grows, so the vectors get more and more horizontal. However, for a fixed value of x, vectors farther from the axis will be more vertical than vectors closer to it.



S-10:

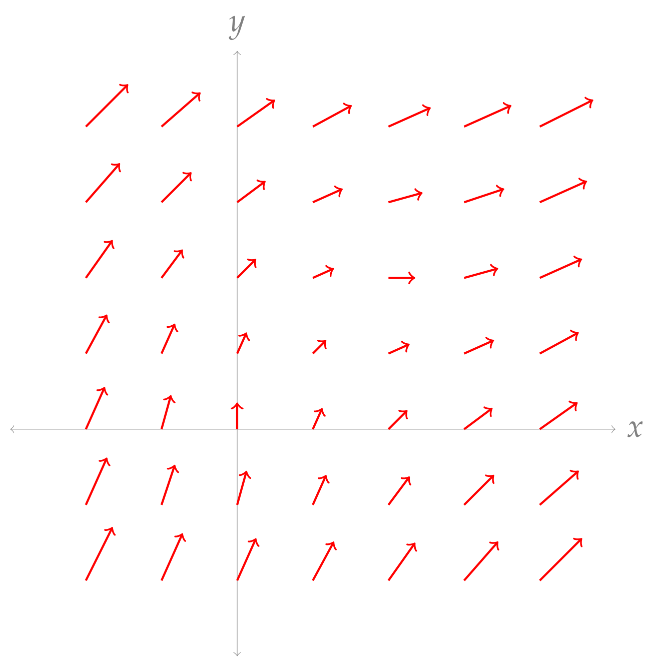
Although ultimately we'll sketch only unit-length vectors, we can still find the direction of $\mathbf{v}(x, y)$ by finding its x- and y components.

Note $\mathbf{v}(x, y) \cdot \hat{\boldsymbol{\imath}}$ is the distance from (x, y) to the origin, while $\mathbf{v}(x, y) \cdot \hat{\boldsymbol{\jmath}}$ is the distance from (x, y) to the point $(1, 1)$. Both these numbers are always nonnegative. This leads to the following sketch:

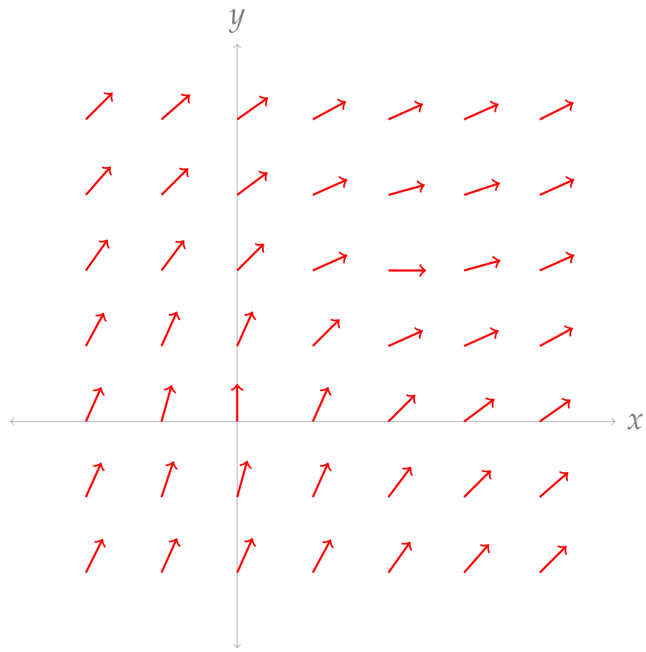


When (x, y) is far from the origin, its distance from $(0, 0)$ is almost the same as its distance from $(1, 0)$. So, we expect $\mathbf{v}(x, y)$ to be approximately a scalar multiple of $(1, 1)$.

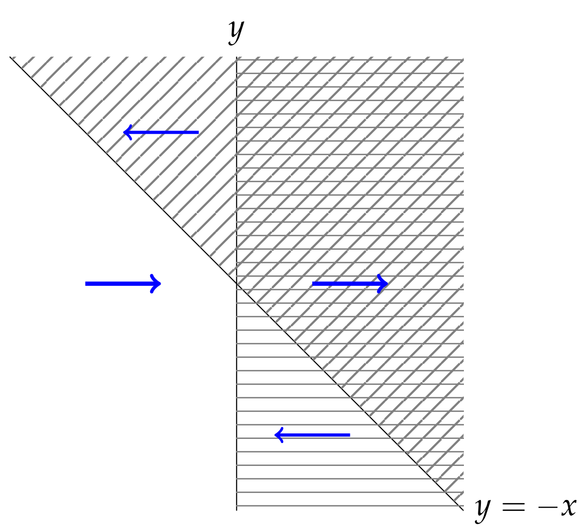
At $(0, 0)$, $v(0, 0) \cdot \hat{\boldsymbol{\imath}} = 0$, so our vector is horizontal; similarly, $v(1, 1) \cdot \hat{\boldsymbol{\jmath}} = 0$ so this vector is horizontal. Vectors very near to $(0, 0)$ are nearly horizontal, while vectors near to $(1, 1)$ are nearly vertical.



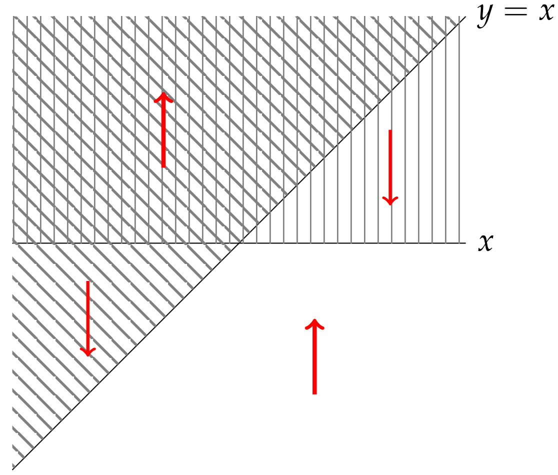
For the direction field, we normalize our vectors to have unit length.



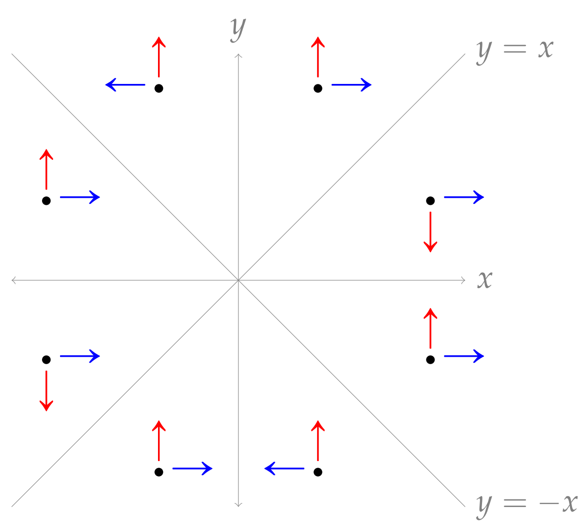
S-11: The sign of $\mathbf{v}(x,y) \cdot \hat{\boldsymbol{\imath}} = x(x+y)$ depends on the signs of x and $x+y$. When they have the same signs, $\mathbf{v}(x,y) \cdot \hat{\boldsymbol{\imath}}$ is positive, so $\mathbf{v}(x,y)$ points to the right; when they have different signs, $\mathbf{v}(x,y)$ points to the left.



Similarly, the sign of $\mathbf{v}(x,y) \cdot \hat{\boldsymbol{\jmath}} = y(y-x)$ depends on the signs of y and $y-x$.

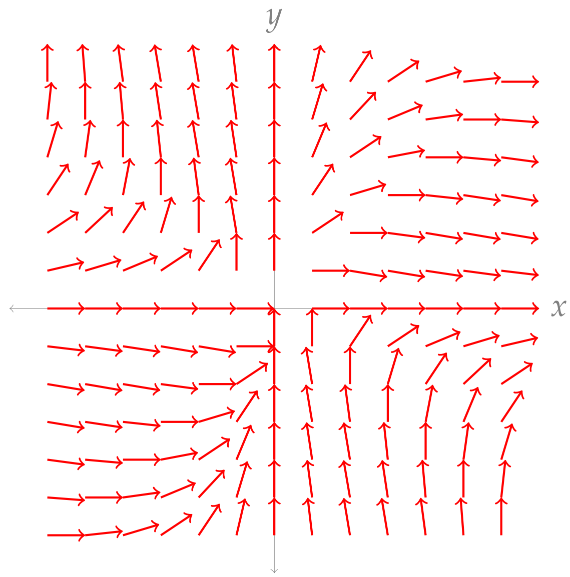


All together:



Refining, we notice that as we move straight up or down, $|\mathbf{v}(x, y) \cdot \hat{\boldsymbol{\imath}}|$ has its minimum along the lines $y = -x$ and $x = 0$. So, the vectors become more strongly vertical as we approach $y = -x$ and $x = 0$ from above or below.

Similarly, $|\mathbf{v}(x, y) \cdot \hat{\boldsymbol{\jmath}}|$ has its minima along the lines $y = x$ and $y = 0$, so the vectors become more strongly horizontal as we approach $y = x$ horizontally.



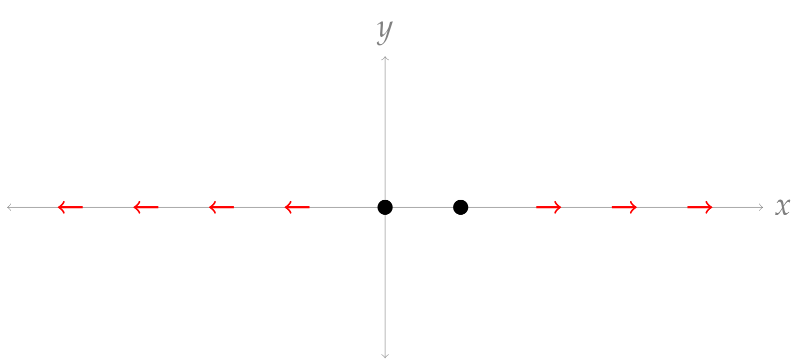
S-12:

The field $\mathbf{v}(x, y)$ is the sum, scaled by $1/3$, of the unit vector pointing away from the origin and the unit vector pointing away from $(1, 0)$. This tells us about a few regions:

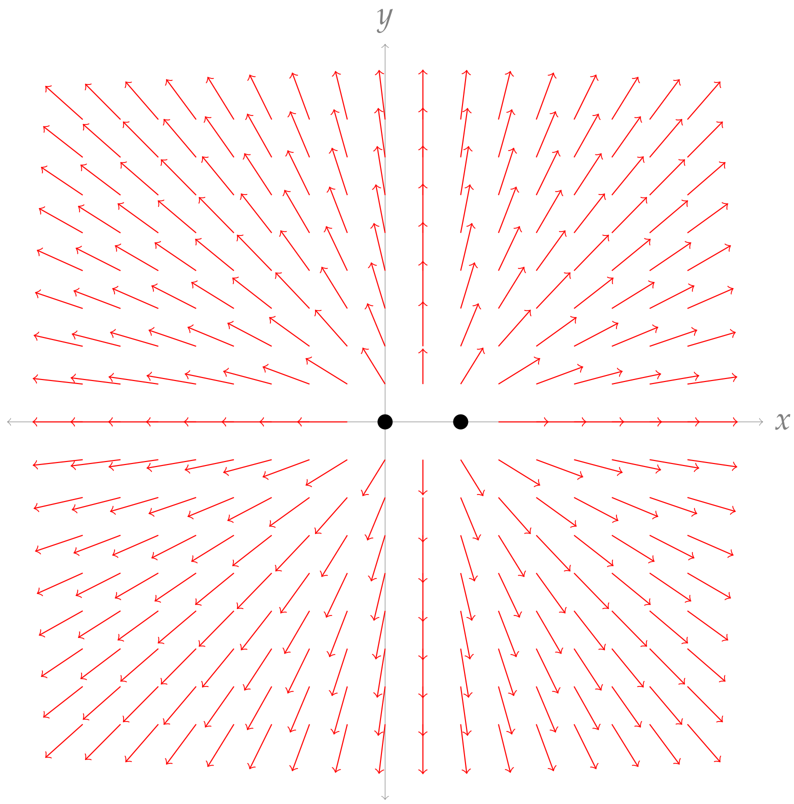
- Along the x axis between $(0, 0)$ and $(1, 0)$, the vectors away from these points are pointing in opposite directions (and have the same length), so they cancel each other out. That is, $v(x, 0) = 0$ for all $x \in (0, 1)$.

- $v(0, 0)$ and $v(1, 0)$ are not defined.

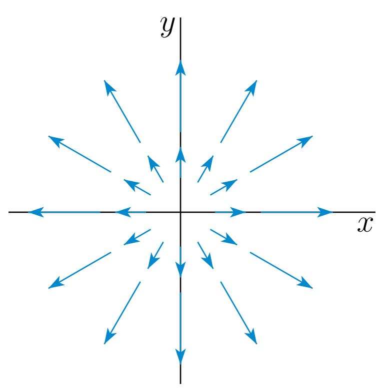
- Along the x-axis outside of $[0, 1]$, the vector pointing away from the point $(0, 0)$ is the same as the vector pointing away from the point $(1, 0)$. So, $v(x, 0) = (-2/3, 0)$ for $x < 0$ and $v(x, 0) = (2/3, 0)$ for $x > 1$.



- As the distance from (x, y) to the origin grows, the vector pointing away from $(0, 0)$ looks more and more like the vector pointing away from $(1, 0)$. So, our vectors far away from the origin look like vectors of length about $2/3$, pointing away from the origin.



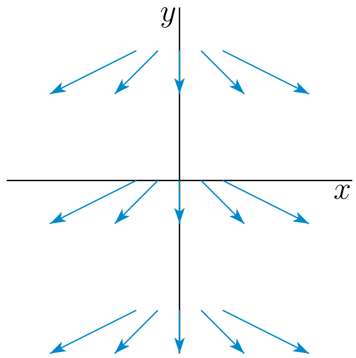
S-13: (a) The vector field $\mathbf{v}(x,y) = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}}$ is the same as the radius vector. It points radially outward and has length growing linearly with the distance from the origin.

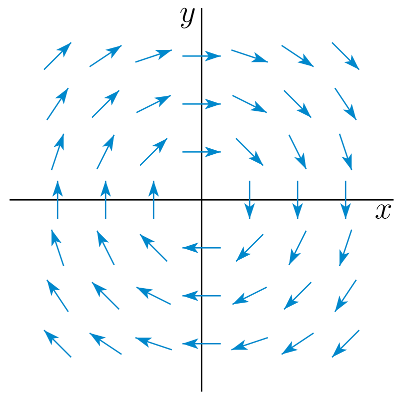


(b) The vertical component of $\mathbf{v}(x,y) = 2x\,\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}$ is always -1. Its horizontal component is $2x$, so that

- $\mathbf{v}(x,y)$ is rightward pointing when $x > 0$ and leftward pointing when $x < 0$, and
- the magnitude of the horizontal component grows linearly with the distance from the y-axis.

It is sketched in the figure on the left below.





(c) For every (x, y) the vector $\mathbf{v}(x, y) = \frac{y\,\hat{\imath} - x\,\hat{\jmath}}{\sqrt{x^2+y^2}}$

- is of length 1 and
- is perpendicular to the radius vector $x\,\hat{\imath} + y\,\hat{\jmath}$.
- $\mathbf{v}(x, y)$ is rightward pointing when $y > 0$ and leftward pointing when $y < 0$, and
- $\mathbf{v}(x, y)$ is downward pointing when $x > 0$ and upward pointing when $x < 0$.

It is sketched in the figure on the right above.

S-14: A particle of unit mass at position (x, y) has distance $D_1 = \sqrt{x^2 + y^2}$ from the 5kg mass, so that mass exerts a force of magnitude $\frac{G(5)}{x^2+y^2}$ on the particle. This force has direction $(-x, -y)$. So, the force exerted by the 5kg mass is $\mathbf{f}_1(x, y) = \frac{-5G}{(x^2+y^2)^{3/2}}(x, y)$.

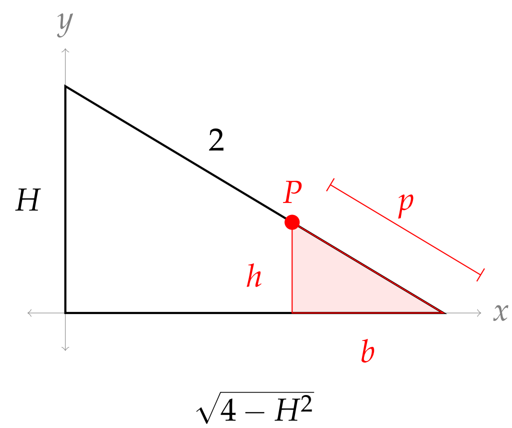
Similarly, the 3 kg mass at $(2, 3)$ exerts a force of $\mathbf{f}_2(x, y) = \frac{3G}{((x-2)^2+(y-3)^2)^{3/2}}(2 - x, 3 - y)$; and the 7 kg mass at $(4, 0)$ exerts a force of $\mathbf{f}_3(x, y) = \frac{7G}{((x-4)^2+y^2)^{3/2}}(4 - x, -y)$.

The net force on a unit mass is therefore

$$\begin{aligned}
\mathbf{f}(x, y) &= \mathbf{f}_1(x, y) + \mathbf{f}_2(x, y) + \mathbf{f}_3(x, y) \\
&= \frac{-5G(x, y)}{(x^2 + y^2)^{3/2}} + \frac{3G(2 - x, 3 - y)}{((x - 2)^2 + (y - 3)^2)^{3/2}} + \frac{7G(4 - x, -y)}{((x - 4)^2 + y^2)^{3/2}}
\end{aligned}$$

S-15:

a. Consider a point P on the pole that is a distance p away from the bottom end. Use this point to make a smaller right triangle, as in the picture below.



Using similar triangles:

$$h = \frac{p}{2}H \qquad\qquad b = \frac{p}{2}\sqrt{4-H^2}$$

If P is at position (x, y), then:

$$y = h = \frac{p}{2}H \qquad\qquad x = \sqrt{4-H^2} - b = \left(1-\frac{p}{2}\right)\sqrt{4-H^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{p}{2}\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{p}{4} \qquad\qquad \frac{\mathrm{d}x}{\mathrm{d}t} = \left(1-\frac{p}{2}\right)\frac{-H}{\sqrt{4-H^2}}\frac{\mathrm{d}H}{\mathrm{d}t} = \left(1-\frac{p}{2}\right)\frac{H}{2\sqrt{4-H^2}}$$

When $H = 1$:

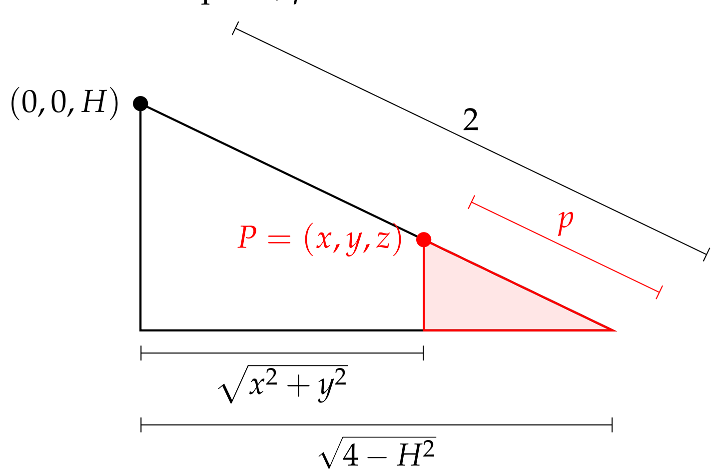
$$\left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{H=1} = -\frac{p}{4} \qquad\qquad \left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{H=1} = \left(1-\frac{p}{2}\right)\frac{1}{2\sqrt{3}}$$

Therefore,

$$\mathbf{v}(p) = \left.\left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}\right)\right|_{H=1} = \left(\left(1-\frac{p}{2}\right)\frac{1}{2\sqrt{3}}, \ -\frac{p}{4}\right)$$

For our model, we set the domain of this function to be $[0, 2]$.

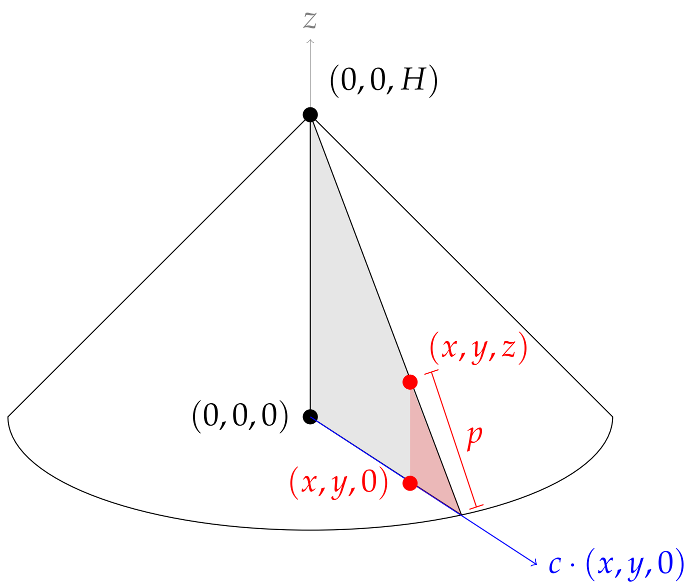
b. Let's start by seeing what we can salvage from our work on part a. As in part a., consider a point P on one of the poles, p metres from the bottom end.



Let P have position (x, y, z). Noting that $\frac{\mathrm{d}H}{\mathrm{d}t}$ is now positive, not negative, if we stick to this two-dimensional slice,

$$\mathbf{V}(p) = \left(\left(1 - \frac{p}{2}\right) \frac{-1}{2\sqrt{3}} \, , \, \frac{p}{4} \right)$$

where the second coordinate is z and the first coordinate refers to the (horizontal) line in the direction of the vector $(x, y, 0)$.



So, we know $\left.\frac{\mathrm{d}z}{\mathrm{d}t}\right|_{H=1} = \frac{p}{4}$, and we know $\left.\left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}\right)\right|_{H=1} = (x, y)c$ for some negative constant c with $|(x, y)c| = \left(1 - \frac{p}{2}\right) \frac{1}{2\sqrt{3}}$. Since we have the direction and the magnitude of the vector, we can find the vector:

$$\left.\left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}\right)\right|_{H=1} = (x, y)c = -\frac{\left(1 - \frac{p}{2}\right)}{2\sqrt{3}\sqrt{x^2 + y^2}}(x, y)$$

We want our equation to be in terms of x, y, and z, so we need to get rid of p. Using similar triangles, $\frac{p}{2} = \frac{\sqrt{4-H^2} - \sqrt{x^2+y^2}}{\sqrt{4-H^2}}$. When $H = 1$, then $1 - \frac{p}{2} = \frac{\sqrt{x^2+y^2}}{\sqrt{3}}$. So:

$$\left.\left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}\right)\right|_{H=1} = -\frac{1}{6}(x, y)$$

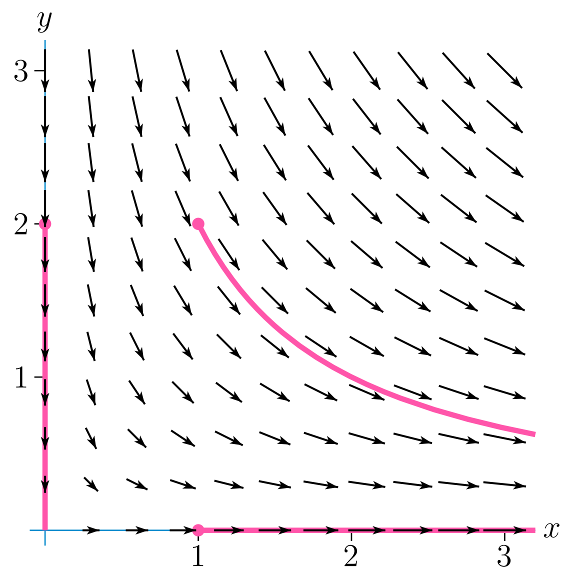
Finally:

$$\mathbf{V}(x, y, z) = \left.\left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right)\right|_{H=1} = \left(-\frac{1}{6}x \, , \, -\frac{1}{6}y \, , \, \frac{1}{2}z\right)$$

Not all values of (x, y, z) are on the frame. But, for those values of (x, y, z) that *are* on the frame, this equation holds.

Solutions to Exercises 2.2 — Jump to TABLE OF CONTENTS

S-1: (a) At every point of the positive y-axis, the velocity vector $\mathbf{v}(0, y)$ points straight down. So a rubber ducky placed in the water at $(0, 2)$ just floats straight down the positive y-axis towards the origin.



(b) At every point of the positive x-axis, the velocity vector $\mathbf{v}(x, 0)$ points straight to the right. So a rubber ducky placed in the water at $(1, 0)$ just floats rightward along the positive x-axis.

(c) At every point of the first quadrant away from the axes, the velocity vector $\mathbf{v}(x, y)$ points downwards and towards the right. So a rubber ducky placed in the water at $(1, 2)$ always floats down and to the right. The closer the ducky gets to the x–axis the more rightward its motion becomes.

S-2: The derivatives

$$\begin{aligned} x'(t) &= -e^{-t}\cos t - e^{-t}\sin t = -x(t) - y(t) \\ y'(t) &= -e^{-t}\sin t + e^{-t}\cos t = -y(t) + x(t) \end{aligned}$$

So $\big(x(t), y(t)\big)$ is a solution of the system of differential equations

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= v_1(x, y) = -x - y \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= v_2(x, y) = \quad x - y \end{aligned}$$

So the vector field is $\mathbf{v}(x, y) = \big(v_1(x, y)\,,\, v_2(x, y)\big) = (-x - y\,,\, x - y)$.

S-3: (a) The field lines of $\mathbf{F}(x,y) = \nabla f = y\,\hat{\boldsymbol{\imath}} + x\,\hat{\boldsymbol{\jmath}}$ obey

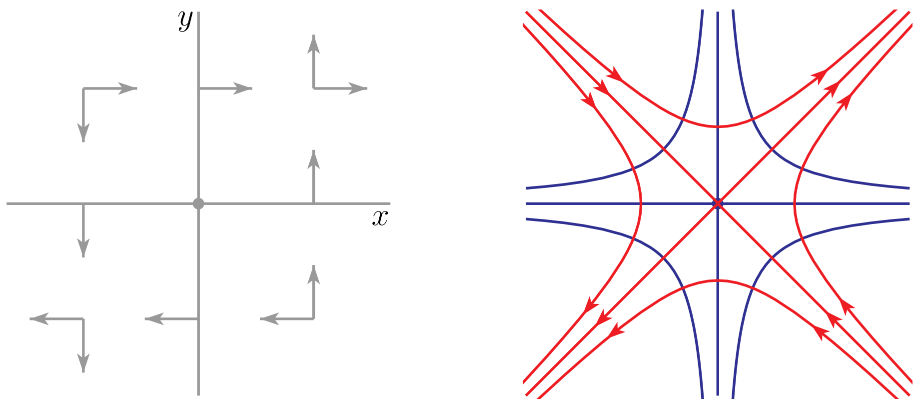
$$\frac{\mathrm{d}x}{y} = \frac{\mathrm{d}y}{x} \iff x\,\mathrm{d}x = y\,\mathrm{d}y \iff \frac{x^2}{2} = \frac{y^2}{2} + C$$

for any constant C.

(b) The sign data

$$\hat{\boldsymbol{\imath}} \cdot \mathbf{F}(x,y) = y \begin{Bmatrix} > 0 & \text{if } y > 0 \\ = 0 & \text{if } y = 0 \\ < 0 & \text{if } y < 0 \end{Bmatrix} \qquad \hat{\boldsymbol{\jmath}} \cdot \mathbf{F}(x,y) = x \begin{Bmatrix} > 0 & \text{if } x > 0 \\ = 0 & \text{if } x = 0 \\ < 0 & \text{if } x < 0 \end{Bmatrix}$$

is visually displayed in the figure on the left below. The arrows in the figure on the left gives us the direction of motion along the field lines $\frac{x^2}{2} = \frac{y^2}{2} + C$ (in red) in the figure on the right below. Some equipotential curves $xy = C$ are also sketched (in blue) in the figure on the right below.



S-4: The field lines obey

$$\frac{\mathrm{d}x}{2y} = \frac{\mathrm{d}y}{x/y^2} = \frac{\mathrm{d}z}{e^y} \qquad \text{if } x, y \neq 0$$

In particular

$$\frac{\mathrm{d}x}{2y} = \frac{y^2\,\mathrm{d}y}{x} \implies x\,\mathrm{d}x = 2y^3\,\mathrm{d}y \implies \frac{1}{2}x^2 = \frac{1}{2}y^4 + C$$

Since $y = 1$ when $x = 1$, $C = 0$. So $x = y^2$ and

$$\frac{\mathrm{d}y}{x/y^2} = \frac{\mathrm{d}z}{e^y} \implies e^y dy = dz \implies z = e^y + D$$

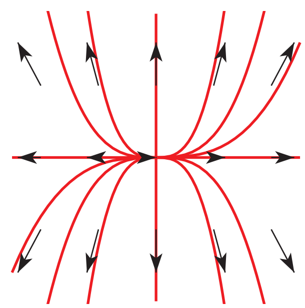
Since $z = e$ when $y = 1$, $D = 0$. So the field line is

$$x = y^2 \qquad z = e^y$$

S-5: The field lines obey

$$\begin{aligned}
\frac{\mathrm{d}x}{x} &= \frac{\mathrm{d}y}{3y} \qquad \text{if } x, y \neq 0 \\
&\implies 3\ln|x| = \ln|y| + C \\
&\implies |x|^3 = e^C|y| \\
&\implies y = \pm e^{-C}x^3 \\
&\implies y = C'x^3
\end{aligned}$$

with C' a nonzero constant. $x = 0$ and $y = 0$ are also field lines, since on the y-axis $\mathbf{F} \parallel \hat{\boldsymbol{\jmath}}$ and on the x-axis $\mathbf{F} \parallel \hat{\boldsymbol{\imath}}$.



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S-1: False, in general.

In the context of Equation 1.7.1, the only forces acting on the particle are gravity, $-mg\hat{\boldsymbol{\jmath}}$, and the normal force, $W\hat{\mathbf{N}}$.

We make no such constraints on the force in Example 2.3.3. Certainly $\mathbf{F}$ *could* arise from gravity and the normal force of a track, but there's nothing saying it has to. For example, suppose φ is an equation that does not depend on m and/or g. Alternately, suppose the y-coordinate of our three-dimensional system is not "up."

S-2: Remember that the screening test can only rule out conservativity — it can never, by itself, guarantee conservativity. So, A is *never* the case.

a.

$$\begin{aligned}
\mathbf{F} &= x\hat{\boldsymbol{\imath}} + z\hat{\boldsymbol{\jmath}} + y\hat{\mathbf{k}} \\
\boldsymbol{\nabla} \times \mathbf{F} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\boldsymbol{\imath}} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\mathbf{k}} \\
&= (1-1)\hat{\boldsymbol{\imath}} + (0-0)\hat{\boldsymbol{\jmath}} + (0-0)\hat{\mathbf{k}} = \mathbf{0}
\end{aligned}$$

This field passes the screening test. That means the screening test doesn't rule out the possibility of $\mathbf{F}$ being conservative. So, we have option C.

b.

$$\begin{aligned}
\mathbf{F} &= y^2z\hat{\boldsymbol{\imath}} + x^2z\hat{\boldsymbol{\jmath}} + x^2y\hat{\mathbf{k}} \\
\boldsymbol{\nabla} \times \mathbf{F} &= \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\hat{\boldsymbol{\imath}} + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\hat{\boldsymbol{\jmath}} + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\hat{\mathbf{k}} \\
&= (x^2 - x^2)\hat{\boldsymbol{\imath}} + (y^2 - 2xy)\hat{\boldsymbol{\jmath}} + (2xz - 2yz)\hat{\mathbf{k}} \neq \mathbf{0}
\end{aligned}$$

So, $\mathbf{F}$ fails the screening test — it's not conservative. That's option B.

c.

$$\begin{aligned}
\mathbf{F} &= (ye^{xy} + 1)\hat{\boldsymbol{\imath}} + (xe^{xy} + z)\hat{\boldsymbol{\jmath}} + \left(\frac{1}{z} + y\right)\hat{\mathbf{k}} \\
\boldsymbol{\nabla} \times \mathbf{F} &= \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\hat{\boldsymbol{\imath}} + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\hat{\boldsymbol{\jmath}} + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\hat{\mathbf{k}} \\
&= (1 - 1)\hat{\boldsymbol{\imath}} + (0 - 0)\hat{\boldsymbol{\jmath}} + (e^{xy}(xy + 1) - e^{xy}(xy + 1))\hat{\mathbf{k}} = \mathbf{0}
\end{aligned}$$

$\mathbf{F}$ passes the screening test, so it may or may not be conservative. That is Option C.

d.

$$\begin{aligned}
\mathbf{F} &= y\cos(xy)\hat{\boldsymbol{\imath}} + x\sin(xy)\hat{\boldsymbol{\jmath}} \\
\frac{\partial F_2}{\partial x} &= xy\cos(xy) + \sin(xy) \\
\frac{\partial F_1}{\partial y} &= -xy\sin(xy) + \cos(xy) \\
\frac{\partial F_2}{\partial x} &\neq \frac{\partial F_1}{\partial y}
\end{aligned}$$

$\mathbf{F}$ fails the screening test, so it is not conservative. That is Option B.

S-3: Let φ be a potential for $\mathbf{F}$. Define $\phi = \varphi + ax + by + cz$. Then $\boldsymbol{\nabla}\phi = \boldsymbol{\nabla}\varphi + (a, b, c) = \mathbf{F} + (a, b, c)$. So, $\mathbf{F} + (a, b, c)$ is also conservative.

S-4:

a. If $\mathbf{F} + \mathbf{G}$ is *conservative* for any particular $\mathbf{F}$ and $\mathbf{G}$, then by definition, there exists a potential φ with $\mathbf{F} + \mathbf{G} = \boldsymbol{\nabla}\varphi$.

   Since $\mathbf{F}$ is conservative, there also exists a potential ψ with $\mathbf{F} = \boldsymbol{\nabla}\psi$.

   But now $\mathbf{G} = (\mathbf{F} + \mathbf{G}) - \mathbf{F} = \boldsymbol{\nabla}\varphi - \boldsymbol{\nabla}\psi = \boldsymbol{\nabla}(\varphi - \psi)$. That means the function $(\varphi - \psi)$ is a potential for G. However, this is impossible: since $\mathbf{G}$ is non-conservative, no function with this property exists.

   So it is not possible that $\mathbf{F} + \mathbf{G}$ is conservative. It must be non-conservative.

b. Counterexample: if $\mathbf{F} = -\mathbf{G}$, then $\mathbf{F} + \mathbf{G} = \mathbf{0} = \boldsymbol{\nabla}c$ for any constant c.

c. Since both fields are conservative, they both have potentials, say $\mathbf{F} = \boldsymbol{\nabla}\varphi$ and $\mathbf{G} = \boldsymbol{\nabla}\psi$. Then $\mathbf{F} + \mathbf{G} = \boldsymbol{\nabla}\varphi + \boldsymbol{\nabla}\psi = \boldsymbol{\nabla}(\varphi + \psi)$. That is, $(\varphi + \psi)$ is a potential for $\mathbf{F} + \mathbf{G}$, so $\mathbf{F} + \mathbf{G}$ is conservative.

S-5: Set $\varphi(x,y) = \arctan\frac{y}{x}$ (using the standard arctan that takes values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$). Note that $\varphi(x,y)$ is well-defined, with all partial derivatives continuous, on D since $x > 1$ there. Then

$$\frac{\partial\varphi}{\partial x}(x,y) = \frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2} = -\frac{y}{x^2+y^2}$$
$$\frac{\partial\varphi}{\partial y}(x,y) = \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2} = \frac{x}{x^2+y^2}$$

so that $\mathbf{F} = \nabla\varphi$.

S-6: If φ is a potential for $\mathbf{F}$, then:

- $\frac{\partial\varphi}{\partial x} = x+y$, so $\varphi = \frac{1}{2}x^2 + xy + \psi_1(y)$
- $\frac{\partial\varphi}{\partial y} = x-y$, so $\varphi = xy - \frac{1}{2}y^2 + \psi_2(x)$

So, for instance, $\varphi = \frac{1}{2}x^2 + xy - \frac{1}{2}y^2$ is a potential for $\mathbf{F}$.

S-7: If φ is a potential for $\mathbf{F}$, then:

- $\frac{\partial\varphi}{\partial x} = \frac{1}{x} - \frac{1}{y}$, so $\varphi = \ln|x| - \frac{x}{y} + \psi_1(y)$
- $\frac{\partial\varphi}{\partial y} = \frac{x}{y^2}$, so $\varphi = -\frac{x}{y} + \psi_2(x)$

So, for instance, $\varphi = \ln|x| - \frac{x}{y}$ is a potential for $\mathbf{F}$.

S-8: None exists: $\frac{\partial F_2}{\partial z} = \frac{1}{3}x^3$, while $\frac{\partial F_3}{\partial y} = \frac{1}{3}x^3 + 1$, so $\mathbf{F}$ fails the screening test, Theorem 2.3.9.

S-9: If φ is a potential for $\mathbf{F}$, then:

- $\frac{\partial\varphi}{\partial x} = \frac{x}{x^2+y^2+z^2}$, so $\varphi = \frac{1}{2}\ln(x^2+y^2+z^2) + \psi_1(y,z)$
- $\frac{\partial\varphi}{\partial y} = \frac{y}{x^2+y^2+z^2}$, so $\varphi = \frac{1}{2}\ln(x^2+y^2+z^2) + \psi_2(x,z)$
- $\frac{\partial\varphi}{\partial z} = \frac{z}{x^2+y^2+z^2}$, so $\varphi = \frac{1}{2}\ln(x^2+y^2+z^2) + \psi_2(x,y)$

So, for instance, $\varphi = \frac{1}{2}\ln(x^2+y^2+z^2)$ is a potential for $\mathbf{F}$.

S-10: (a) We shall show that $\mathbf{F}(x,y,z)$ is conservative by finding a potential for it.

$\varphi(x,y,z)$ is a potential for this $\mathbf{F}$ if and only if

$$\begin{aligned}
\frac{\partial \varphi}{\partial x}(x,y,z) &= x \\
\frac{\partial \varphi}{\partial y}(x,y,z) &= -2y \\
\frac{\partial \varphi}{\partial z}(x,y,z) &= 3z
\end{aligned}$$

Integrating the first of these equations gives

$$\varphi(x,y,z) = \frac{x^2}{2} + f(y,z)$$

Substituting this into the second equation gives

$$\frac{\partial f}{\partial y}(y,z) = -2y$$

which integrates to

$$f(y,z) = -y^2 + g(z)$$

Finally, substituting $\varphi(x,y,z) = \frac{x^2}{2} - y^2 + g(z)$ into the last equation gives

$$g'(z) = 3z$$

which integrates to

$$g(z) = \frac{3}{2}z^2 + C$$

with C being an arbitrary constant. So, $\mathbf{F}(x,y,z)$ is conservative and $\varphi(x,y,z) = \frac{1}{2}x^2 - y^2 + \frac{3}{2}z^2$ is one allowed potential.

(b) The field $\mathbf{F} = F_1\,\hat{\boldsymbol{\imath}} + F_2\,\hat{\boldsymbol{\jmath}}$ can be conservative only if it passes the screening test

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

In this case

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\Big(\frac{x}{x^2+y^2}\Big) = -\frac{2xy}{(x^2+y^2)}$$

is different from

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\Big(\frac{-y}{x^2+y^2}\Big) = \frac{2xy}{(x^2+y^2)}$$

for all (x,y) with x and y both nozero. So $\mathbf{F}$ is not conservative.

S-11: By Theorem 2.4.7 in the CLP-4 text, the field $\mathbf{F} = F_1\,\hat{\boldsymbol{\imath}} + F_2\,\hat{\boldsymbol{\jmath}} + F_3\,\hat{\mathbf{k}}$ is conservative only if it passes the screening test $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$. That is, if and only if

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

or,

$$\frac{\partial}{\partial y}\big(e^{(z^2)}\big) = \frac{\partial}{\partial x}\big(2Byz^3\big) \qquad \Longleftrightarrow \qquad 0 = 0$$

$$\frac{\partial}{\partial z}\big(e^{(z^2)}\big) = \frac{\partial}{\partial x}\big(Axze^{(z^2)} + 3By^2z^2\big) \qquad \Longleftrightarrow \qquad 2ze^{(z^2)} = Aze^{(z^2)}$$

$$\frac{\partial}{\partial z}\big(2Byz^3\big) = \frac{\partial}{\partial y}\big(Axze^{(z^2)} + 3By^2z^2\big) \qquad \Longleftrightarrow \qquad 6Bye^{(z^2)} = 6Bye^{(z^2)}$$

Hence only $A = 2$ works. We shall see in part (b) that any B works.

(b) When $A = 2$, and B is any real number.

$$\mathbf{F} = e^{(z^2)}\,\hat{\boldsymbol{\imath}} + 2Byz^3\,\hat{\boldsymbol{\jmath}} + \big(2xze^{(z^2)} + 3By^2z^2\big)\,\hat{\mathbf{k}}$$

$\varphi(x,y,z)$ is a potential for this $\mathbf{F}$ if and only if

$$\frac{\partial\varphi}{\partial x}(x,y,z) = e^{(z^2)}$$
$$\frac{\partial\varphi}{\partial y}(x,y,z) = 2Byz^3$$
$$\frac{\partial\varphi}{\partial z}(x,y,z) = 2xze^{(z^2)} + 3By^2z^2$$

Integrating the first of these equations gives

$$\varphi(x,y,z) = xe^{(z^2)} + f(y,z)$$

Substituting this into the second equation gives

$$\frac{\partial f}{\partial y}(y,z) = 2Byz^3$$

which integrates to

$$f(y,z) = By^2z^3 + g(z)$$

Finally, substituting $\varphi(x,y,z) = xe^{(z^2)} + By^2z^3 + g(z)$ into the last equation gives

$$2xze^{(z^2)} + 3By^2z^2 + g'(z) = 2xze^{(z^2)} + 3By^2z^2 \qquad \text{or} \quad g'(z) = 0$$

which integrates to

$$g(z) = C$$

with C being an arbitrary constant. So, for each real number B, $\varphi(x,y,z) = xe^{(z^2)} + By^2z^3$ is one allowed potential.

S-12: In each second $2\pi m$ cm^2 of fluid crosses each circle of radius r (and hence circumference $2\pi r$) centred on the origin. So the speed of flow at radius r is $\frac{m}{r}$. As the direction of flow is radially outward

$$\mathbf{v} = m\frac{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}}{x^2 + y^2}$$

$\varphi(x, y)$ is a potential for this **F** if and only if

$$\frac{\partial \varphi}{\partial x}(x, y) = m\frac{x}{x^2 + y^2}$$
$$\frac{\partial \varphi}{\partial y}(x, y) = m\frac{y}{x^2 + y^2}$$

Integrating the first of these equations gives

$$\varphi(x, y) = \tfrac{1}{2}m \ln(x^2 + y^2) + f(y)$$

Substituting this into the second equation gives

$$m\frac{y}{x^2 + y^2} + f'(y) = m\frac{y}{x^2 + y^2} \qquad \text{or} \quad f'(y) = 0$$

which integrates to

$$f(y) = C$$

with C an arbitrary constant. So one possible potential is

$$\varphi = \tfrac{1}{2}m \ln(x^2 + y^2)$$

S-13:

Following Example 2.3.3, the particle can never escape the region $\{(x, y, z) : \varphi(x, y, z) \geqslant -E\}$. So, we should find E, then figure out the region.

The kinetic energy of the particle is $\frac{1}{2}m|\mathbf{v}|^2$, so the total energy of the system (also the kinetic energy when the potential energy is 0) is $\frac{1}{2}(10)(2^2) = 20$ J.

Therefore, a region it can never escape is

$$\{\ (x, y, z) \mid \varphi(x, y, z) \geqslant -20\ \}$$

that is,

$$\{\ (x, y, z) \mid x^2 + y^2 + z^2 \leqslant 20\ \}$$

So, it can never escape the sphere centred at the origin with radius $\sqrt{20}$.

S-14: Example 2.3.3 tells us $\frac{1}{2}m|\mathbf{v}(t)|^2 - \varphi\big(x(t), y(t), z(t)\big) = E$ is a constant quantity, provided **F** is conservative with potential φ. So, it would be nice if **F** were conservative.

If $\mathbf{F} = \boldsymbol{\nabla}\varphi$, then

- $\frac{\partial \varphi}{\partial x} = 0$, so $\varphi = \psi_1(y, z)$
- $\frac{\partial \varphi}{\partial y} = 1$, so $\varphi = y + \psi_2(x, z)$
- $\frac{\partial \varphi}{\partial z} = 3z^{1/3}$, so $\varphi = \frac{9}{4}z^{4/3} + \psi_3(x, y)$

We can choose $\varphi(x,y,z) = y + \frac{9}{4}z^{4/3}$. So, $\frac{1}{2}m|\mathbf{v}(t)|^2 - \varphi\big(x(t), y(t), z(t)\big) = E$ is a constant quantity, as desired. Using the information that the particle has mass 1/2, and speed 1 when it is at the origin:

$$E = \frac{1}{2} \cdot \frac{1}{2}|1|^2 - \varphi\big(0,0,0\big) = \frac{1}{4}$$

When the particle is at $(1,1,1)$:

$$\begin{aligned}\frac{1}{4} &= \frac{1}{2} \cdot \frac{1}{2}|\mathbf{v}|^2 - \varphi(1,1,1) = \frac{|\mathbf{v}|^2}{4} - \left(1 + \frac{9}{4}\right) \\ |\mathbf{v}| &= \sqrt{14}\end{aligned}$$

So, at the point $(1,1,1)$, the particle has speed $\sqrt{14}$.

S-15:

We can start with the screening test, Theorem 2.3.9.

$$\begin{aligned}\nabla \times \mathbf{F} &= \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\hat{\imath} + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\hat{\jmath} + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\hat{\mathbf{k}} \\ &= \Big(g'(y)h'(z) - g'(y)h'(z)\Big)\hat{\imath} + \Big(0-0\Big)\hat{\jmath} + \Big(0-0\Big)\hat{\mathbf{k}} = \mathbf{0}\end{aligned}$$

So, it's possible that the field is conservative. Remember, this test alone isn't enough to tell us it's conservative. (Had the test come out differently, though, we'd be done.)

Suppose $\mathbf{F} = \nabla\varphi(x,y,z)$. Then:

- $\frac{\partial \varphi}{\partial x} = 2f(x)f'(x)$. By inspection, we see $\varphi = f^2(x) + \psi_1(y,z)$. (We could also find this by evaluating $\int 2f(x)f'(x)\mathrm{d}x$ with the substitution $u = f(x)$.)
- $\frac{\partial \varphi}{\partial y} = g'(y)h(z)$, so $\varphi = g(y)h(z) + \psi_2(x,z)$.
- $\frac{\partial \varphi}{\partial z} = g(y)h'(z)$, so $\varphi = g(y)h(z) + \psi_2(x,y)$.

All together, we can choose $\varphi(x,y,z) = f^2(x) + g(y)h(z)$.

S-16: Following Definition 2.3.8, The curl of a vector field is defined by

$$\nabla \times \mathbf{F} = \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\hat{\imath} + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\hat{\jmath} + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\hat{\mathbf{k}}$$

When $\mathbf{F} = \langle xy, xz, y^2 + z\rangle$,

$$\nabla \times \mathbf{F} = (2y - x)\hat{\imath} + (0-0)\hat{\jmath} + (z - x)\hat{\mathbf{k}}$$

When the curl is $0\hat{\imath} + 0\hat{\jmath} + 0\hat{\mathbf{k}}$, we have $x = 2y$ and $x = z$. That is, our points are of the form $(2c, c, 2c)$ for any constant c. So, the region in question is the line through the origin in the direction of the vector $(2,1,2)$.

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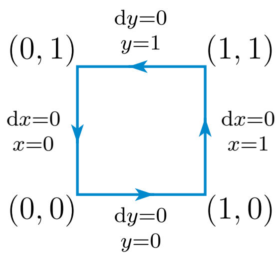
S-1: The square has four sides, each of which is a line segment.

- On the first side, $y = 0$ and $dy = 0$. That is, we may parametrize the first side by $\mathbf{r}(x) = x\,\hat{\imath}$ with $0 \leqslant x \leqslant 1$.
- On the second side, $x = 1$ and $dx = 0$. We may parametrize the second side by $\mathbf{r}(y) = \hat{\imath} + y\,\hat{\jmath}$ with $0 \leqslant y \leqslant 1$.
- On the third side, $y = 1$ and $dy = 0$. We may parametrize the third side by $\mathbf{r}(x) = x\,\hat{\imath} + \hat{\jmath}$ with x running from 1 to 0.
- On the final side, $x = 0$ and $dx = 0$. We may parametrize the fourth side by $\mathbf{r}(y) = y\,\hat{\jmath}$ with y running from 1 to 0.



So

$$\begin{aligned}\int_C x^2y^2\,dx + x^3y\,dy &= \int_0^1 x^2 \times 0^2\,dx + \int_0^1 1^3 \times y\,dy + \int_1^0 x^2 \times 1^2\,dx + \int_1^0 0^3 \times y\,dy \\ &= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}\end{aligned}$$

S-2: Every $\mathbf{F}$ in this problem is defined and has continuous first-order partial derivatives on all of $\mathbb{R}^2$ or $\mathbb{R}^3$. The characterization in Theorem 2.4.7 tells us that our fields will be conservative if and only if they pass the screening test, i.e. have curl 0.

a.

$$\begin{aligned}\mathbf{F} &= x\hat{\imath} + z\hat{\jmath} + y\hat{\mathbf{k}} \\ \nabla \times \mathbf{F} &= \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\hat{\imath} + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\hat{\jmath} + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\hat{\mathbf{k}} \\ &= (1-1)\hat{\imath} + (0-0)\hat{\jmath} + (0-0)\hat{\mathbf{k}} = \mathbf{0}\end{aligned}$$

This field passes the screening test. Since $\mathbf{F}$ is defined and has continuous first-order partial derivatives on all of $\mathbb{R}^3$, it is conservative. So, we have option A.

b.

$$\begin{aligned}\mathbf{F} &= y^2z\hat{\imath} + x^2z\hat{\jmath} + x^2y\hat{\mathbf{k}} \\ \nabla \times \mathbf{F} &= \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\hat{\imath} + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\hat{\jmath} + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\hat{\mathbf{k}} \\ &= (x^2 - x^2)\hat{\imath} + (y^2 - 2xy)\hat{\jmath} + (2xz - 2yz)\hat{\mathbf{k}} \neq \mathbf{0}\end{aligned}$$

So, $\mathbf{F}$ fails the screening test. So, it's not conservative. That's option B.

c.

$$\begin{aligned}\mathbf{F} &= (xe^{xy}+1)\hat{\imath} + (ye^{xy}+z)\hat{\jmath} + \left(\frac{1}{z}+y\right)\hat{\mathbf{k}} \\ \nabla\times\mathbf{F} &= \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\hat{\imath} + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\hat{\jmath} + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\hat{\mathbf{k}} \\ &= (1-1)\hat{\imath} + (0-0)\hat{\jmath} + (e^{xy}(xy+1) - xye^{xy}(xy+1))\hat{\mathbf{k}} = \mathbf{0}\end{aligned}$$

$\mathbf{F}$ passes the screening test. Since $\mathbf{F}$ is defined and has continuous first-order partial derivatives on all of $\mathbb{R}^3$, it is conservative. So, we have option A.

d.

$$\begin{aligned}\mathbf{F} &= y\cos(xy)\hat{\imath} + x\sin(xy)\hat{\jmath} \\ \frac{\partial F_2}{\partial x} &= xy\cos(xy) + \sin(xy) \\ \frac{\partial F_1}{\partial y} &= -xy\sin(xy) + \cos(xy) \\ \frac{\partial F_2}{\partial x} &\neq \frac{\partial F_1}{\partial y}\end{aligned}$$

$\mathbf{F}$ fails the screening test, so it is not conservative. That is Option B.

S-3: Since $\mathbf{F}$ is conservative, $\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = 0$ over any closed curve C. The given curve is closed, so the integral is simply zero.

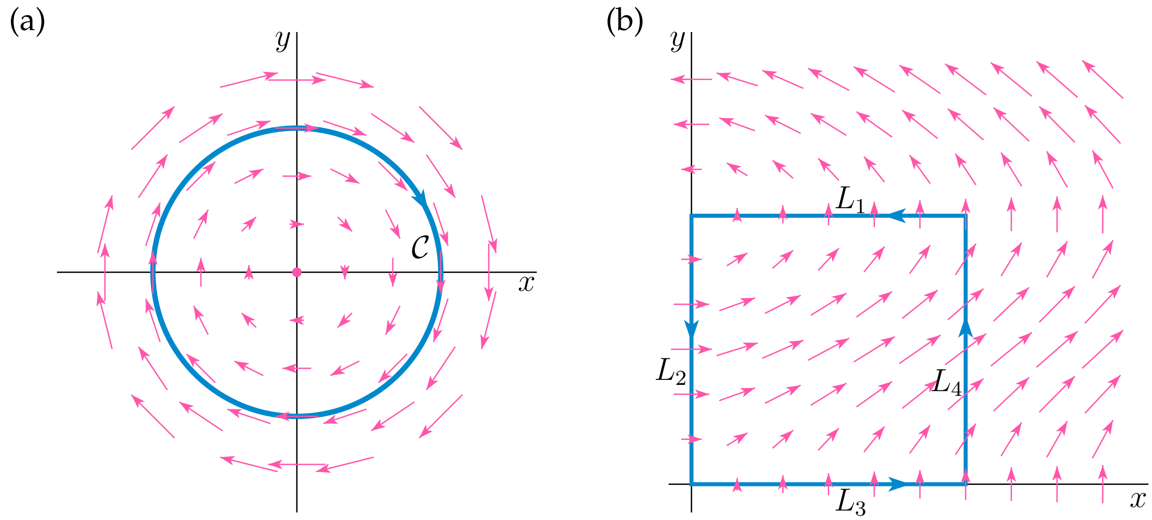
S-4: Since $\mathbf{F}$ is conservative, and A and B start and end at the same points, by path-independence $\int_B \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_A \mathbf{F}\cdot \mathrm{d}\mathbf{r} = 5$.

S-5: By Theorem 2.4.6, the condition that "$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = 0$ for all closed paths C" is equivalent to the condition that "$\mathbf{F}$ is conservative", which, since $\mathbf{F}$ is defined on all of $\mathbb{R}^3$, is equivalent to the condition that $\mathbf{F}$ pass the screening test

$$\mathbf{0} = \nabla\times\mathbf{F} = \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x\sin y & ae^x\cos y + bz & cx\end{bmatrix} = -b\,\hat{\imath} - c\,\hat{\jmath} + \big(ae^x\cos y - e^x\cos y\big)\,\hat{\mathbf{k}}$$

which is the case if and only if $b = c = 0$ and $a = 1$.

S-6: (a) Consider the circle $\mathcal{C}$ in the figure (a) on the left below, oriented *clockwise*. The vector field $\mathbf{F}$ is in the same direction as $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$ at every point of the curve. So $\mathbf{F}\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} > 0$ at every point of $\mathcal{C}$ and $\mathcal{C}$ is a closed curve with $\oint_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} > 0$. As a consequence $\mathbf{F}$ is not conservative.



(b) Consider the square in the figure (b) on the right above, oriented *counterclockwise*. It consists of the four line segments L_1, L_2, L_3 and L_4. On all of L_1, L_2, L_3 we have that $\mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t) = 0$ because the vector field is perpendicular to the line segment. On L_4 we have $\mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t) > 0$. So

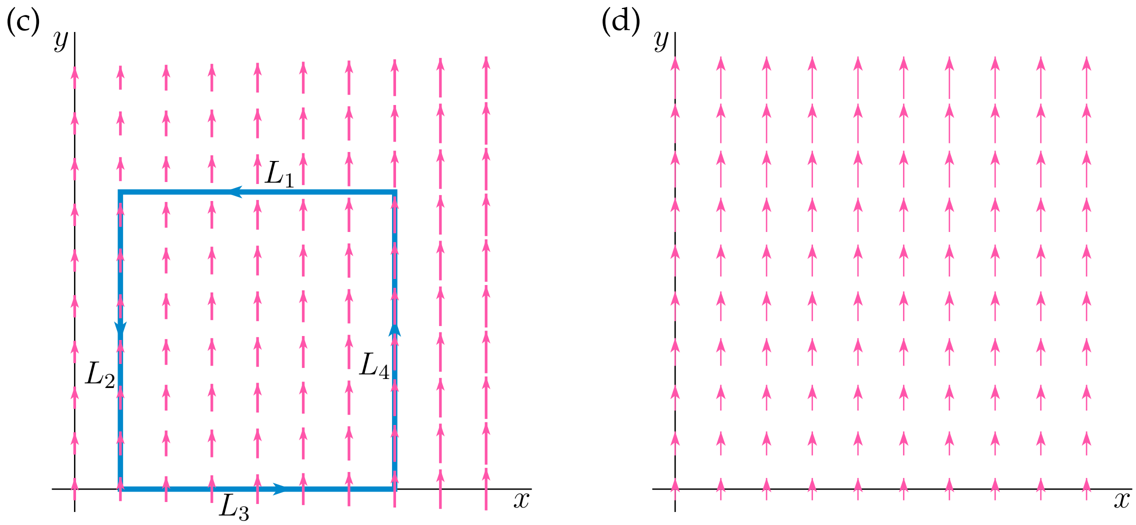
$$\begin{aligned}\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \int_{L_1} \mathbf{F} \cdot d\mathbf{r} + \int_{L_2} \mathbf{F} \cdot d\mathbf{r} + \int_{L_3} \mathbf{F} \cdot d\mathbf{r} + \int_{L_4} \mathbf{F} \cdot d\mathbf{r} \\ &= 0 + 0 + 0 + \int_{L_4} \mathbf{F} \cdot d\mathbf{r} > 0\end{aligned}$$

So $\mathcal{C}$ is a closed curve with $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} > 0$ and $\mathbf{F}$ is not conservative.

(c) Consider the square in the figure (c) on the left below, oriented *counterclockwise*. It consists of the four line segments L_1, L_2, L_3 and L_4. On L_1 and L_3 we have that the dot product $\mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t) = 0$ because the vector field is perpendicular to the line segment. On L_2 we have $\mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t) < 0$ while on L_4 we have $\mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t) > 0$. The vector field $\mathbf{F}$ is longer on L_4 than on L_2. So $\mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t)$ has a larger magnitude on L_4 than L_2 and

$$\begin{aligned}\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \int_{L_1} \mathbf{F} \cdot d\mathbf{r} + \int_{L_2} \mathbf{F} \cdot d\mathbf{r} + \int_{L_3} \mathbf{F} \cdot d\mathbf{r} + \int_{L_4} \mathbf{F} \cdot d\mathbf{r} \\ &= 0 + \int_{L_2} \mathbf{F} \cdot d\mathbf{r} + 0 + \int_{L_4} \mathbf{F} \cdot d\mathbf{r} > 0\end{aligned}$$

So $\mathcal{C}$ is a closed curve with $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} > 0$ and $\mathbf{F}$ is not conservative.



(d) We are told that one of the four vector fields is conservative. Only the vector field in (d) is left, so it is conservative.

Remark: We can verify that vector field (d) is indeed conservative by observing (look at the figure (d) on the right above) that the $\hat{\imath}$ component of the vector field is exactly zero and that the $\hat{\jmath}$ component depends only on y. So the vector field is of the form

$$\mathbf{F}(x,y) = a(y)\,\hat{\jmath}$$

for some function $a(y)$. If $A(y)$ is any antiderivative of $a(y)$, we have $\mathbf{F} = \boldsymbol{\nabla} A$, so that $\mathbf{F}$ is conservative with potential $A(y)$.

S-7:

(a) The (largest possible) domain is $D = \left\{\, (x,y,z) \,\middle|\, x^2+y^2 \neq 0 \,\right\}$. That is, all of $\mathbb{R}^3$ except the points lying along the z-axis.

(b) As preliminary computations, let's find

$$\frac{\partial}{\partial y}\left(\frac{x-2y}{x^2+y^2}\right) = \frac{-2}{x^2+y^2} - \frac{2y(x-2y)}{(x^2+y^2)^2} = \frac{-2x^2+2y^2-2xy}{(x^2+y^2)^2}$$
$$\frac{\partial}{\partial x}\left(\frac{2x+y}{x^2+y^2}\right) = \frac{2}{x^2+y^2} - \frac{2x(2x+y)}{(x^2+y^2)^2} = \frac{-2x^2+2y^2-2xy}{(x^2+y^2)^2}$$

So the curl of $\mathbf{F}$ is

$$\boldsymbol{\nabla}\times\mathbf{F} = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x-2y}{x^2+y^2} & \frac{2x+y}{x^2+y^2} & z \end{bmatrix} = \left(\frac{-2x^2+2y^2-2xy}{(x^2+y^2)^2} - \frac{-2x^2+2y^2-2xy}{(x^2+y^2)^2}\right)\hat{\mathbf{k}} = \mathbf{0}$$

on the domain of $\mathbf{F}$.

(c) Parametrize the circle by

$$\mathbf{r}(t) = 2\cos t\,\hat{\imath} + 2\sin t\,\hat{\jmath} + 3\,\hat{\mathbf{k}} \qquad \mathbf{r}'(t) = -2\sin t\,\hat{\imath} + 2\cos t\,\hat{\jmath}$$

with $0 \leqslant \theta \leqslant 2\pi$. So the integral is

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} \bigg\{ \overbrace{\frac{2\cos t - 4\sin t}{4}}^{\frac{x-2y}{x^2+y^2}} \hat{\boldsymbol{\imath}} + \overbrace{\frac{4\cos t + 2\sin t}{4}}^{\frac{2x+y}{x^2+y^2}} \hat{\boldsymbol{\jmath}} + \overbrace{3}^{z} \hat{\mathbf{k}} \bigg\} \cdot \Big\{ \overbrace{-2\sin t\,\hat{\boldsymbol{\imath}} + 2\cos t\,\hat{\boldsymbol{\jmath}}}^{\mathbf{r}'(t)} \Big\} \mathrm{d}t$$

$$= \int_0^{2\pi} \frac{-4\sin t\cos t + 8\sin^2 t + 8\cos^2 t + 4\sin t\cos t}{4}\,\mathrm{d}t$$

$$= 2\int_0^{2\pi} \mathrm{d}t = 4\pi$$

(d) As the integral of $\mathbf{F}$ around the simple closed curve C is not zero, $\mathbf{F}$ cannot be conservative. See Theorem 2.4.6 and Examples 2.3.14 and 4.3.8 in the CLP-4 text.

S-8: The point here is that $\mathbf{F}$ is conservative, as $\mathbf{F} = \nabla\phi$ with

$$\phi = \frac{x^2}{2} + yx - yz + \frac{z^2}{2}$$

So, for all paths from $\mathbf{r}(t_0) = (1, 0, -1)$ to $\mathbf{r}(t_1) = (0, -2, 3)$,

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \phi\big(\mathbf{r}(t_1)\big) - \phi\big(\mathbf{r}(t_0)\big) &= \phi(0, -2, 3) - \phi(1, 0, -1) \\ &= \left[0 + 0 + 6 + \frac{9}{2}\right] - \left[\frac{1}{2} + 0 - 0 + \frac{1}{2}\right] \\ &= 9\frac{1}{2}\end{aligned}$$

.

S-9: Note that:

- Along the line segment from $(0, 0)$ to $(1, 0)$, x increases from 0 to 1, while y is held fixed at $y = 0$. So we may parametrize this segment by $\mathbf{r}(x) = x\,\hat{\boldsymbol{\imath}}$, $0 \leqslant x \leqslant 1$.
- Along the line segment from $(1, 0)$ to $(1, \pi)$, y increases from 0 to π, while x is held fixed at $x = 1$. So we may parametrize this segment by $\mathbf{r}(x) = \hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}}$, $0 \leqslant y \leqslant \pi$.
- Along the line segment from $(1, \pi)$ to $(0, \pi)$, x decreases from 1 to 0, while y is held fixed at $y = \pi$. So we may parametrize this segment by $\mathbf{r}(x) = x\,\hat{\boldsymbol{\imath}} + \pi\,\hat{\boldsymbol{\jmath}}$ with x running from 1 to 0.

Hence

$$\begin{aligned}\int_C \mathbf{V} \cdot \mathrm{d}\mathbf{r} &= \int_0^1 \mathbf{V}(x, 0) \cdot \hat{\boldsymbol{\imath}}\,\mathrm{d}x + \int_0^{\pi} \mathbf{V}(1, y) \cdot \hat{\boldsymbol{\jmath}}\,\mathrm{d}y + \int_1^0 \mathbf{V}(x, \pi) \cdot \hat{\boldsymbol{\imath}}\,\mathrm{d}x \\ &= \int_0^1 (e^x + x^2)\,\mathrm{d}x + \int_0^{\pi} (y + 3)\,\mathrm{d}y + \int_1^0 (-e^x + x^2)\,\mathrm{d}x \\ &= 2\int_0^1 e^x\,\mathrm{d}x + \int_0^{\pi} (y + 3)\,\mathrm{d}y \\ &= 2(e - 1) + \frac{\pi^2}{2} + 3\pi\end{aligned}$$

S-10: (a) We may parametrize the curve by $\mathbf{r}(t) = t\,\hat{\boldsymbol{\imath}} + t^2\,\hat{\boldsymbol{\jmath}}$ with $0 \leqslant t \leqslant 1$. Then $\mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t) = \hat{\boldsymbol{\imath}} + 2t\,\hat{\boldsymbol{\jmath}}$ and $\mathbf{F}\big(x(t), y(t)\big) = t^3\,\hat{\boldsymbol{\imath}} - t^2\,\hat{\boldsymbol{\jmath}}$ so

$$\begin{aligned}\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_0^1 \mathbf{F}\big(x(t), y(t)\big)\cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\;\mathrm{d}t = \int_0^1 \big[t^3\,\hat{\boldsymbol{\imath}} - t^2\,\hat{\boldsymbol{\jmath}}\big]\cdot\big[\hat{\boldsymbol{\imath}} + 2t\,\hat{\boldsymbol{\jmath}}\big]\;\mathrm{d}t = \int_0^1 \big[-t^3\big]\;\mathrm{d}t\\ &= -\frac{1}{4}\end{aligned}$$

(b) The path is the union of three line segments.

- On the first segment of the path $y = z = 0$ so $\mathbf{F}$ simplifies to $x\,\hat{\boldsymbol{\imath}} - x\,\hat{\mathbf{k}}$ and $\mathrm{d}\mathbf{r} = \hat{\boldsymbol{\imath}}\,\mathrm{d}x$ (i.e. we can parametrize the first segment of the path by $\mathbf{r}(x) = x\,\hat{\boldsymbol{\imath}}$ with $0 \leqslant x \leqslant 1$), so $\mathbf{F}\cdot\mathrm{d}\mathbf{r} = x\,\mathrm{d}x$.
- On the second segment of the path $x = 1, z = 0$ so $\mathbf{F}$ simplifies to $\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} - (1+y)\hat{\mathbf{k}}$ and $\mathrm{d}\mathbf{r} = \hat{\boldsymbol{\jmath}}\,\mathrm{d}y$ (parametrize the second segment of the path by $\mathbf{r}(y) = \hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}}$ with $0 \leqslant y \leqslant 1$), so $\mathbf{F}\cdot\mathrm{d}\mathbf{r} = y\,\mathrm{d}y$.
- On the final segment of the path $x = y = 1$ so $\mathbf{F}$ simplifies to $(1-z)\hat{\boldsymbol{\imath}} + (1-z)\hat{\boldsymbol{\jmath}} - 2\hat{\mathbf{k}}$ and $\mathrm{d}\mathbf{r} = \hat{\mathbf{k}}\,\mathrm{d}z$ (parametrize the third segment of the path by $\mathbf{r}(z) = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}}$ with $0 \leqslant z \leqslant 1$), so $\mathbf{F}\cdot\mathrm{d}\mathbf{r} = -2\,\mathrm{d}z$.

So

$$\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_0^1 x\,\mathrm{d}x + \int_0^1 y\,\mathrm{d}y + \int_0^1 (-2)\,\mathrm{d}z = \frac{1}{2} + \frac{1}{2} - 2 = -1$$

S-11: Parametrize the curve using y as a parameter. Then $y = t$, $x = 2y = 2t$ and $z = \frac{8}{xy} = \frac{8}{2t^2}$ so that:

$$\begin{aligned}\mathbf{r}(t) &= 2t\,\hat{\boldsymbol{\imath}} + t\,\hat{\boldsymbol{\jmath}} + \frac{4}{t^2}\,\hat{\mathbf{k}}, \qquad 1 \leqslant t \leqslant 2\\ \mathbf{r}'(t) &= 2\,\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \frac{8}{t^3}\,\hat{\mathbf{k}}\\ \mathbf{F}(\mathbf{r}(t)) &= 4t^2\,\hat{\boldsymbol{\imath}} + 4t^3\,\hat{\mathbf{k}}\\ \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) &= 8t^2 - 32\end{aligned}$$

Then

$$\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_1^2 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\;\mathrm{d}t = \int_1^2 \big(8t^2 - 32\big)\;\mathrm{d}t = \left[\frac{8}{3}t^3 - 32t\right]_1^2 = -\frac{40}{3}$$

S-12: Note $\mathbf{F}$ is defined and continuous on all of $\mathbb{R}^3$. By Theorem 2.4.6, the integral $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = 0$ for all closed paths C if and only if $\mathbf{F}$ is conservative. Furthermore, $\mathbf{F}$ has continuous first-order partial derivatives on all of $\mathbb{R}^3$. Using Theorem 2.4.7, **F** is conservative if and only if $\nabla \times \mathbf{F} = \mathbf{0}$:

$$\begin{aligned}\mathbf{0} = \nabla \times \mathbf{F} &= \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \sin y & ae^x \cos y + bz & cx \end{bmatrix} \\ &= (0-b)\hat{\imath} - (c-0)\hat{\jmath} + (ae^x \cos y - e^x \cos y)\hat{\mathbf{k}}\end{aligned}$$

So $a = 1,\ b = c = 0$.

S-13: (a), (b) The curls of **F** and **G** are

$$\begin{aligned}\nabla \times \mathbf{F} &= \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6x^2yz^2 & 2x^3z^2 + 2y - xz & 4x^3yz \end{bmatrix} \\ &= (4x^3z - 4x^3z + x)\,\hat{\imath} - (12x^2yz - 12x^2yz)\,\hat{\jmath} + (6x^2z^2 - z - 6x^2z^2)\,\hat{\mathbf{k}} \\ &= x\,\hat{\imath} - z\,\hat{\mathbf{k}} \\ \nabla \times \mathbf{G} &= \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 0 & xy \end{bmatrix} \\ &= x\hat{\imath} - z\,\hat{\mathbf{k}}\end{aligned}$$

Hence the screening test for

$$\nabla \times \big(\mathbf{F} + \lambda\mathbf{G}\big) = (x + \lambda x)\,\hat{\imath} - (z + \lambda z)\,\hat{\mathbf{k}}$$

passes for $\lambda = -1$. Furthermore

$$\begin{aligned}\mathbf{F} - \mathbf{G} &= (6x^2yz^2 - yz)\,\hat{\imath} + (2x^3z^2 + 2y - xz)\,\hat{\jmath} + (4x^3yz - xy)\,\hat{\mathbf{k}} \\ &= \nabla\big(2x^3yz^2 - xyz + y^2\big)\end{aligned}$$

The potential was found by guessing. Alternatively, we can find it by using that $\phi(x, y, z)$ is a potential for $\mathbf{F} - \mathbf{G}$ if and only if

$$\begin{aligned}\frac{\partial \phi}{\partial x}(x, y, z) &= 6x^2yz^2 - yz \\ \frac{\partial \phi}{\partial y}(x, y, z) &= 2x^3z^2 + 2y - xz \\ \frac{\partial \phi}{\partial z}(x, y, z) &= 4x^3yz - xy\end{aligned}$$

Integrating the first of these equations gives

$$\phi(x, y, z) = 2x^3yz^2 - xyz + f(y, z)$$

Substituting this into the second equation gives

$$2x^3z^2 - xz + \frac{\partial f}{\partial y}(y, z) = 2x^3z^2 + 2y - xz \qquad \text{or} \qquad \frac{\partial f}{\partial y}(y, z) = 2y$$

which integrates to

$$f(y,z) = y^2 + g(z)$$

Finally, substituting $\phi(x,y,z) = 2x^3yz^2 - xyz + y^2 + g(z)$ into the last equation gives

$$4x^3yz - xy + g'(z) = 4x^3yz - xy \qquad \text{or} \qquad g'(z) = 0$$

which integrates to

$$g(z) = K$$

with K being an arbitrary constant. Choosing $K = 0$ gives the potential $\phi(x,y,z) = 2x^3yz^2 - xyz + y^2$ as in the guess above.

(c) Any point (x,y,z) on the curve must have $z = x$ and $y = e^{xz} = e^{x^2}$. So we may parametrize the curve by $\mathbf{r}(x) = x\,\hat{\boldsymbol{\imath}} + e^{x^2}\,\hat{\boldsymbol{\jmath}} + x\,\hat{\mathbf{k}}$, $0 \leqslant x \leqslant 1$. Hence

$$\begin{aligned}\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_C (\mathbf{F}-\mathbf{G})\cdot \mathrm{d}\mathbf{r} + \int_C \mathbf{G}\cdot \mathrm{d}\mathbf{r} \\ &= \Big[2x^3yz^2 - xyz + y^2\Big]_{(0,1,0)}^{(1,e,1)} + \int_0^1 \overbrace{[xe^{x^2}\hat{\boldsymbol{\imath}} + xe^{x^2}\hat{\mathbf{k}}]}^{\mathbf{G}(\mathbf{r}(x))}\cdot \overbrace{[\hat{\boldsymbol{\imath}} + 2xe^{x^2}\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}]}^{\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}x}}\ \mathrm{d}x \\ &= e + e^2 - 1 + \int_0^1 2xe^{x^2}\ \mathrm{d}x = e + e^2 - 1 + \Big[e^{x^2}\Big]_0^1 = e^2 + 2e - 2\end{aligned}$$

S-14: Parametrize the line segment by

$$\mathbf{r}(t) = (0,0,1) + t\big\{(2,1,0) - (0,0,1)\big\} = \big(2t, t, 1-t\big) \qquad 0 \leqslant t \leqslant 1$$

so that $\mathbf{r}(0) = (0,0,1)$ is the initial point of the line segment and $\mathbf{r}(1) = (2,1,0)$ is the final point of the segment. Then

$$\mathbf{r}'(t) = (2,1,-1)$$

and the work is

$$\begin{aligned}\int \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_0^1 \mathbf{F}\big(\mathbf{r}(t)\big)\cdot \mathbf{r}'(t)\,\mathrm{d}t = \int_0^1 \big(2t - t^2\,,\ t - (1-t)^2\,,\ (1-t) - 4t^2\big)\cdot (2,1,-1)\,\mathrm{d}t \\ &= \int_0^1 \big(4t - 2t^2\ +\ t - 1 + 2t - t^2\ -\ 1 + t + 4t^2\big)\,\mathrm{d}t = \int_0^1 \big(t^2 + 8t - 2\big)\,\mathrm{d}t \\ &= \frac{1}{3} + 4 - 2 = \frac{7}{3}\end{aligned}$$

S-15: On P, $z = \ln\frac{1}{x} = -\ln(x)$. So parametrize the curve P by

$$\mathbf{r}(\theta) = \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}} - \ln(\cos\theta)\,\hat{\mathbf{k}} \qquad 0 \leqslant \theta \leqslant \frac{\pi}{4}$$

Then

$$\begin{aligned}\mathbf{r}'(\theta) &= -\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}} + \tan\theta\,\hat{\mathbf{k}} \\ \mathbf{F}\big(\mathbf{r}(\theta)\big) &= \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}} + \cos^3\theta\,\hat{\mathbf{k}} \\ \mathbf{F}\big(\mathbf{r}(\theta)\big)\cdot\mathbf{r}'(\theta) &= \sin\theta\cos^2\theta\end{aligned}$$

so that

$$\begin{aligned}\text{Work} &= \int_P \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/4} \mathbf{F}\big(\mathbf{r}(\theta)\big)\cdot\mathbf{r}'(\theta)\,d\theta = \int_0^{\pi/4} \sin\theta\cos^2\theta\,d\theta = -\frac{1}{3}\cos^3\theta\Big|_0^{\pi/4} \\ &= \frac{1}{3}\Big[1 - \frac{1}{2^{3/2}}\Big] \approx 0.2155\end{aligned}$$

S-16: Hmmm. **F** looks suspiciously complicated. Let's guess that **F** is conservative and look for a potential for it. $\phi(x,y,z)$ is a potential for this **F** if and only if

$$\begin{aligned}\frac{\partial\varphi}{\partial x}(x,y,z) &= yz\cos x \\ \frac{\partial\varphi}{\partial y}(x,y,z) &= z\sin x + 2yz \\ \frac{\partial\varphi}{\partial z}(x,y,z) &= y\sin x + y^2 - \sin z\end{aligned}$$

Integrating the first of these equations gives

$$\varphi(x,y,z) = yz\sin x + f(y,z)$$

Substituting this into the second equation gives

$$z\sin x + \frac{\partial f}{\partial y}(y,z) = z\sin x + 2yz \qquad \text{or} \qquad \frac{\partial f}{\partial y}(y,z) = 2yz$$

which integrates to

$$f(y,z) = y^2 z + g(z)$$

Finally, substituting $\varphi(x,y,z) = yz\sin x + y^2 z + g(z)$ into the last equation gives

$$y\sin x + y^2 + g'(z) = y\sin x + y^2 - \sin z \qquad \text{or} \quad g'(z) = -\sin z$$

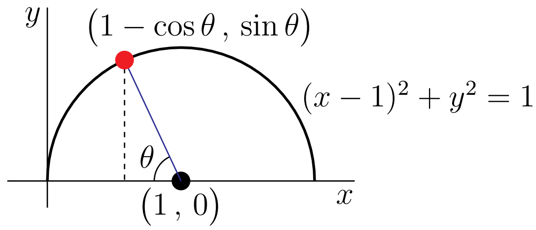
which integrates to

$$g(z) = \cos z + C$$

with C being an arbitrary constant. So $\phi(x,y,z) = yz\sin x + y^2 z + \cos z$ is one allowed scalar potential and the specified integral is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \varphi(\mathbf{r})\Big|_{\mathbf{r}(0)}^{\mathbf{r}(\pi/2)} = \varphi(\pi/2\,,\ \pi/2\,,\ \pi/2) - \varphi(0,0,0) = \frac{\pi^3}{8} + \frac{\pi^2}{4} - 1$$

S-17: **Solution 1:**
We are being asked to evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ with C being the specified semi-circle and $\mathbf{F} = xy\,\hat{\boldsymbol{\jmath}}$. As $\nabla \times \mathbf{F} \neq \mathbf{0}$, the vector field $\mathbf{F}$ is not conservative. So we'll evaluate the integral directly. First, using the figure,



we parametrize C by

$$\mathbf{r}(\theta) = \big(x(\theta)\,,\,y(\theta)\big) = (1-\cos\theta)\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}} \qquad 0 \leqslant \theta \leqslant \pi$$

So the integral is

$$\int_C xy\,dy = \int_0^\pi x(\theta)\,y(\theta)\,y'(\theta)\,d\theta = \int_0^\pi (1-\cos\theta)\,\sin\theta\,\cos\theta\,d\theta$$

Making the substitution $u = \cos\theta$, $du = -\sin\theta\,d\theta$, $u(0) = 1$, $u(\pi) = -1$,

$$\int_C xy\,dy = \int_1^{-1} (1-u)\,u\,(-du) = \int_{-1}^1 (u-u^2)\,du = -2\int_0^1 u^2\,du = -2\frac{1^3}{3} = -\frac{2}{3}$$

Solution 2:
We can write x in terms of y over C in two pieces:

- Let C_1 be the quarter-circle $x = 1-\sqrt{1-y^2}$ as y goes from 0 to 1, and
- Let C_2 be the quarter-circle $x = 1+\sqrt{1-y^2}$ as y goes from 1 to 0.

Then:

$$\begin{aligned}
\int_C xy\mathrm{d}y &= \int_{C_1} xy\mathrm{d}y + \int_{C_2} xy\mathrm{d}y \\
&= \int_0^1 \left(1-\sqrt{1-y^2}\right) y\,dy + \int_1^0 \left(1+\sqrt{1-y^2}\right) y\,dy \\
&= \int_0^1 y\,dy - \int_0^1 y\sqrt{1-y^2}\,dy + \int_1^0 y\,dy + \int_1^0 y\sqrt{1-y^2}\,dy \\
&= -2\int_0^1 y\sqrt{1-y^2}\,dy
\end{aligned}$$

Using the substitution $u = 1-y^2$, $du = -2y\,dy$:

$$= \int_1^0 u^{1/2}du = -\frac{2}{3}$$

S-18:

The line integral is $\int_C \mathbf{F}\cdot d\mathbf{r}$ with $\mathbf{F} = (ye^x + \sin y)\,\hat{\boldsymbol{\imath}} + (e^x + \sin y + x\cos y)\,\hat{\boldsymbol{\jmath}}$. We are to show that it is independent of path. That is the case if and only if $\mathbf{F}$ is conservative. So let's look for a potential φ for $\mathbf{F}$. That is, let's look for a function φ that obeys

$$\frac{\partial\varphi}{\partial x}(x,y) = ye^x + \sin y$$
$$\frac{\partial\varphi}{\partial y}(x,y) = e^x + \sin y + x\cos y$$

Integrating the first of these equations gives

$$\varphi(x,y) = ye^x + x\sin y + f(y)$$

Substituting this into the second equation gives

$$e^x + x\cos y + f'(y) = e^x + \sin y + x\cos y \qquad \text{or} \quad f'(y) = \sin y$$

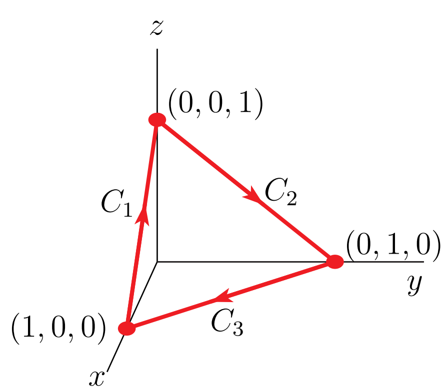
which integrates to

$$f(y) = -\cos y + C$$

So $\mathbf{F}$ is indeed conservative with one potential being $\varphi(x,y) = ye^x + x\sin y - \cos y$ and the line integral is

$$\begin{aligned}\int_C (ye^x + \sin y)\,dx + (e^x + \sin y + x\cos y)\,dy &= \int_C \mathbf{F}\cdot d\mathbf{r} = \varphi(x,y)\Big|_{(1,0)}^{(0,\pi/2)}\\ &= \Big[ye^x + x\sin y - \cos y\Big]_{(1,0)}^{(0,\pi/2)}\\ &= 1 + \frac{\pi}{2}\end{aligned}$$

S-19: Here is a sketch of C.



Note that

- $y = 0$ on the line segment from $(1,0,0)$ to $(0,0,1)$ so that the integral reduces to $\int zx\,dz$ on that line segment and

- $x = 0$ on the line segment from $(0,0,1)$ to $(0,1,0)$ so that the integral reduces to $\int yz\,\mathrm{d}y$ on that line segment and
- $z = 0$ on the line segment from $(0,1,0)$ to $(1,0,0)$ so that the integral reduces to $\int xy\,\mathrm{d}x$ on that line segment.

So it looks feasible to evaluate the integral directly. Label the sides of the triangle C_1, C_2 and C_3 as in the sketch above.

- We parametrize C_1 by $\mathbf{r}(t) = (1,0,0) + t[(0,0,1) - (1,0,0)] = (1-t\,,\,0\,,\,t)$, $0 \leqslant t \leqslant 1$. So

$$\begin{aligned}\int_{C_1} xy\,\mathrm{d}x + yz\,\mathrm{d}y + zx\,\mathrm{d}z &= \int_{C_1} zx\,\mathrm{d}z = \int_0^1 \overbrace{(1-t)}^{x}\,\overbrace{(t)}^{z}\,\overbrace{(1)}^{z'(t)}\,\mathrm{d}t = \int_0^1 (t-t^2)\,\mathrm{d}t \\ &= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}\end{aligned}$$

- We parametrize C_2 by $\mathbf{r}(t) = (0,0,1) + t[(0,1,0) - (0,0,1)] = (0\,,\,t\,,\,1-t)$, $0 \leqslant t \leqslant 1$. So

$$\begin{aligned}\int_{C_2} xy\,\mathrm{d}x + yz\,\mathrm{d}y + zx\,\mathrm{d}z &= \int_{C_2} yz\,\mathrm{d}y = \int_0^1 \overbrace{(t)}^{y}\,\overbrace{(1-t)}^{z}\,\overbrace{(1)}^{y'(t)}\,\mathrm{d}t = \int_0^1 (t-t^2)\,\mathrm{d}t \\ &= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}\end{aligned}$$

- We parametrize C_3 by $\mathbf{r}(t) = (0,1,0) + t[(1,0,0) - (0,1,0)] = (t\,,\,1-t\,,\,0)$, $0 \leqslant t \leqslant 1$. So

$$\begin{aligned}\int_{C_3} xy\,\mathrm{d}x + yz\,\mathrm{d}y + zx\,\mathrm{d}z &= \int_{C_3} xy\,\mathrm{d}x = \int_0^1 \overbrace{(t)}^{x}\,\overbrace{(1-t)}^{y}\,\overbrace{(1)}^{x'(t)}\,\mathrm{d}t = \int_0^1 (t-t^2)\,\mathrm{d}t \\ &= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}\end{aligned}$$

All together

$$\int_C xy\,\mathrm{d}x + yz\,\mathrm{d}y + zx\,\mathrm{d}z = \sum_{\ell=1}^{3} \int_{C_\ell} xy\,\mathrm{d}x + yz\,\mathrm{d}y + zx\,\mathrm{d}z = 3 \times \frac{1}{6} = \frac{1}{2}$$

S-20: We are told that $\mathbf{F}$ is conservative. Let's find a potential φ obeying $\nabla\varphi = \mathbf{F}$. That is,

$$\begin{aligned}\frac{\partial\varphi}{\partial x} &= y + ze^x \\ \frac{\partial\varphi}{\partial y} &= x + e^y \sin z \\ \frac{\partial\varphi}{\partial z} &= z + e^x + e^y \cos z\end{aligned}$$

The first equation forces $\varphi(x,y,z) = xy + ze^x + \psi(y,z)$. Substituting this into the second equation gives $x + \frac{\partial\psi}{\partial y}(y,z) = x + e^y \sin z$ or $\frac{\partial\psi}{\partial y}(y,z) = e^y \sin z$ which forces $\psi(y,z) = e^y \sin z + \zeta(z)$. So far, we have $\varphi(x,y,z) = xy + ze^x + e^y \sin z + \zeta(z)$. Substituting this into the third equation gives $e^x + e^y \cos z + \zeta'(z) = z + e^x + e^y \cos z$ or $\zeta'(z) = z$ which forces $\zeta(z) = \frac{z^2}{2} + C$, for some constant C, which we take to be zero. So our potential is

$$\varphi(x,y,z) = xy + ze^x + e^y \sin z + \frac{z^2}{2}$$

So the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi\big(\mathbf{r}(\pi)\big) - \varphi\big(\mathbf{r}(0)\big) = \varphi\big(\pi, e^\pi, 0\big) - \varphi\big(0,1,0\big) = \pi e^\pi$$

S-21: (a) Note $\mathbf{F}$ is defined and continuous on all of $\mathbb{R}^3$. Furthermore, $\mathbf{F}$ has continuous first-order partial derivatives on all of $\mathbb{R}^3$. Using Theorem 2.4.7, $\mathbf{F}$ is conservative if and only if it has zero curl:

$$\begin{aligned} 0 = \nabla \times \mathbf{F} &= \nabla \times \big(\alpha e^y\, \hat{\imath} + (xe^y + \beta\cos z)\, \hat{\jmath} - \gamma y \sin z\, \hat{\mathbf{k}}\big) \\ &= (-\gamma \sin z + \beta \sin z)\hat{\imath} + (e^y - \alpha e^y)\hat{\mathbf{k}} \end{aligned}$$

which is the case if and only if $\alpha = 1$, $\beta = \gamma$.

(b) We use Theorem 2.4.2: if φ is a potential for $\mathbf{F}$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(P_1) - \varphi(P_0)$$

where C runs from P_0 to P_1. So, we find φ.

Assume that $\alpha = 1$, $\beta = \gamma$. We find a potential φ for $\mathbf{F}$ by antidifferentiating.

$$\begin{aligned} \frac{\partial\varphi}{\partial x}(x,y,z) &= e^y &&\implies \varphi(x,y,z) = xe^y + \psi_1(y,z) \\ \frac{\partial\varphi}{\partial y}(x,y,z) &= xe^y + \beta\cos z &&\implies \varphi(x,y,z) = xe^y + \beta y \cos z + \psi_2(x,z) \\ \frac{\partial\varphi}{\partial z}(x,y,z) &= -\beta y \sin z &&\implies \varphi(x,y,z) = \beta y \cos z + \psi_3(x,y) \end{aligned}$$

for some functions $\psi_1(y,z)$, $\psi_2(x,z)$ and $\psi_3(x,y)$ to be determined.

We'd like a single function $\varphi(x,y,z)$ that simultaneously obeys all three of these equations, for some ψ_j's. An initial guess is simply the sum of all of the distinct terms, other that the ψ_j's, that appear in the three equations above. The term xe^y appears in the ψ_1 and ψ_2 equations and the term $\beta y \cos z$ appears in the ψ_2 and ψ_3 equations. So we guess

$$\varphi(x,y,z) \stackrel{?}{=} xe^y + \beta y \cos z$$

If we let $\psi_1(y,z) = \beta y \cos z$, $\psi_2(x,z) = 0$, and $\psi_3(x,y) = xe^y$, then we see this function $\varphi(x,y,z)$ does indeed obey all three equations and so is a potential for $\mathbf{F}$.

The curve C runs from $P_0 = (0^2, e^0, \pi \cdot 0) = (0,1,0)$ to $P_1 = (1^2, e^1, \pi \cdot 1) = (1, e, \pi)$. Using Theorem 2.4.2:

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \varphi(1,e,\pi) - \varphi(0,1,0) = \big(e^e - \beta e\big) - \beta = e^e - \beta(e+1)$$

S-22: (a) The curl of $\mathbf{F}$ is

$$\boldsymbol{\nabla} \times \mathbf{F} = \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos x & 2 + \sin y & e^z \end{bmatrix} = \mathbf{0}$$

Because F_1 is a function only of x, F_2 is a function only of y, and F_3 is a function only of z, that all partial derivatives used in computing the curl are 0.

(b) The vector field $\mathbf{F}$ passes the screening test on all of $\mathbb{R}^3$ and so is conservative by Theorem 2.4.7 in the text. Alternatively, we can see that

$$\mathbf{F} = \boldsymbol{\nabla}\big(\sin x + 2y - \cos y + e^z\big)$$

by inspection. Alternatively, f can be found by antidifferentiating its partial derivatives:

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y,z) &= \cos x && \Longrightarrow\ f(x,y,z) = \sin x + \psi_1(y,z) \\ \frac{\partial f}{\partial y}(x,y,z) &= 2 + \sin y && \Longrightarrow\ f(x,y,z) = 2y - \cos y + \psi_2(x,z) \\ \frac{\partial f}{\partial z}(x,y,z) &= e^z && \Longrightarrow\ f(x,y,z) = e^z + \psi_3(x,y) \end{aligned}$$

We'd like a single function $f(x,y,z)$ that simultaneously obeys all three of these equations, for some ψ_j's. An initial guess is simply the sum of all of the distinct terms, other than the ψ_j's, that appear in the three equations. The term $\sin x$ appears in the ψ_1 equation, the terms $2y$ and $-\cos y$ appears in the ψ_2 equation, and the term e^z appears in the ψ_3 equation. So we guess

$$f(x,y,z) \stackrel{?}{=} \sin x + 2y - \cos y + e^z$$

If we let $\psi_1(y,z) = 2y - \cos y + e^z$, $\psi_2(x,z) = \sin x + e^z$, and $\psi_3(x,y) = \sin x + 2y - \cos y$, then we see this function $f(x,y,z)$ is indeed a potential for $\mathbf{F}$.

(c) Since $\mathbf{F} = \boldsymbol{\nabla} f$,

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= f\big(\mathbf{r}(3\pi)\big) - f\big(\mathbf{r}(0)\big) \\ &= f(3\pi, -1, 0) - f(0,1,0) \\ &= \big(0 - 2 - \cos(-1) + 1\big) - \big(0 + 2 - \cos 1 + 1\big) \end{aligned}$$

Since cosine is an even function, $\cos(-1) = \cos 1$.

$$= -4$$

S-23: (a) The curl is

$$\boldsymbol{\nabla}\times\mathbf{F}=\det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ z+e^y & xe^y-e^z\sin y & 1+x+e^z\cos y\end{bmatrix}=\mathbf{0}$$

so $\mathbf{F}$ passes the screening test. Since its first-order partial derivatives are continuous on all of $\mathbb{R}^3$, it is conservative by Theorem 2.4.7 in the text.

By inspection, the potential is $\varphi(x,y,z)=xz+xe^y+e^z\cos y+z$ — this is another way to verify that $\mathbf{F}$ is conservative. Alternatively, φ can be found by antidifferentiating its partial derivatives.

$$\begin{aligned}
\frac{\partial\varphi}{\partial x}(x,y,z)&=z+e^y &&\implies \varphi(x,y,z)=zx+xe^y+\psi_1(y,z)\\
\frac{\partial\varphi}{\partial y}(x,y,z)&=xe^y-e^z\sin y &&\implies \varphi(x,y,z)=xe^y+e^z\cos y+\psi_2(x,z)\\
\frac{\partial\varphi}{\partial z}(x,y,z)&=1+x+e^z\cos y &&\implies \varphi(x,y,z)=z+zx+e^z\cos y+\psi_3(x,y)
\end{aligned}$$

We'd like a single function $\varphi(x,y,z)$ that simultaneously obeys all three of these equations. An initial guess is simply the sum of the distinct terms (without the ψ_j's) that appear in the equations above:

$$\varphi(x,y,z)\stackrel{?}{=}zx+xe^y+e^z\cos y+z$$

If we let $\psi_1(y,z)=e^z\cos y+z$, $\psi_2(x,z)=zx+z$, and $\psi_3(x,y)=xe^y$, then we see this function $\varphi(x,y,z)$ is indeed a potential for $\mathbf{F}$.

(b) Since $\mathbf{F}=\boldsymbol{\nabla}\varphi$, with $\varphi=xz+xe^y+e^z\cos y+z$,

$$\begin{aligned}
\int_C\mathbf{F}\cdot d\mathbf{r}&=\varphi\big(\mathbf{r}(\pi)\big)-\varphi\big(\mathbf{r}(0)\big)=\Big[xz+xe^y+e^z\cos y+z\Big]_{\mathbf{r}(0)}^{\mathbf{r}(\pi)}\\
&=\Big[xz+xe^y+e^z\cos y+z\Big]_{(0,0,1)}^{(\pi^2,0,1)}\\
&=\Big(\pi^2+\pi^2+e+1\Big)-(0+0+e+1)=2\pi^2
\end{aligned}$$

S-24: (a) For $\mathbf{F}$ to be conservative, it must pass the screening test

$$\begin{aligned}
\mathbf{0}=\boldsymbol{\nabla}\times\mathbf{F}&=\det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ (x-a)ye^x & xe^x+z^3 & byz^2\end{bmatrix}\\
&=\big(bz^2-3z^2\big)\,\hat{\boldsymbol{\imath}}-\big(0-0\big)\,\hat{\boldsymbol{\jmath}}+\big(e^x+xe^x-(x-a)e^x\big)\,\hat{\mathbf{k}}
\end{aligned}$$

This is the case if and only if $b=3$ and $a=-1$

(b) Set $a = -1$ and $b = 3$. For f to be a potential for $\mathbf{F}$, it must obey

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y,z) &= (x+1)ye^x \\ \frac{\partial f}{\partial y}(x,y,z) &= xe^x + z^3 \\ \frac{\partial f}{\partial z}(x,y,z) &= 3yz^2\end{aligned}$$

Integrating the second of these equations gives

$$f(x,y,z) = xye^x + yz^3 + g(x,z)$$

Substituting this into the last equation gives

$$3yz^2 + \frac{\partial g}{\partial z}(x,z) = 3yz^2 \qquad \text{or} \qquad \frac{\partial g}{\partial z}(x,z) = 0$$

which forces

$$g(x,z) = h(x)$$

Finally, substituting $f(x,y,z) = xye^x + yz^3 + h(x)$ into the first equation gives

$$xye^x + ye^x + h'(x) = (x+1)ye^x \quad \text{or} \quad h'(x) = 0$$

So $h(x) = C$ and hence $f(x,y,z) = xye^x + yz^3 + C$ works for any constant C.

(c) Since $\mathbf{F} = \boldsymbol{\nabla} f$,

$$\begin{aligned}\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_C \boldsymbol{\nabla} f \cdot \mathrm{d}\mathbf{r} &= f\big(\mathbf{r}(\pi)\big) - f\big(\big(\mathbf{r}(0)\big) = f(\pi,1,-1) - f(0,1,1) \\ &= \big[\pi e^\pi - 1\big] - \big[1\big] = \pi e^\pi - 2\end{aligned}$$

(d) Since

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_C (x+1)ye^x\,\mathrm{d}x + (xe^x + z^3)\,\mathrm{d}y + 3yz^2\,\mathrm{d}z$$

we have

$$\begin{aligned}I &= \int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} + \int_C yz^2\mathrm{d}z \\ &= \pi e^\pi - 2 + \int_0^\pi \overbrace{(\cos 2t)}^{y}\,\overbrace{\cos^2 t}^{z^2}\,\overbrace{(-\sin t)\,\mathrm{d}t}^{\mathrm{d}z} \\ &= \pi e^\pi - 2 + \int_0^\pi (2\cos^2 t - 1)\cos^2 t(-\sin t)\,\mathrm{d}t \\ &= \pi e^\pi - 2 + \int_1^{-1} (2u^2 - 1)u^2\,\mathrm{d}u \qquad \text{with } u = \cos t,\ \mathrm{d}u = -\sin t\,\mathrm{d}t \\ &= \pi e^\pi - 2 + \Big[\frac{2u^5}{5} - \frac{u^3}{3}\Big]_1^{-1} \\ &= \pi e^\pi - 2 + \Big[-\frac{4}{5} + \frac{2}{3}\Big] \\ &= \pi e^\pi - \frac{32}{15}\end{aligned}$$

S-25: (a) The vector field $\mathbf{F}$ is conservative if and only if it passes the screening test $\nabla \times \mathbf{F} = \mathbf{0}$. That is, if and only if,

$$\mathbf{0} = \nabla \times \mathbf{F} = \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2e^{3z} + Axy^3 & 2xye^{3z} + 3x^2y^2 & Bxy^2e^{3z} \end{vmatrix}$$

$$= \left(2Bxye^{3z} - 6xye^{3z}\right)\hat{\imath} - \left(By^2e^{3z} - 3y^2e^{3z}\right)\hat{\jmath} + \left(2ye^{3z} + 6xy^2 - 2ye^{3z} - 3Axy^2\right)\hat{\mathbf{k}}$$

So $\mathbf{F}$ is conservative if and only if $A = 2$ and $B = 3$.

(b) Let $A = 2$ and $B = 3$. We find a potential φ for $\mathbf{F}$ by antidifferentiating its partial derivatives.

$$\begin{aligned}
\frac{\partial \varphi}{\partial x}(x,y,z) &= y^2e^{3z} + 2xy^3 && \Longrightarrow \varphi(x,y,z) = xy^2e^{3z} + x^2y^3 + \psi_1(y,z) \\
\frac{\partial \varphi}{\partial y}(x,y,z) &= 2xye^{3z} + 3x^2y^2 && \Longrightarrow \varphi(x,y,z) = xy^2e^{3z} + x^2y^3 + \psi_2(x,z) \\
\frac{\partial \varphi}{\partial z}(x,y,z) &= 3xy^2e^{3z} && \Longrightarrow \varphi(x,y,z) = xy^2e^{3z} + \psi_3(x,y)
\end{aligned}$$

Let's guess that

$$\varphi(x,y,z) = xy^2e^{3z} + x^2y^3$$

(This was obtained by summing the distinct terms in the above three equations, without the ψ_i's.) If we set $\psi_1(y,z) = \psi_2(x,z) = 0$ and $\psi_3(x,y) = x^2y^3$, we see our choice of φ is indeed a potential for $\mathbf{F}$.

(c) Set $A = 2$ and $B = 3$. We are asked the evaluate $\int_C \mathbf{G} \cdot \mathrm{d}\mathbf{r}$ with

$$\mathbf{G} = (y^2e^{3z} + xy^3)\,\hat{\imath} + (2xye^{3z} + 3x^2y^2)\,\hat{\jmath} + 3xy^2e^{3z}\,\hat{\mathbf{k}} = \mathbf{F} - xy^3\,\hat{\imath}$$

So

$$\begin{aligned}
\int_C (y^2e^{3z} + xy^3)\,\mathrm{d}x + (2xye^{3z} + 3x^2y^2)\,\mathrm{d}y + 3xy^2e^{3z}\,\mathrm{d}z &= \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} - \int_C xy^3\,\mathrm{d}\mathbf{r} \\
&= \varphi\big(\mathbf{r}(1)\big) - \varphi\big(\mathbf{r}(0)\big) - \int_0^1 \overbrace{e^{2t}\left(e^{-t}\right)^3\hat{\imath}}^{xy^3\,\hat{\imath}} \cdot \overbrace{\left(2e^{2t}\,\hat{\imath} - e^{-t}\,\hat{\jmath} + \frac{1}{1+t}\,\hat{\mathbf{k}}\right)}^{\mathbf{r}'(t)}\,\mathrm{d}t \\
&= \varphi\big(e^2, 1/e, \ln 2\big) - \varphi\big(1,1,0\big) - \int_0^1 2e^t\,\mathrm{d}t \\
&= \left\{e^2\left(1/e\right)^2 e^{3\ln 2} + e^4\left(1/e\right)^3\right\} - \big(1+1\big) - 2(e-1) \\
&= 2^3 + e - 2 - 2e + 2 \\
&= 8 - e
\end{aligned}$$

S-26: (a) The field is conservative only if

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

That is,

$$\begin{aligned}
\frac{\partial}{\partial y}\left(2x\sin(\pi y) - e^z\right) &= \frac{\partial}{\partial x}\Big(ax^2\cos(\pi y) - 3e^z\Big) &&\iff& 2\pi x\cos(\pi y) &= 2ax\cos(\pi y)\\
\frac{\partial}{\partial z}\left(2x\sin(\pi y) - e^z\right) &= -\frac{\partial}{\partial x}\left(x+by\right)e^z &&\iff& -e^z &= -e^z\\
\frac{\partial}{\partial z}\Big(ax^2\cos(\pi y) - 3e^z\Big) &= -\frac{\partial}{\partial y}\left(x+by\right)e^z &&\iff& -3e^z &= -be^z
\end{aligned}$$

Hence only $a = \pi,\ b = 3$ works.

(b) When $a = \pi,\ b = 3$

$$\begin{aligned}
\mathbf{F} &= \left(2x\sin(\pi y) - e^z\right)\hat{\imath} + \Big(\pi x^2\cos(\pi y) - 3e^z\Big)\hat{\jmath} - (x+3y)\,e^z\hat{\mathbf{k}}\\
&= \nabla\big(x^2\sin(\pi y) - xe^z - 3ye^z + C\big)
\end{aligned}$$

so $\varphi(x,y,z) = x^2\sin(\pi y) - xe^z - 3ye^z + C$ for any constant C. Here φ was guessed. Alternatively, it can be found by antidifferentiating the partial derivatives of $\mathbf{F}$.

$$\begin{aligned}
\frac{\partial\varphi}{\partial x}(x,y,z) &= 2x\sin(\pi y) - e^z &&\implies \varphi(x,y,z) = x^2\sin(\pi y) - xe^z + \psi_1(y,z)\\
\frac{\partial\varphi}{\partial y}(x,y,z) &= \pi x^2\cos(\pi y) - 3e^z &&\implies \varphi(x,y,z) = x^2\sin(\pi y) - 3ye^z + \psi_2(x,z)\\
\frac{\partial\varphi}{\partial z}(x,y,z) &= -(x+3y)e^z &&\implies \varphi(x,y,z) = -xe^z - 3ye^z + \psi_3(x,y)
\end{aligned}$$

Summing the distinct terms on the right hand sides of the three equations above, we guess

$$\varphi(x,y,z) = x^2\sin(\pi y) - xe^z - 3ye^z$$

is a potential for $\mathbf{F}$. Setting $\psi_1(y,z) = -3ye^z$, $\psi_2(x,z) = -xe^z$, and $\psi_3(x,y) = x^2\sin(\pi y)$ convinces us that our guess is indeed a valid potential.

(c) By part (b),

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \varphi(1,1,\ln 2) - \varphi(0,0,0) = \Big(\sin\pi - e^{\ln 2} - 3e^{\ln 2}\Big) - (\sin(0) - 0 - 0) = -8$$

(d) Observe that $\mathbf{G} = \mathbf{F} + 3ye^z\,\hat{\mathbf{k}}$, with $\mathbf{F}$ evaluated with $a = \pi,\ b = 3$. Hence

$$\int_C \mathbf{G}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot d\mathbf{r} + \int_C 3ye^z\,\hat{\mathbf{k}}\cdot d\mathbf{r} = -8 + \int_C 3ye^z\,\hat{\mathbf{k}}\cdot d\mathbf{r}$$

To evaluate the remaining integral, parametrize the curve by $\mathbf{r}(t) = t\hat{\imath} + t\hat{\jmath} + \ln(1+t)\hat{\mathbf{k}}$ with $0 \leqslant t \leqslant 1$. Then $\mathbf{r}'(t) = \hat{\imath} + \hat{\jmath} + \frac{1}{1+t}\hat{\mathbf{k}}$ and $3ye^z\hat{\mathbf{k}} = 3t(1+t)\hat{\mathbf{k}}$ so that $3ye^z\,\hat{\mathbf{k}}\cdot d\mathbf{r} = 3t\,dt$. Subbing in

$$\int_C \mathbf{G}\cdot d\mathbf{r} = -8 + \int_0^1 3t\,dt = -8 + \frac{3}{2} = -\frac{13}{2}$$

S-27: (a) The potential f must obey

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y,z) &= -2y\cos x\sin x\\
\frac{\partial f}{\partial y}(x,y,z) &= \cos^2 x + (1+yz)e^{yz}\\
\frac{\partial f}{\partial z}(x,y,z) &= y^2e^{yz}
\end{aligned}$$

Integrating the last of these equations with respect to z gives

$$f(x,y,z) = ye^{yz} + g(x,y)$$

Substituting this into the second equation gives

$$e^{yz} + yze^{yz} + \frac{\partial g}{\partial y}(x,y) = \cos^2 x + (1+yz)e^{yz} \qquad \text{or} \qquad \frac{\partial g}{\partial y}(x,y) = \cos^2 x$$

which forces

$$g(x,y) = y\cos^2 x + h(x)$$

Finally, substituting $f(x,y,z) = ye^{yz} + y\cos^2 x + h(x)$ into the first equation gives

$$-2y\sin x\cos x + h'(x) = -2y\cos x\sin x \quad \text{or} \quad h'(x) = 0$$

So $h(x) = C$ and hence $f(x,y,z) = ye^{yz} + y\cos^2 x + C$ works for any constant C.

(b) By part (a)

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C \nabla f\cdot d\mathbf{r} = f\big(\pi, e^\pi, 0\big) - f\big(0,1,-\pi^2\big) = \Big[ye^{yz} + y\cos^2 x\Big]_{(0,1,-\pi^2)}^{(\pi,e^\pi,0)}\\
&= \big(2e^\pi\big) - \big(e^{-\pi^2} + 1\big)
\end{aligned}$$

S-28: (a) The curl of $\mathbf{F}$ is zero because F_1 is a function only of x, F_2 is a function only of y, and F_3 is a function only of z. That is:

$$\nabla\times\mathbf{F} = \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2x & 2y & 2z\end{bmatrix} = (0-0)\hat{\imath} + (0-0)\hat{\jmath} + (0-0)\hat{\mathbf{k}} = \mathbf{0}$$

(b) All first-order partial derivative of $\mathbf{F}$ are continuous on all of $\mathbb{R}^3$. By part (a), $\mathbf{F}$ passes the screening test and is conservative by Theorem 2.4.7 in the text. By inspection, a potential is $\varphi = x^2 + y^2 + z^2$. Since $\mathbf{F} = \nabla\varphi$,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \Big[x^2+y^2+z^2\Big]_{(0,0,0)}^{(a_1,a_2,a_3)} = a_1^2 + a_2^2 + a_3^2 = \mathbf{a}\cdot\mathbf{a}$$

S-29: (a) The curl of $\mathbf{F}$ is

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{F} &= \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{yz} & xze^{yz} + ze^{y} & xye^{yz} + e^{y} \end{bmatrix} \\
&= \big[(xe^{yz} + xyze^{yz} + e^{y}) - (xe^{yz} + xyze^{yz} + e^{y})\big]\,\hat{\boldsymbol{\imath}} - \big[ye^{yz} - ye^{yz}\big]\,\hat{\boldsymbol{\jmath}} + \big[ze^{yz} - ze^{yz}\big]\,\hat{\mathbf{k}} \\
&= \mathbf{0}
\end{aligned}$$

(b) $\mathbf{F}$ is defined on all of $\mathbb{R}^3$ and passes the conservative field screening test $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$. So $\mathbf{F}$ is conservative. We find a potential φ for $\mathbf{F}$ by antidifferentiating its partial derivatives.

$$\begin{aligned}
\frac{\partial\varphi}{\partial x}(x,y,z) &= e^{yz} && \implies \varphi(x,y,z) = xe^{yz} + \psi_1(y,z) \\
\frac{\partial\varphi}{\partial y}(x,y,z) &= xze^{yz} + ze^{y} && \implies \varphi(x,y,z) = xe^{yz} + ze^{y} + \psi_2(x,z) \\
\frac{\partial\varphi}{\partial z}(x,y,z) &= xye^{yz} + e^{y} && \implies \varphi(x,y,z) = xe^{yz} + ze^{y} + \psi_3(x,y)
\end{aligned}$$

All together, $\varphi(x,y,z) = xe^{yz} + ze^{y} + C$ works for any constant C. So the specified work integral is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi\big(\mathbf{r}(\pi/2)\big) - \varphi\big(\mathbf{r}(0)\big) = \varphi\big(0,1,\pi/2\big) - \varphi\big(1,0,0\big) = \frac{\pi e}{2} - 1$$

S-30: (a), (b) The function $f(x,y)$ is a potential for $\mathbf{F}(x,y)$ if and only if it obeys

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y) &= 2xy\cos(x^2) \\
\frac{\partial f}{\partial y}(x,y) &= \sin(x^2) - \sin(y)
\end{aligned}$$

Integrating the first of these equations gives

$$f(x,y) = y\sin(x^2) + g(y)$$

Substituting this into the second equation gives

$$\sin(x^2) + g'(y) = \sin(x^2) - \sin(y) \qquad \text{or} \qquad g'(y) = -\sin(y)$$

which integrates to

$$g(y) = \cos(y) + C$$

with C an arbitrary constant. Hence $f(x,y) = y\sin(x^2) + \cos(y) + C$ is a potential for any constant C. Because $\mathbf{F}$ has a potential, it is conservative.

(c) We may parametrize C by

$$\mathbf{r}(t) = \sin(t)\,\hat{\boldsymbol{\imath}} + t\,\hat{\boldsymbol{\jmath}} \qquad \frac{\pi}{2} \leqslant t \leqslant \pi$$

As $f(x,y) = y\sin(x^2) + \cos(y)$ is a potential for $\mathbf{F}$

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = f\big(\mathbf{r}(\pi)\big) - f\big(\mathbf{r}(\pi/2)\big) = f\big(0,\pi\big) - f\big(1,\pi/2\big) = \big(-1\big) - \Big(\frac{\pi}{2}\sin(1)\Big)$$
$$= -1 - \frac{\pi}{2}\sin(1)$$

S-31: (a) The stated integral property is characteristic of conservative fields (Theorem 2.4.6). Since all partial derivatives of $\mathbf{F}$ are defined on all of $\mathbb{R}^3$, an equivalent property is

$$\mathbf{0} = \nabla\times\mathbf{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (mxyz + z^2 - ny^2) & (x^2z - 4xy) & (x^2y + pxz + qz^3) \end{vmatrix}$$
$$= \hat{\boldsymbol{\imath}}\left(x^2 - x^2\right) - \hat{\boldsymbol{\jmath}}\,(2xy + pz - mxy - 2z) + \hat{\mathbf{k}}\,(2xz - 4y - mxz + 2ny).$$

This requires $p = 2$, $m = 2$, and $n = 2$, but leaves $q \in \mathbb{R}$ completely free.

(b) **Solution 1:**
The choices from (a) give

$$\mathbf{F} = (2xyz + z^2 - 2y^2)\,\hat{\boldsymbol{\imath}} + (x^2z - 4xy)\,\hat{\boldsymbol{\jmath}} + (x^2y + 2xz + qz^3)\,\hat{\mathbf{k}}.$$

We find a potential φ for $\mathbf{F}$ by antidifferentiating its partial derivatives.

$$\frac{\partial\varphi}{\partial x}(x,y,z) = 2xyz + z^2 - 2y^2 \qquad \Longrightarrow \varphi(x,y,z) = x^2yz + xz^2 - 2xy^2 + \psi_1(y,z)$$
$$\frac{\partial\varphi}{\partial y}(x,y,z) = x^2z - 4xy \qquad \Longrightarrow \varphi(x,y,z) = x^2yz - 2xy^2 + \psi_2(x,z)$$
$$\frac{\partial\varphi}{\partial z}(x,y,z) = x^2y + 2xz + qz^3 \qquad \Longrightarrow \varphi(x,y,z) = x^2yz + xz^2 + \frac{q}{4}z^4 + \psi_3(x,y)$$

All together, $\mathbf{F} = \nabla\varphi$ for

$$\varphi(x,y,z) = x^2yz + xz^2 - 2xy^2 + \frac{1}{4}qz^4 + C$$

where C is any constant.

Rearranging the sphere's equation to $x^2 + y^2 + (z-1)^2 = 1$ reveals that its bottom is at $\mathbf{r}_0 = (0,0,0)$, and its top is at $\mathbf{r}_1 = (0,0,2)$. Hence the work done is

$$W = \int_{\mathcal{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_{\mathcal{C}} \nabla\varphi\cdot\mathrm{d}\mathbf{r} = \varphi(0,0,2) - \varphi(0,0,0) = 4q$$

Solution 2:
Since the integral is path-independent, all paths from $\mathbf{r}_0$ to $\mathbf{r}_1$ produce the same result. A simple choice is

$$\mathcal{C}: \qquad \mathbf{r} = (0,0,t), \quad 0 \leqslant t \leqslant 2$$

Here $\mathbf{r}'(t) = (0, 0, 1)$, so direct calculation gives

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{t=0}^{2} \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t)\, \mathrm{d}t = \int_{t=0}^{2} qt^3\, \mathrm{d}t = \Big[\frac{1}{4}qt^4\Big]_{t=0}^{2} = 4q$$

S-32: (a) Parametrize C by x. When the first component of a point on the curve is x, then the second component, y, must be x^2 and the third component, z, must be x^3. So

$$\begin{aligned} \mathbf{r}(x) &= x\,\hat{\boldsymbol{\imath}} + x^2\,\hat{\boldsymbol{\jmath}} + x^3\,\hat{\mathbf{k}} \qquad 0 \leqslant x \leqslant 1 \\ \mathbf{r}'(x) &= \hat{\boldsymbol{\imath}} + 2x\,\hat{\boldsymbol{\jmath}} + 3x^2\,\hat{\mathbf{k}} \\ \frac{\mathrm{d}s}{\mathrm{d}x}(x) &= \sqrt{1 + 4x^2 + 9x^4} \end{aligned}$$

and

$$\begin{aligned} \rho(x)\,\frac{\mathrm{d}s}{\mathrm{d}x}(x) &= \big(8x + 36x^3\big)\sqrt{1 + 4x^2 + 9x^4} \\ \int_C \rho\, \mathrm{d}s &= \int_0^1 \big(8x + 36x^3\big)\sqrt{1 + 4x^2 + 9x^4}\, \mathrm{d}x \end{aligned}$$

Substituting $u = 1 + 4x^2 + 9x^4$, $\mathrm{d}u = \big(8x + 36x^3\big)\, \mathrm{d}x$, $u(0) = 1$, $u(1) = 14$,

$$\begin{aligned} \int_C \rho\, \mathrm{d}s &= \int_1^{14} \sqrt{u}\, \mathrm{d}u = \frac{2}{3}u^{3/2}\Big|_1^{14} \\ &= \frac{2}{3}\big[14^{3/2} - 1\big] \approx 34.26 \end{aligned}$$

(b) Since $\mathbf{F}(x, y, z) = \nabla f(x, y, z)$ with $f(x, y, z) = x \sin y + yz + \frac{1}{2}z^2$,

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = f(1, 1, 1) - f(0, 0, 0) = \sin 1 + \frac{3}{2} \approx 2.3415$$

The potential f was just guessed. Alternatively, it can be found by solving

$$\begin{aligned} \frac{\partial f}{\partial x}(x, y, z) &= \sin y \\ \frac{\partial f}{\partial y}(x, y, z) &= x \cos y + z \\ \frac{\partial f}{\partial z}(x, y, z) &= y + z \end{aligned}$$

Integrating the first of these equations gives

$$f(x, y, z) = x \sin y + g(y, z)$$

Substituting this into the second equation gives

$$x \cos y + \frac{\partial g}{\partial y}(y, z) = x \cos y + z \qquad \text{or} \qquad \frac{\partial g}{\partial y}(x, z) = z$$

which forces

$$g(y,z) = yz + h(z)$$

Finally, substituting $f(x,y,z) = x\sin y + yz + h(z)$ into the last equation gives

$$y + h'(z) = y + z \quad \text{or} \quad h'(z) = z$$

So $h(x) = \frac{z^2}{2} + C$ and hence $f(x,y,z) = x\sin y + yz + \frac{z^2}{2} + C$ for any constant C.

S-33: First, we'll parametrize (x,y), which wraps once, counterclockwise, aroung the circle $x^2 + y^2 = 1$. So $x(t) = \cos t$, $y(t) = \sin t$, $0 \leqslant t \leqslant 2\pi$ works. As (x,y) wraps around the circle, z has to start at 0 (when $t = 0$) and end at 1 (when $t = 2\pi$). So $z(t) = \frac{t}{2\pi}$ works and our parametrization is

$$\mathbf{r}(t) = \cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}} + \frac{t}{2\pi}\,\hat{\mathbf{k}}$$

(Compare to Example 1.4.4 in the CLP-4 text.) With this parametrization

$$\begin{aligned}
\mathbf{r}'(t) &= -\sin t\,\hat{\boldsymbol{\imath}} + \cos t\,\hat{\boldsymbol{\jmath}} + \frac{1}{2\pi}\,\hat{\mathbf{k}} \\
\mathbf{F}\big(x(t),y(t),z(t)\big) &= -\sin t\,\hat{\boldsymbol{\imath}} + \cos t\,\hat{\boldsymbol{\jmath}} + \frac{t^2}{4\pi^2}\hat{\mathbf{k}} \\
\mathbf{F}\big(x(t),y(t),z(t)\big)\cdot\mathbf{r}'(t) &= 1 + \frac{t^2}{8\pi^3}
\end{aligned}$$

and

$$\begin{aligned}
\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_0^{2\pi} \mathbf{F}\big(x(t),y(t),z(t)\big)\cdot\mathbf{r}'(t)\,\mathrm{d}t = \int_0^{2\pi}\left(1 + \frac{t^2}{8\pi^3}\right)\,\mathrm{d}t \\
&= 2\pi + \frac{1}{3}
\end{aligned}$$

S-34: (a) Let's evaluate the integral directly using the parametrization

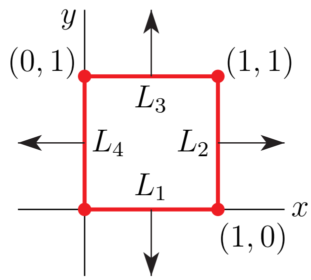
$$\mathbf{r}(x) = x\,\hat{\boldsymbol{\imath}} + (9 - x^2)\,\hat{\boldsymbol{\jmath}}$$

with $-3 \leqslant x \leqslant 3$.

Since $\mathbf{r}'(x) = \hat{\boldsymbol{\imath}} - 2x\,\hat{\boldsymbol{\jmath}}$,

$$\begin{aligned}
\int_C (x^2 + y)\,\mathrm{d}x + x\,\mathrm{d}y &= \int_{-3}^{3}\Big(x^2 + \overbrace{9 - x^2}^{y} + x\overbrace{(-2x)}^{\frac{\mathrm{d}y}{\mathrm{d}x}}\Big)\,\mathrm{d}x = \int_{-3}^{3}(9 - 2x^2)\,\mathrm{d}x \\
&= 2\int_0^3 (9 - 2x^2)\,\mathrm{d}x = 2\left(27 - 2\frac{3^3}{3}\right) = 18
\end{aligned}$$

(b) In this solution, we'll evaluate the integral directly. Label the four sides of the square L_1, L_2, L_3 and L_4 as in the figure



The parametrization of L_1 by arc length is $\mathbf{r}(s) = s\,\hat{\boldsymbol{\imath}}$, $0 \leqslant s \leqslant 1$. As the outward pointing normal to L_1 is $-\hat{\boldsymbol{\jmath}}$,

$$\int_{L_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}s = \int_0^1 \mathbf{F}(s,0)\cdot(-\hat{\boldsymbol{\jmath}})\,\mathrm{d}s = \int_0^1 (-0)\,\mathrm{d}s = 0$$

The parametrization of L_2 by arc length is $\mathbf{r}(s) = \hat{\boldsymbol{\imath}} + s\,\hat{\boldsymbol{\jmath}}$, $0 \leqslant s \leqslant 1$. As the outward pointing normal to L_2 is $\hat{\boldsymbol{\imath}}$,

$$\int_{L_2} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}s = \int_0^1 \mathbf{F}(1,s)\cdot\hat{\boldsymbol{\imath}}\,\mathrm{d}s = \int_0^1 2\,\mathrm{d}s = 2$$

The parametrization of L_3 by arc length (starting at $(1,1)$) is $\mathbf{r}(s) = (1-s)\,\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$, $0 \leqslant s \leqslant 1$. As the outward pointing normal to L_3 is $\hat{\boldsymbol{\jmath}}$,

$$\int_{L_3} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}s = \int_0^1 \mathbf{F}(1-s,1)\cdot\hat{\boldsymbol{\jmath}}\,\mathrm{d}s = \int_0^1 e^{1-s}\,\mathrm{d}s = \Big[-e^{1-s}\Big]_0^1 = e-1$$

The parametrization of L_4 by arc length (starting at $(0,1)$) is $\mathbf{r}(s) = (1-s)\,\hat{\boldsymbol{\jmath}}$, $0 \leqslant s \leqslant 1$. As the outward pointing normal to L_4 is $-\hat{\boldsymbol{\imath}}$,

$$\int_{L_2} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}s = \int_0^1 \mathbf{F}(0,1-s)\cdot(-\hat{\boldsymbol{\imath}})\,\mathrm{d}s = \int_0^1 (-0)\,\mathrm{d}s = 0$$

All together

$$\begin{aligned}\int_C \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}s &= \int_{L_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}s + \int_{L_2} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}s + \int_{L_3} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}s + \int_{L_4} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}s \\ &= 0 + 2 + (e-1) + 0 = e+1\end{aligned}$$

S-35: (a) Since $m = 1$, Newton's law of motion gives

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{F}(t) = \hat{\boldsymbol{\jmath}} - \sin t\,\hat{\mathbf{k}}$$

Integating gives

$$\mathbf{v}(t) = t\,\hat{\boldsymbol{\jmath}} + \cos t\,\hat{\mathbf{k}} + \mathbf{c}$$

for some constant vector $\mathbf{c}$. Since $\mathbf{v}(0) = \hat{\boldsymbol{\imath}} + \hat{\mathbf{k}}$, we have $\mathbf{c} = \hat{\boldsymbol{\imath}}$ so that

$$\mathbf{r}'(t) = \mathbf{v}(t) = \hat{\boldsymbol{\imath}} + t\,\hat{\boldsymbol{\jmath}} + \cos t\,\hat{\mathbf{k}}$$

Integating again gives

$$\mathbf{r}(t) = t\,\hat{\boldsymbol{\imath}} + \frac{t^2}{2}\,\hat{\boldsymbol{\jmath}} + \sin t\,\hat{\mathbf{k}} + \mathbf{c}$$

for some (new) constant vector $\mathbf{c}$. Since $\mathbf{r}(0) = \hat{\boldsymbol{\jmath}}$, we have $\mathbf{c} = \hat{\boldsymbol{\jmath}}$ so that

$$\mathbf{r}(t) = t\,\hat{\boldsymbol{\imath}} + \left(1 + \frac{t^2}{2}\right)\,\hat{\boldsymbol{\jmath}} + \sin t\,\hat{\mathbf{k}}$$

(b) The particle has $x = \pi/2$ when $t = \pi/2$ and then

$$\mathbf{r}_1 = \mathbf{r}(\pi/2) = \frac{\pi}{2}\,\hat{\boldsymbol{\imath}} + \left(1 + \frac{\pi^2}{8}\right)\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}$$

(c) The work done between time $t = 0$ and time $t = \pi/2$ is

$$\begin{aligned}
\int_0^{\pi/2} \mathbf{F}(t)\cdot \mathrm{d}\mathbf{r} &= \int_0^{\pi/2} \mathbf{F}(t)\cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\,\mathrm{d}t = \int_0^{\pi/2} [\hat{\boldsymbol{\jmath}} - \sin t\,\hat{\mathbf{k}}]\cdot[\hat{\boldsymbol{\imath}} + t\,\hat{\boldsymbol{\jmath}} + \cos t\,\hat{\mathbf{k}}]\,\mathrm{d}t \\
&= \int_0^{\pi/2} [t - \sin t\cos t]\,\mathrm{d}t = \left[\frac{t^2}{2} + \frac{1}{2}\cos^2 t\right]_0^{\pi/2} = \frac{\pi^2}{8} - \frac{1}{2}
\end{aligned}$$

S-36: (a) We can parametrize L by

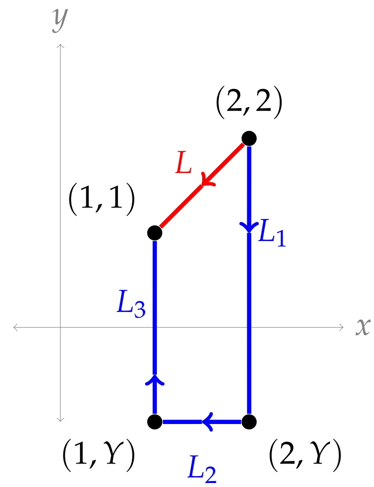
$$\mathbf{r}(t) = \big(x(t), y(t)\big) = (t, t),$$

with t running from 2 to 1. Using this parametrization,

$$\begin{aligned}
\int_L \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_2^1 \mathbf{F}\big(x(t), y(t)\big)\cdot \big(x'(t), y'(t)\big)\,\mathrm{d}t = \int_2^1 \big(3t\,,\, t-1\big)\cdot\big(1,1\big)\,\mathrm{d}t \\
&= \int_2^1 (4t - 1)\,\mathrm{d}t = -5
\end{aligned}$$

(b) First, we note that such a choice of path is even possible: if $\mathbf{F}$ were conservative, then $\int_c \mathbf{F}\cdot \mathrm{d}\mathbf{r}$ would be -5 for every path starting at $(2,2)$ and ending at $(1,1)$, because it would be path independent. Since $\frac{\partial F_1}{\partial y} = 3$ and $\frac{\partial F_2}{\partial x} = 1 \neq \frac{\partial F_1}{\partial y}$, by Theorem 2.4.6, $\mathbf{F}$ is *not* path-independent.

Solution 1:



Let's try a family of polygonal paths C_Y that consist of

- the line segment L_1 from $(2,2)$ to $(2,Y)$ followed by
- the line segment L_2 from $(2,Y)$ to $(1,Y)$ followed by
- the line segment L_3 from $(1,Y)$ to $(1,1)$.

This is a way of characterizing a family of alternate paths with only one parameter, Y. We are hoping that the value of the integral $\int_{C_Y} \mathbf{F}\cdot d\mathbf{r}$ depends on Y and that we can choose a specific value of Y so as to make the value of the integral $\int_{C_Y} \mathbf{F}\cdot d\mathbf{r}$ exactly 4.

Note that

- On L_1, $x = 2$ is a constant (so that $dx = 0$) and y runs from 2 to Y.
- On L_2, $y = Y$ is a constant (so that $dy = 0$) and x runs from 2 to 1.
- On L_3, $x = 1$ is a constant (so that $dx = 0$) and y runs from Y to 1

So,

$$\begin{aligned}\int_{C_Y} \mathbf{F}\cdot d\mathbf{r} &= \int_{L_1}\{(3y\,dx + (x-1)dy\} + \int_{L_2}\{(3y\,dx + (x-1)dy\} + \int_{L_3}\{(3y\,dx + (x-1)dy\} \\ &= \int_2^Y dy + \int_2^1 3Y\,dx + \int_Y^1 0\,dy \\ &= (Y-2) + 3Y(1-2) = -2Y - 2\end{aligned}$$

Since we want our integral to be 4, we set $4 = -2Y - 2$, and find $Y = -3$. That is, the path D consisting of line segments from $(2,2)$ to $(2,-3)$ to $(1,-3)$ to $(1,1)$ gives us $\int_D \mathbf{F}\cdot d\mathbf{r} = 4$.

Solution 2: Choosing three straight line segments was a convenient way to solve this, but not the only way. To emphasize this point, we show that we also could have considered (for example) the family of parabolas that pass through $(2,2)$ and $(1,1)$.

That is, we consider the family of functions $y = ax^2 + bx + c$ with $2 = 4a + 2b + c$ and $1 = a + b + c$. Subtracting the equation $a + b + c = 1$ from the equation $4a + 2b + c = 2$

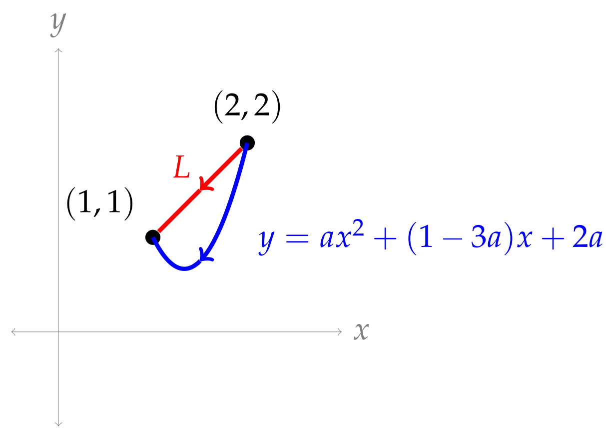
(in order to eliminate c) gives

$$\begin{aligned} (4a+2b+c)-(a+b+c) &= (2)-(1) \\ \implies \qquad 3a+b &= 1 \\ \implies \qquad b &= 1-3a \end{aligned}$$

Using $b = 1 - 3a$,

$$\begin{aligned} a+b+c &= 1 \\ \implies \qquad a+(1-3a)+c &= 1 \\ \implies \qquad c &= 2a \end{aligned}$$

So, the class of functions described by $y = ax^2 + (1-3a)x + 2a$ for some constant a are parabolas that pass through $(1,1)$ and $(2,2)$.



So, we consider paths of the form:

$$\begin{aligned} \mathbf{r}(x) &= \big(x,\ ax^2+(1-3a)x+2a\big) \\ \mathbf{F}(\mathbf{r}(x)) &= \big(3ax^2+3(1-3a)x+6a,\ x-1\big) \\ \mathbf{r}'(x) &= \big(1,\ 2ax+1-3a\big) \\ \mathbf{F}(\mathbf{r}(x))\cdot\mathbf{r}'(x) &= \big(3ax^2+3(1-3a)x+6a\big)\ +\ \big(2ax^2+(1-3a)x-2ax+(3a-1)\big) \\ &= 5ax^2+(4-14a)x+(9a-1) \end{aligned}$$

So, if C is a portion of this parabola from $(2,2)$ to $(1,1)$, then

$$\begin{aligned} \int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_2^1 \big(5ax^2+(4-14a)x+(9a-1)\big)\,\mathrm{d}x \\ &= \left[\frac{5a}{3}x^3+(2-7a)x^2+(9a-1)x\right]_2^1 \\ &= \frac{a}{3}-5 \end{aligned}$$

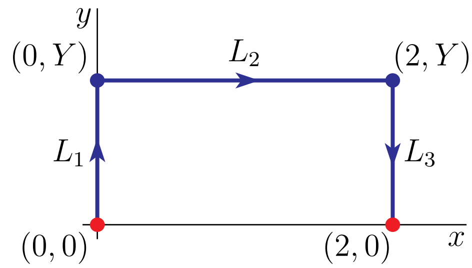
Since we want our integral to have value 4, we set $4 = \frac{a}{3} - 5$, which yields $a = 27$.

If we choose C to be the path from $(2,2)$ to $(1,1)$ along the parabola $27x^2 - 80x + 54$, then $\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = 4$, as desired.

S-37: **Solution 1:**
Let's try a family of polygonal paths C_Y (sketched below) that consist of

- the line segment L_1 from $(0,0)$ to $(0,Y)$ followed by
- the line segment L_2 from $(0,Y)$ to $(2,Y)$ followed by
- the line segment L_3 from $(2,Y)$ to $(2,0)$.

Here Y is a parameter. We are hoping that the value of the integral $\int_{C_Y} \mathbf{F}\cdot d\mathbf{r}$ depends on



Y and that we can choose a specific value of Y so as to make the value of the integral $\int_{C_Y} \mathbf{F}\cdot d\mathbf{r}$ exactly 8. Note that

- on L_1, $x=0$ is a constant (so that $dx=0$) and y runs from 0 to Y and
- on L_2, $y=Y$ is a constant (so that $dy=0$) and x runs from 0 to 2 and
- on L_3, $x=2$ is a constant (so that $dx=0$) and y runs from Y to 0

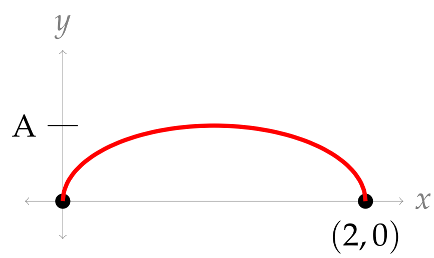
Since $\mathbf{F}\cdot d\mathbf{r} = (2y+2)\,dx$,

$$\begin{aligned}\int_{C_Y} \mathbf{F}\cdot d\mathbf{r} &= \int_{L_1}(2y+2)\,dx + \int_{L_2}(2y+2)\,dx + \int_{L_3}(2y+2)\,dx \\ &= 0 + \int_0^2 (2Y+2)\,dx + 0 \\ &= 2(2Y+2)\end{aligned}$$

So $Y=1$ does the job.

Solution 2:
There's nothing magical about the form of the path from Solution 1. It's just a path that's relatively easy to describe using one constant Y. To emphasize this point, we provide a solution with an alternate path based on an ellipse.

A partial ellipse running from $(0,0)$ to $(2,2)$ can be described by $\mathbf{r}(t) = (\cos t + 1\,,\ A\sin t)$ for a constant A, with t running from π to 0. (To find this: we centre a circle of radius 1 at the point $(1,0)$, then multiply its y-coordinate by A.)



In this case, $\mathbf{F}(\mathbf{r}(t)) = (2A\sin t + 2, 0)$ and $\mathbf{r}'(t) = (-\sin t, A\cos t)$, so

$$\begin{aligned}
\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= -\sin t(2A\sin 2 + 2) = -A(2\sin^2 t) - 2\sin t = -A(1-\cos 2t) - 2\sin t \\
\int \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_\pi^0 \big(A(\cos 2t - 1) - 2\sin t\big)\mathrm{d}t = \left[A\left(\frac{1}{2}\sin(2t) - t\right) + 2\cos t\right]_\pi^0 \\
&= A\pi + 4
\end{aligned}$$

Setting $A\pi + 4 = 8$, we find $A = \frac{4}{\pi}$. So, the half-ellipse $\mathbf{r}(t) = \Big(\cos t + 1\,,\ \frac{4}{\pi}\sin t\Big)$, with t running from π to 0, is another path that gives $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 8$.

S-38: The vector field $\mathbf{F}$ is conservative, with

$$\mathbf{F} = \boldsymbol{\nabla}\varphi \qquad \varphi(x,y) = x + \int_0^y \tilde{y}g(\tilde{y})\,\mathrm{d}\tilde{y}$$

Consquently, for $P = (x_0, 0)$ and $Q = (x_1, 0)$,

$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \varphi(Q) - \varphi(P) = x_1 + \int_0^0 \tilde{y}g(\tilde{y})\,\mathrm{d}\tilde{y} - x_0 - \int_0^0 \tilde{y}g(\tilde{y})\,\mathrm{d}\tilde{y} \\
&= x_1 - x_0
\end{aligned}$$

Thus

$$\left|\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}\right| = |x(Q) - x(P)| = \text{distance between } P \text{ and } Q$$

S-39:

- First notice that the vector field $\tilde{\mathbf{F}}(x,y,z) = z^2\,\hat{\mathbf{k}}$ is conservative (with potential $\frac{1}{3}z^3$), so $\int_{C_1} \tilde{\mathbf{F}} \cdot \mathrm{d}\mathbf{r} = \int_{C_2} \tilde{\mathbf{F}} \cdot \mathrm{d}\mathbf{r}$ for any two curves C_1 and C_2 from P_1 to P_2 (whether or not they are on the surface S). Consequently, the statement "$\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$" is true if and only if the statement "$\int_{C_1} (\mathbf{F} - \tilde{\mathbf{F}}) \cdot \mathrm{d}\mathbf{r} = \int_{C_2} (\mathbf{F} - \tilde{\mathbf{F}}) \cdot \mathrm{d}\mathbf{r}$" is true. So we may replace the vector field $\mathbf{F}$ with the vector field

$$\mathbf{G}(x,y,z) = \mathbf{F}(x,y,z) - \tilde{\mathbf{F}}(x,y,z) = (xz + axy^2)\hat{\boldsymbol{\imath}} + yz\hat{\boldsymbol{\jmath}}$$

- We are to consider only curves on the surface S. For any such curve C, say parametrized by $\mathbf{r}(t)$ with $a \leqslant t \leqslant b$, the integral

$$\int_C \mathbf{G} \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{G}\big(\mathbf{r}(t)\big) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\,\mathrm{d}t$$

depends only on the values of $\mathbf{G}$ on the surface S. In particular, if another vector field $\mathbf{H}$ obeys $\mathbf{H}(x,y,z) = \mathbf{G}(x,y,z)$, for all points (x,y,z) on S, we have

$$\int_C \mathbf{G} \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{G}\big(\mathbf{r}(t)\big) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\,\mathrm{d}t = \int_a^b \mathbf{H}\big(\mathbf{r}(t)\big) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\,\mathrm{d}t = \int_C \mathbf{H} \cdot \mathrm{d}\mathbf{r}$$

So we may replace $\mathbf{G}$ with

$$\begin{aligned}\mathbf{H}(x,y,z) = \mathbf{G}(x,y,2+x^2-3y^2) &= \big[x(2+x^2-3y^2)+axy^2\big]\hat{\imath} + y(2+x^2-3y^2)\hat{\jmath}\\ &= (2x+x^3-3xy^2+axy^2)\hat{\imath} + (2y+yx^2-3y^3)\hat{\jmath}\end{aligned}$$

Note that $\mathbf{H}(x,y,z)$ is defined on all of $\mathbb{R}^3$. It just happens to not depend on z.

- The curl of $\mathbf{H}$ is

$$\begin{aligned}\nabla\times\mathbf{H} &= \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2x+x^3-3xy^2+axy^2 & 2y+yx^2-3y^3 & 0\end{bmatrix}\\ &= \big(2xy-[-6xy+2axy]\big)\hat{\mathbf{k}} = (8-2a)xy\,\hat{\mathbf{k}}\end{aligned}$$

This is zero if $a=4$. As $\mathbf{H}$ has continuous first order partial derivatives on all of $\mathbb{R}^3$, Theorem 2.4.7 of the CLP-4 text tells us that, when $a=4$, $\mathbf{H}$ is conservative and that $\int_{C_1}\mathbf{H}\cdot d\mathbf{r} = \int_{C_2}\mathbf{H}\cdot d\mathbf{r}$ for any two curves C_1 and C_2 from P_1 to P_2

So $a=4$ does the job.

S-40: (a) The curl of $\mathbf{F}$ is

$$\begin{aligned}\nabla\times\mathbf{F} &= \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ (1+ax^2)ye^{3x^2}-bxz\cos(x^2z) & xe^{3x^2} & x^2\cos(x^2z)\end{bmatrix}\\ &= 0\,\hat{\imath} + [-bx\cos(x^2z)+bx^3z\sin(x^2z)-2x\cos(x^2z)+2x^3z\sin(x^2z)]\,\hat{\jmath}\\ &\qquad + [e^{3x^2}+6x^2e^{3x^2}-(1+ax^2)e^{3x^2}]\,\hat{\mathbf{k}}\\ &= [-(b+2)x\cos(x^2z)+(b+2)x^3z\sin(x^2z)]\,\hat{\jmath} + (6-a)x^2e^{3x^2}\,\hat{\mathbf{k}}\end{aligned}$$

(b) For $\mathbf{F}$ to be conservative it is necessary that $\nabla\times\mathbf{F}=\mathbf{0}$. This is the case when $b=-2$ and $a=6$.

(c) For f to be a potential, when $b=-2$ and $a=6$, we need

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y,z) &= (1+6x^2)ye^{3x^2}+2xz\cos(x^2z)\\ \frac{\partial f}{\partial y}(x,y,z) &= xe^{3x^2}\\ \frac{\partial f}{\partial z}(x,y,z) &= x^2\cos(x^2z)\end{aligned}$$

Integrating the second of these equations gives

$$f(x,y,z) = xye^{3x^2}+g(x,z)$$

Substituting this into the last equation gives

$$\frac{\partial g}{\partial z}(x,z) = x^2\cos(x^2z)$$

which integrates to

$$g(x,z) = \sin(x^2 z) + h(x)$$

Finally, substituting $f(x,y,z) = xye^{3x^2} + \sin(x^2z) + h(x)$ into the first equation gives

$$(1+6x^2)ye^{3x^2} + 2xz\cos(x^2z) + h'(x) = (1+6x^2)ye^{3x^2} + 2xz\cos(x^2z) \quad \text{or} \quad h'(x) = 0$$

So $h(x) = C$ and hence $f(x,y,z) = xye^{3x^2} + \sin(x^2z) + C$ works for any constant C.

(d) Note that the integral is $\int_C \big(\mathbf{F}_{a=6,b=-2} - 6x^2ye^{3x^2}\hat{\boldsymbol{\imath}}\big) \cdot \mathrm{d}\mathbf{r}$. So

$$\begin{aligned}
\int_C \Big(ye^{3x^2} + 2xz\cos(x^2z)\Big)\mathrm{d}x + xe^{3x^2}\mathrm{d}y + x^2\cos(x^2z)\mathrm{d}z &= \int_C \boldsymbol{\nabla} f \cdot \mathrm{d}\mathbf{r} - 6\int_C x^2ye^{3x^2}\,\mathrm{d}x \\
&= f(1,1,1) - f(0,0,0) - 6\int_0^1 t^3e^{3t^2}\,\mathrm{d}t \\
&= e^3 + \sin 1 - \frac{1}{3}\int_0^3 ue^u\,\mathrm{d}u \qquad \text{with } u = 3t^2,\ \mathrm{d}u = 6t\,\mathrm{d}t \\
&= e^3 + \sin 1 - \frac{1}{3}\Big[ue^u - e^u\Big]_0^3 \qquad \text{integration by parts} \\
&= \frac{1}{3}e^3 + \sin 1 - \frac{1}{3}
\end{aligned}$$

S-41: (a) Parametrize C by x. Then

$$\begin{aligned}
\mathbf{r}(x) &= x\,\hat{\boldsymbol{\imath}} + x^2\,\hat{\boldsymbol{\jmath}} + x^3\,\hat{\mathbf{k}} \qquad 0 \leqslant x \leqslant 1 \\
\mathbf{r}'(x) &= \hat{\boldsymbol{\imath}} + 2x\,\hat{\boldsymbol{\jmath}} + 3x^2\,\hat{\mathbf{k}} \\
\mathbf{F}\big(\mathbf{r}(x)\big) \cdot \mathbf{r}'(x) &= \big((x^4 - x^2)\,\hat{\boldsymbol{\imath}} + (x + x^3)\,\hat{\boldsymbol{\jmath}} + x^2\,\hat{\mathbf{k}}\big) \cdot \big(\hat{\boldsymbol{\imath}} + 2x\,\hat{\boldsymbol{\jmath}} + 3x^2\,\hat{\mathbf{k}}\big) \\
&= x^4 - x^2 + 2x^2 + 2x^4 + 3x^4 = x^2 + 6x^4 \\
\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_0^1 \big[x^2 + 6x^4\big]\,\mathrm{d}x = \Big[\frac{x^3}{3} + \frac{6x^5}{5}\Big]_0^1 = \frac{23}{15} = 1.5\dot{3}
\end{aligned}$$

(b) Parametrize C by x as in part (a). Then

$$\begin{aligned}
\frac{\mathrm{d}s}{\mathrm{d}x} &= \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}x}\right| = \sqrt{1 + 4x^2 + 9x^4} \\
\rho(x, x^2, x^3)\,\frac{\mathrm{d}s}{\mathrm{d}x} &= \big(8x + 36x^3\big)\sqrt{1 + 4x^2 + 9x^4} \\
\int_C \rho\,\mathrm{d}s &= \int_0^1 \big(8x + 36x^3\big)\sqrt{1 + 4x^2 + 9x^4}\,\mathrm{d}x
\end{aligned}$$

Using the substitution $u = 1 + 4x^2 + 9x^4$, $\mathrm{d}u = (8x + 36x^3)\,\mathrm{d}x$:

$$\begin{aligned}
&= \frac{2}{3}\big[1 + 4x^2 + 9x^4\big]^{3/2}\Big|_0^1 \\
&= \frac{2}{3}\big[14^{3/2} - 1\big] \approx 34.26
\end{aligned}$$

(c) Since $\mathbf{F} = \nabla f$ with $f = x\sin y + yz + \frac{1}{2}z^2$,

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = f(1,1,1) - f(0,0,0) = \sin 1 + \frac{3}{2} \approx 2.3415$$

The potential f was just guessed. Alternatively, it can be found by antidifferentiating:

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y,z) &= \sin y &&\implies f(x,y,z) = x\sin y + \psi_1(y,z)\\
\frac{\partial f}{\partial y}(x,y,z) &= x\cos y + z &&\implies f(x,y,z) = x\sin y + yz + \psi_2(x,z)\\
\frac{\partial f}{\partial z}(x,y,z) &= y + z &&\implies f(x,y,z) = yz + \frac{1}{2}z^2 + \psi_3(x,y)
\end{aligned}$$

All together, $f(x,y,z) = x\sin y + yz + \frac{z^2}{2} + C$ works for any constant C.

S-42: (a) This field is conservative if and only if it passes the screening test $\nabla \times \mathbf{F} = \mathbf{0}$. That is, if and only if,

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

That is,

$$\begin{aligned}
\frac{\partial}{\partial y}(Ax^3y^2z) &= \frac{\partial}{\partial x}\big(z^3 + Bx^4yz\big) &&\iff & 2Ax^3yz &= 4Bx^3yz\\
\frac{\partial}{\partial z}(Ax^3y^2z) &= \frac{\partial}{\partial x}\big(3yz^2 - x^4y^2\big) &&\iff & Ax^3y^2 &= -4x^3y^2\\
\frac{\partial}{\partial z}\big(z^3 + Bx^4yz\big) &= \frac{\partial}{\partial y}\big(3yz^2 - x^4y^2\big) &&\iff & 3z^2 + Bx^4y &= 3z^2 - 2x^4y
\end{aligned}$$

Hence only $A = -4$, $B = -2$ works.

(b) When $A = -4,\ B = -2$

$$\mathbf{F} = -4x^3y^2z\,\hat{\imath} + \big(z^3 - 2x^4yz\big)\,\hat{\jmath} + \big(3yz^2 - x^4y^2\big)\,\hat{\mathbf{k}}$$

We find a potential function $\varphi(x,y,z)$ for this $\mathbf{F}$ by antidifferentiating.

$$\begin{aligned}
\frac{\partial \varphi}{\partial x}(x,y,z) &= -4x^3y^2z &&\implies \varphi(x,y,z) = -x^4y^2z + \psi_1(y,z)\\
\frac{\partial \varphi}{\partial y}(x,y,z) &= z^3 - 2x^4yz &&\implies \varphi(x,y,z) = yz^3 - x^4y^2z + \psi_2(x,z)\\
\frac{\partial \varphi}{\partial z}(x,y,z) &= 3yz^2 - x^4y^2 &&\implies \varphi(x,y,z) = yz^3 - x^4y^2z + \psi_3(x,y)
\end{aligned}$$

All together, $\varphi(x,y,z) = -x^4y^2z + yz^3 + C$ with C being an arbitrary constant.

(c) $I = \varphi(1,-1,1) - \varphi(0,0,0) = -2$.

(d) Note that $J = \int_{\mathcal{C}} \mathbf{G} \cdot \mathrm{d}\mathbf{r}$ with

$$\begin{aligned}\mathbf{G} &= (z - 4x^3y^2z)\hat{\boldsymbol{\imath}} + (z^3 - x^4yz)\hat{\boldsymbol{\jmath}} + (3yz^2 - x^4y^2)\hat{\mathbf{k}} \\ &= \mathbf{F} + z\,\hat{\boldsymbol{\imath}} + x^4yz\,\hat{\boldsymbol{\jmath}}\end{aligned}$$

so that

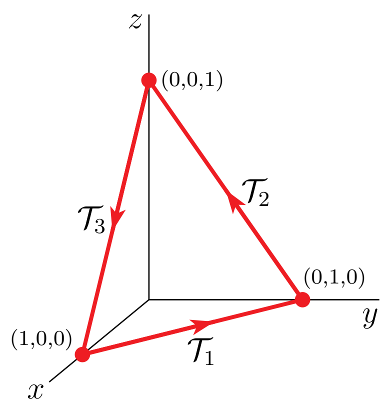
$$J = \int_{\mathcal{C}} (z\hat{\boldsymbol{\imath}} + x^4yz\,\hat{\boldsymbol{\jmath}} + \mathbf{F}) \cdot \mathrm{d}\mathbf{r} = -2 + \int_{\mathcal{C}} (z\hat{\boldsymbol{\imath}} + x^4yz\,\hat{\boldsymbol{\jmath}}) \cdot \mathrm{d}\mathbf{r}$$

Parametrize $\mathcal{C}$ by $\mathbf{r}(x) = x\,\hat{\boldsymbol{\imath}} - x\,\hat{\boldsymbol{\jmath}} + x^2\,\hat{\mathbf{k}}$ with $0 \leqslant x \leqslant 1$. As $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}x} = \hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + 2x\,\hat{\mathbf{k}}$

$$\int_{\mathcal{C}} (z\hat{\boldsymbol{\imath}} + x^4yz\,\hat{\boldsymbol{\jmath}}) \cdot \mathrm{d}\mathbf{r} = \int_0^1 (x^2\hat{\boldsymbol{\imath}} - x^7\,\hat{\boldsymbol{\jmath}}) \cdot (\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + 2x\,\hat{\mathbf{k}})\ \mathrm{d}x = \int_0^1 (x^2 + x^7)\ \mathrm{d}x = \frac{1}{3} + \frac{1}{8} = \frac{11}{24}$$

$$\implies J = -2 + \frac{11}{24} = -\frac{37}{24} \approx -1.5417$$

(e) $\mathcal{T}$ is a closed path and $\mathbf{F}$ is conservative, so $\int_{\mathcal{T}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$. Let $\mathcal{T}_1$ be the line segment from $(1,0,0)$ to $(0,1,0)$, $\mathcal{T}_2$ be the line segment from $(0,1,0)$ to $(0,0,1)$ and $\mathcal{T}_3$ be the line segment from $(0,0,1)$ to $(1,0,0)$.



On $\mathcal{T}_1$, $z = 0$, so $\int_{\mathcal{T}_1} z\hat{\boldsymbol{\imath}} \cdot \mathrm{d}\mathbf{r} = 0$. On $\mathcal{T}_2$, $x = 0$, so $\hat{\boldsymbol{\imath}} \cdot \mathrm{d}\mathbf{r} = \mathrm{d}x = 0$ and $\int_{\mathcal{T}_2} z\hat{\boldsymbol{\imath}} \cdot \mathrm{d}\mathbf{r} = 0$. Parametrize $\mathcal{T}_3$ by $\mathbf{r}(t) = t\hat{\boldsymbol{\imath}} + (1-t)\hat{\mathbf{k}}$, $0 \leqslant t \leqslant 1$. Then $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \hat{\boldsymbol{\imath}} - \hat{\mathbf{k}}$ and the z-coordinate of the path is parametrized by $1 - t$. So,

$$\begin{aligned}\int_{\mathcal{T}} (z\hat{\boldsymbol{\imath}} + \mathbf{F}) \cdot \mathrm{d}\mathbf{r} &= \int_{\mathcal{T}_3} z\hat{\boldsymbol{\imath}} \cdot \mathrm{d}\mathbf{r} = \int_0^1 \overbrace{(1-t)\hat{\boldsymbol{\imath}}}^{z\hat{\boldsymbol{\imath}}} \cdot \overbrace{(\hat{\boldsymbol{\imath}} - \hat{\mathbf{k}})\ \mathrm{d}t}^{\mathrm{d}\mathbf{r}} \\ &= \int_0^1 (1-t)\ \mathrm{d}t = \frac{1}{2}\end{aligned}$$

S-43: (a) By Newton's law of motion

$$m\mathbf{a} = \mathbf{F} \quad \implies \quad 2\mathbf{v}'(t) = \big(4t\,,\, 6t^2\,,\, -4t\big) \quad \implies \quad \mathbf{v}'(t) = \big(2t\,,\, 3t^2\,,\, -2t\big)$$

So

$$\mathbf{v}(t) = \mathbf{v}(0) + \int_0^t \mathbf{v}'(u)\,\mathrm{d}u = (0,0,0) + \int_0^t \big(2u\,,\, 3u^2\,,\, -2u\big)\,\mathrm{d}u = \big(t^2\,,\, t^3\,,\, -t^2\big)$$

(b) From part (a), $\mathbf{r}'(t) = \mathbf{v}(t) = \big(t^2\,,\, t^3\,,\, -t^2\big)$. So

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}(0) + \int_0^t \mathbf{r}'(u)\,\mathrm{d}u = (1,2,3) + \int_0^t \big(u^2\,,\, u^3\,,\, -u^2\big)\,\mathrm{d}u \\ &= (1,2,3) + \big(t^3/3\,,\, t^4/4\,,\, -t^3/3\big) = \left(\frac{t^3}{3}+1\,,\, \frac{t^4}{4}+2\,,\, -\frac{t^3}{3}+3\right)\end{aligned}$$

(c) From parts (a) and (b)

$$|\mathbf{r}'(t)| = \big|t^2(1,t,-1)\big| = t^2\sqrt{2+t^2}$$

and

$$\begin{aligned}\mathbf{r}'(t)\times\mathbf{r}''(t) &= \det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ t^2 & t^3 & -t^2 \\ 2t & 3t^2 & -2t\end{bmatrix} \\ &= \big(-2t^4+3t^4\big)\,\hat{\boldsymbol{\imath}} - \big(-2t^3+2t^3\big)\,\hat{\boldsymbol{\jmath}} + \big(3t^4-2t^4\big)\,\hat{\mathbf{k}} \\ &= t^4\,\hat{\boldsymbol{\imath}} + t^4\,\hat{\mathbf{k}} \\ \implies \big|\mathbf{r}'(t)\times\mathbf{r}''(t)\big| &= \sqrt{2}\,t^4\end{aligned}$$

The curvature is (see §1.5 of the CLP-4 text)

$$\begin{aligned}\kappa(t) &= \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{2}\,t^4}{\big(t^2\sqrt{2+t^2}\big)^3} \\ &= \frac{\sqrt{2}}{t^2\big(2+t^2\big)^{3/2}}\end{aligned}$$

(d) $W = \int \mathbf{F}\cdot\mathrm{d}\mathbf{r}$:

$$\begin{aligned}\int_{t=0}^{t=T} \mathbf{F}(t)\cdot\mathrm{d}\mathbf{r} &= \int_0^T \mathbf{F}(t)\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\,\mathrm{d}t = \int_0^T \big(4t\,,\, 6t^2\,,\, -4t\big)\cdot\big(t^2\,,\, t^3\,,\, -t^2\big)\,\mathrm{d}t \\ &= \int_0^T \big(8t^3+6t^5\big)\,\mathrm{d}t = 2T^4 + T^6\end{aligned}$$

S-44: (a) For the specified curve

$$\begin{aligned}\mathbf{r}(t) &= \Big(\frac{4\sqrt{2}}{3}t^{3/2}, \frac{4\sqrt{2}}{3}t^{3/2}, t(2-t)\Big) \\ \mathbf{v}(t) = \mathbf{r}'(t) &= \big(2\sqrt{2}t^{1/2}, 2\sqrt{2}t^{1/2}, 2-2t\big) \\ |\mathbf{v}| &= \sqrt{8t+8t+4-8t+4t^2} = \sqrt{4(1+2t+t^2)} = 2|1+t| = 2(1+t)\end{aligned}$$

So the distance travelled is

$$\int_0^2 |\mathbf{v}(t)|\,\mathrm{d}t = \int_0^2 2(1+t)\,\mathrm{d}t = 2\Big[t + \frac{t^2}{2}\Big]_0^2 = 8$$

(b) As

$$\begin{aligned}
\mathbf{v}(t) &= \mathbf{r}'(t) = \big(2\sqrt{2}t^{1/2}, 2\sqrt{2}t^{1/2}, 2-2t\big) & \mathbf{v}(1) &= 2\sqrt{2}\big(1,1,0\big)\\
\mathbf{a}(t) &= \mathbf{v}'(t) = \big(\sqrt{2}t^{-1/2}, \sqrt{2}t^{-1/2}, -2\big) & \mathbf{a}(1) &= \sqrt{2}\big(1,1,-\sqrt{2}\big)\\
\mathbf{v}(1)\times\mathbf{a}(1) &= 4\big(-\sqrt{2},\sqrt{2},0\big) & |\mathbf{v}(1)| &= 4
\end{aligned}$$

the curvature

$$\kappa(1) = \frac{|\mathbf{v}(1)\times\mathbf{a}(1)|}{|\mathbf{v}(1)|^3} = \frac{8}{4^3} = \frac{1}{8}$$

(c) $\mathbf{G} = \nabla\varphi$ with $\varphi(x,y,z) = -Mgz$, so that gravity is conservative. The work done is

$$\varphi\big(\mathbf{r}(2)\big) - \varphi\big(\mathbf{r}(0)\big) = \varphi\big({}^{16}\!/_3,\ {}^{16}\!/_3,\ 0\big) - \varphi(0,0,0) = 0$$

Friction is not conservative, so we have to compute the work long hand.

$$\begin{aligned}
\int_0^2 \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \int_0^2 \mathbf{F}(t)\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\,\mathrm{d}t = -\int_0^2 |\mathbf{v}(t)|^2\mathbf{v}(t)\cdot\mathbf{v}(t)\,\mathrm{d}t = -\int_0^2 |\mathbf{v}(t)|^4\,\mathrm{d}t\\
&= -2^4\int_0^2 (1+t)^4\,\mathrm{d}t = -\frac{16}{5}(1+t)^5\Big|_0^2\\
&= -\frac{16}{5}(3^5-1) \approx -774.4
\end{aligned}$$

(d) **Solution 1:** We know, from Theorem 1.3.3.c in the text, that

$$\mathbf{a}(t) = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\hat{\mathbf{T}} + \kappa\Big(\frac{\mathrm{d}s}{\mathrm{d}t}\Big)^2\hat{\mathbf{N}}$$

We have also been told that, at the apex, $\hat{\mathbf{N}} = -\hat{\mathbf{k}}$ and that $\frac{\mathrm{d}s}{\mathrm{d}t}(t) = 3$ for all t. So $\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = 0$. As $\kappa = \frac{1}{8}$ at the apex

$$\mathbf{a}(1) = 0\hat{\mathbf{T}} + \frac{1}{8}(3)^2(-\hat{\mathbf{k}}) = -\frac{9}{8}\hat{\mathbf{k}}$$

Solution 2: The bird follows the parametrized path

$$\mathbf{r}(u) = \Big(\frac{4\sqrt{2}}{3}u^{3/2}, \frac{4\sqrt{2}}{3}u^{3/2}, u(2-u)\Big)$$

This is the same path as the plane, but the parameter u is not time. Let's denote by $\mathbf{R}(t)$ the position of the bird at time t. At time t the bird is at some point on the parametrized path, so there is some $u(t)$ with

$$\mathbf{R}(t) = \mathbf{r}\big(u(t)\big)$$

We saw in part (a) that $\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\right| = 2(1+u)$. Since the bird always has speed 3,

$$3 = \left|\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t}(t)\right| = \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}(u(t))\frac{\mathrm{d}u}{\mathrm{d}t}\right| = 2\big(1+u(t)\big)\frac{\mathrm{d}u}{\mathrm{d}t}$$
$$\implies \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{3}{2(1+u(t))} \implies \frac{\mathrm{d}^2u}{\mathrm{d}t^2} = -\frac{3}{2(1+u(t))^2}\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{9}{4(1+u(t))^3}$$

At the apex $u = 1$ so that $\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{3}{4}$ and $\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = -\frac{9}{32}$. The bird's acceleration is

$$\frac{\mathrm{d}^2\mathbf{R}}{\mathrm{d}t^2}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t}(t)\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}(u(t))\frac{\mathrm{d}u}{\mathrm{d}t}(t)\Big) = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}u^2}\Big(\frac{\mathrm{d}u}{\mathrm{d}t}\Big)^2 + \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\frac{\mathrm{d}^2u}{\mathrm{d}t^2}$$

From part (a)

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u} = \big(2\sqrt{2}u^{1/2}, 2\sqrt{2}u^{1/2}, 2-2u\big)$$
$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}u^2} = \big(\sqrt{2}u^{-1/2}, \sqrt{2}u^{-1/2}, -2\big)$$

At the apex, when $u = 1$,

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u} = \big(2\sqrt{2}, 2\sqrt{2}, 0\big)$$
$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}u^2} = \big(\sqrt{2}, \sqrt{2}, -2\big)$$

and the acceleration is

$$\frac{\mathrm{d}^2\mathbf{R}}{\mathrm{d}t^2} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}u^2}\Big(\frac{\mathrm{d}u}{\mathrm{d}t}\Big)^2 + \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = (\sqrt{2}, \sqrt{2}, -2)\Big(\frac{3}{4}\Big)^2 + (2\sqrt{2}, 2\sqrt{2}, 0)\Big(-\frac{9}{32}\Big)$$
$$= \Big(0, 0, -\frac{9}{8}\Big)$$

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S-1: This parametrization is almost trivial. We know it will have the form $\mathbf{r}(x,y) = \psi_1(x,y)\hat{\boldsymbol{\imath}} + \psi_2(x,y)\hat{\boldsymbol{\jmath}} + \psi_3(x,y)\hat{\mathbf{k}}$ where ψ_1 gives the x-component (i.e. x), ψ_2 gives the y-component (i.e. y), and ψ_3 gives the z-component (i.e. $e^{x+1} + xy$). So, $\mathbf{r}(x,y) = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + (e^{x+1} + xy)\hat{\mathbf{k}}$

S-2: Our parametrization is

$$x(u,v) = u + v$$
$$y(u,v) = u^2 + v^2$$
$$z(u,v) = u - v$$

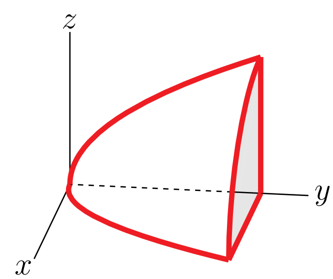
- Adding $x(u,v)$ and $z(u,v)$ gives $x(u,v)+z(u,v)=2u$.
- Subtracting $z(u,v)$ from $x(u,v)$ gives $x(u,v)-z(u,v)=2v$.

So $u=\frac{1}{2}\big(x(u,v)+z(u,v)\big)$ and $v=\frac{1}{2}\big(x(u,v)-z(u,v)\big)$. So on our surface

$$\begin{aligned} y(u,v) = u^2+v^2 &= \frac{1}{4}\big(x(u,v)+z(u,v)\big)^2+\frac{1}{4}\big(x(u,v)-z(u,v)\big)^2 \\ &= \frac{1}{2}x(u,v)^2+\frac{1}{2}z(u,v)^2 \end{aligned}$$

All points of our surface lie on $2y=x^2+z^2$. This is a parabolic bowl:

- no points have $y<0$ and
- the $y=Y$ (with $Y>0$) cross-section is the circle $x^2+z^2=2Y$, $y=Y$
- the $x=0$ cross-section is the parabola $2y=z^2$, $x=0$
- the $z=0$ cross-section is the parabola $2y=x^2$, $z=0$



S-3: Note that, since $x^2+y^2=1+2z^2$ on S, the condition $z\geqslant 1$ is equivalent to $x^2+y^2\geqslant 3$, $z\geqslant 0$. So the hyperboloid is $\{\,(x,y,z)\mid x^2+y^2=1+2z^2,\ 3\leqslant x^2+y^2\leqslant 9, z\geqslant 0\,\}$.

(a) No. Under this parametrization, the condition $3\leqslant x^2+y^2\leqslant 9$ is $3\leqslant u^2+v^2\leqslant 9$, not $2\leqslant u^2+v^2\leqslant 9$.

(b) Yes. Under this parametrization, $x=u\sin v$, $y=-u\cos v$ and $z=\sqrt{\frac{u^2}{2}-\frac{1}{2}}$. So

- $x^2+y^2-2z^2=u^2-2\left(\frac{u^2}{2}-\frac{1}{2}\right)=1$, as desired.
- The condition $x^2+y^2\leqslant 9$ is equivalent to $u\leqslant 3$, since $u\geqslant 0$.
- The condition $x^2+y^2\geqslant 3$ is equivalent to $u\geqslant\sqrt{3}$, since $u\geqslant 0$.
- $z=\sqrt{\frac{u^2}{2}-\frac{1}{2}}\geqslant 0$

(c) Yes. Under this parametrization, $x=\sqrt{1+2v^2}\cos u$, $y=\sqrt{1+2v^2}\sin u$ and $z=v$. So

- $x^2+y^2-2z^2=1+2v^2-2v^2=1$, as desired.
- The condition $x^2+y^2\leqslant 9$ is equivalent to $1+2v^2\leqslant 9$, which is equivalent to $v\leqslant 2$, since $v\geqslant 0$.
- The condition $x^2+y^2\geqslant 3$ is equivalent to $1+2v^2\geqslant 3$, which is equivalent to $v\geqslant 1$, since $v\geqslant 0$.
- $z=v\geqslant 0$

(d) Yes. Under this parametrization, $x=\sqrt{1+u}\sin v$, $y=\sqrt{1+u}\cos v$ and $z=\sqrt{u/2}$. So

- $x^2 + y^2 - 2z^2 = 1 + u - 2(u/2) = 1$, as desired.
- The condition $x^2 + y^2 \leqslant 9$ is equivalent to $1 + u \leqslant 9$, which is equivalent to $u \leqslant 8$.
- The condition $x^2 + y^2 \geqslant 3$ is equivalent to $1 + u \geqslant 3$, which is equivalent to $u \geqslant 2$.
- $z = \sqrt{u/2} \geqslant 0$

(e) No. Under this parametrization, $x = \sqrt{u}\cos v$, $y = -\sqrt{u}\sin v$ and $z = \sqrt{(u+1)/2}$. So

- $x^2 + y^2 - 2z^2 = u - 2(u+1)/2 = -1$, not $+1$

S-4: (a) No. $z = \sin\phi\sin\theta$ is negative when $0 < \phi \leqslant \frac{\pi}{4}$, $\pi < \theta < 2\pi$.

(b) Yes. Note that $x^2 + \big(-y\big)^2 + \big(\sqrt{2 - x^2 - y^2}\big)^2 = 2$ and that, for $x^2 + y^2 \leqslant 1$, we have both $x^2 + (-y)^2 \leqslant 1$ and $\sqrt{2 - x^2 - y^2} \geqslant 0$.

(c) No. $(u\sin\theta)^2 + (u\cos\theta)^2 = u^2 > 1$ for $1 < u \leqslant 2$. Also $\sqrt{2 - u^2}$ is not defined for $\sqrt{2} < u \leqslant 2$.

(d) Yes. Note that

- $\big(\sqrt{2}\sin\phi\cos\theta\big)^2 + \big(\sqrt{2}\sin\phi\sin\theta\big)^2 + \big(\sqrt{2}\cos\phi\big)^2 = 2$
- For $0 \leqslant \phi \leqslant \frac{\pi}{4}$, we have $z = \sqrt{2}\cos\phi > 0$.
- As ϕ runs from 0 to $\frac{\pi}{4}$, $r(\phi) = \sqrt{2}\sin\phi$ runs from 0 to 1, so that $\big(x = r(\phi)\cos\theta\,,\ y = r(\phi)\sin\theta\big)$ covers all of $x^2 + y^2 \leqslant 1$ as ϕ runs from 0 to $\frac{\pi}{4}$ and θ runs from 0 to 2π.

(e) Yes. Note that

- $\big(-\sqrt{2 - z^2}\sin\phi\big)^2 + \big(\sqrt{2 - z^2}\cos\phi\big)^2 + \big(z\big)^2 = 2$
- For $1 \leqslant z \leqslant \sqrt{2}$, we have obviously have $z > 0$.
- As z runs from 1 to $\sqrt{2}$, $r(z) = \sqrt{2 - z^2}$ runs from 1 to 0, so that $\big(x = -r(z)\sin\phi\,,\ y = r(z)\cos\phi\big)$ covers all of $x^2 + y^2 \leqslant 1$ as z runs from 1 to $\sqrt{2}$ and ϕ runs from 0 to 2π.

S-5: (a) No. When $u = v = 0$, $z = 4$ is not between 0 and 1.

(b) Yes. Note that when $x = \sqrt{4-u}\ \cos v$, $y = \sqrt{4-u}\ \sin v$ and $z = u$ with $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant 2\pi$,

- $z + x^2 + y^2 = 4$
- $0 \leqslant z = u \leqslant 1$
- For each fixed $z = u$ between 0 and 1, (x, y) runs once around the circle $x^2 + y^2 = 4 - z = 4 - u$ as v runs from 0 to 2π.

(c) Yes. Note that when $x = u\ \cos v$, $y = u\ \sin v$ and $z = 4 - u^2$, with $\sqrt{3} \leqslant u \leqslant 2$, $0 \leqslant v \leqslant 2\pi$

- $z + x^2 + y^2 = 4$
- $0 \leqslant z = 4 - u^2 \leqslant 1$
- For each fixed $z = 4 - u^2$ between 0 and 1, (x, y) runs once around the circle $x^2 + y^2 = 4 - z = u^2$ as v runs from 0 to 2π.

S-6: First note that,

- for A, B and C, $\mathbf{r}(\theta,\phi) = x(\theta,\phi)\,\hat{\boldsymbol{\imath}} + y(\theta,\phi)\,\hat{\boldsymbol{\jmath}} + z(\theta,\phi)\,\hat{\mathbf{k}}$ obeys
$$x(\theta,\phi)^2 + y(\theta,\phi)^2 + z(\theta,\phi)^2 = 4$$
and so lies on S_1
- for D, E and F, $\mathbf{r}(\theta,z) = x(\theta,z)\,\hat{\boldsymbol{\imath}} + y(\theta,z)\,\hat{\boldsymbol{\jmath}} + z(\theta,z)\,\hat{\mathbf{k}}$ obeys
$$x(\theta,z)^2 + y(\theta,z)^2 = 4 - z(\theta,z)^2$$
and so lies on S_1
- for G, H and I, $\mathbf{r}(\theta,z) = x(\theta,z)\,\hat{\boldsymbol{\imath}} + y(\theta,z)\,\hat{\boldsymbol{\jmath}} + z(\theta,z)\,\hat{\mathbf{k}}$ obeys
$$x(\theta,z)^2 + y(\theta,z)^2 = z(\theta,z)^2$$
and so lies on S_3
- for J, K and L, $\mathbf{r}(x,y) = x(x,y)\,\hat{\boldsymbol{\imath}} + y(x,y)\,\hat{\boldsymbol{\jmath}} + z(x,y)\,\hat{\mathbf{k}}$ obeys
$$x(x,y)^2 + y(x,y)^2 = z(x,y)^2$$
and so lies on S_3

(a) To get a part of S_1, we need to use one of the parametrizations A, B, C, D, E, F. In the cases of A, B, C, for $\mathbf{r}(\theta,\phi) = x(\theta,\phi)\,\hat{\boldsymbol{\imath}} + y(\theta,\phi)\,\hat{\boldsymbol{\jmath}} + z(\theta,\phi)\,\hat{\mathbf{k}}$ to lie inside S_2 we need (recalling that all points of S_1 have $z(\theta,\phi) \geqslant 0$ and hence $0 \leqslant \phi \leqslant \pi/2$)

$$x(\theta,\phi)^2 + y(\theta,\phi)^2 \leqslant 1 \iff 4\sin^2\phi \leqslant 1 \iff \sin\phi \leqslant \frac{1}{2} \iff 0 \leqslant \phi \leqslant \frac{\pi}{6}$$

In the cases of D, E, F, for $\mathbf{r}(\theta,z) = x(\theta,z)\,\hat{\boldsymbol{\imath}} + y(\theta,z)\,\hat{\boldsymbol{\jmath}} + z(\theta,z)\,\hat{\mathbf{k}}$ to lie inside S_2 we need (recalling that all points of S_1 have $z(\theta,z) \geqslant 0$ and hence $z \geqslant 0$)

$$x(\theta,z)^2 + y(\theta,z)^2 \leqslant 1 \iff 4 - z^2 \leqslant 1 \iff z \geqslant \sqrt{3}$$

So parametrizations A and F work.

(b) To get a part of S_1, we need to use one of the parametrizations A, B, C, D, E, F. In the cases of A, B, C, for $\mathbf{r}(\theta,\phi) = x(\theta,\phi)\,\hat{\boldsymbol{\imath}} + y(\theta,\phi)\,\hat{\boldsymbol{\jmath}} + z(\theta,\phi)\,\hat{\mathbf{k}}$ to lie inside S_3 we need (recalling that all points of S_1 have $z(\theta,\phi) \geqslant 0$ and hence $0 \leqslant \phi \leqslant \pi/2$)

$$x(\theta,\phi)^2 + y(\theta,\phi)^2 \leqslant z(\theta,\phi)^2 \iff 4\sin^2\phi \leqslant 4\cos^2\phi \iff \tan\phi \leqslant 1 \iff 0 \leqslant \phi \leqslant \frac{\pi}{4}$$

In the cases of D, E, F, for $\mathbf{r}(\theta,z) = x(\theta,z)\,\hat{\boldsymbol{\imath}} + y(\theta,z)\,\hat{\boldsymbol{\jmath}} + z(\theta,z)\,\hat{\mathbf{k}}$ to lie inside S_3 we need (recalling that all points of S_1 have $z(\theta,z) \geqslant 0$ and hence $z \geqslant 0$)

$$x(\theta,z)^2 + y(\theta,z)^2 \leqslant z(\theta,z)^2 \iff 4 - z^2 \leqslant z^2 \iff z \geqslant \sqrt{2}$$

So parametrizations B and E work.

(c) To get a part of S_3, we need to use one of the parametrizations G, H, I, J, K, L. In the cases of G, H, I, for $\mathbf{r}(\theta,z) = x(\theta,z)\,\hat{\boldsymbol{\imath}} + y(\theta,z)\,\hat{\boldsymbol{\jmath}} + z(\theta,z)\,\hat{\mathbf{k}}$ to lie inside S_2 we need (recalling that all points of S_3 have $z \geqslant 0$)

$$x(\theta,z)^2 + y(\theta,z)^2 \leqslant 1 \iff z^2 \leqslant 1 \iff 0 \leqslant z \leqslant 1$$

In the cases of J, K, L, for $\mathbf{r}(x,y) = x(x,y)\,\hat{\boldsymbol{\imath}} + y(x,y)\,\hat{\boldsymbol{\jmath}} + z(x,y)\,\hat{\mathbf{k}}$ to lie inside S_3 we need

$$x(x,y)^2 + y(x,y)^2 \leqslant 1 \iff x^2 + y^2 \leqslant 1$$

So parametrizations G and J work.

(d) To get a part of S_3, we need to use one of the parametrizations G, H, I, J, K, L. In the cases of G, H, I, for $\mathbf{r}(\theta,z) = x(\theta,z)\,\hat{\boldsymbol{\imath}} + y(\theta,z)\,\hat{\boldsymbol{\jmath}} + z(\theta,z)\,\hat{\mathbf{k}}$ to lie inside S_1 we need (recalling that all points of S_3 have $z \geqslant 0$)

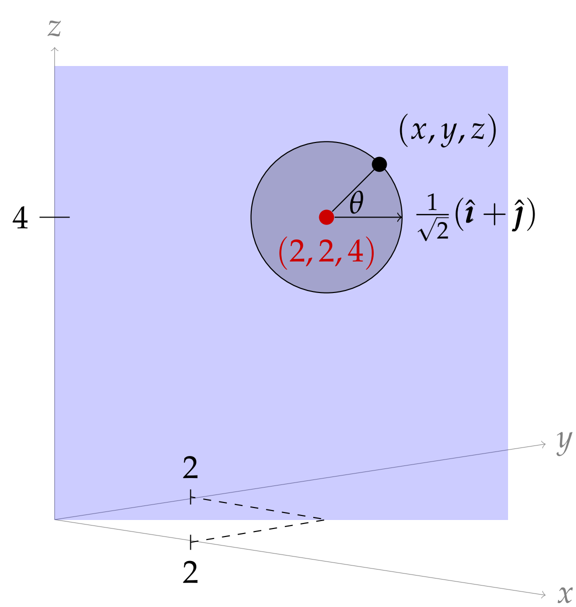
$$x(\theta,z)^2 + y(\theta,z)^2 + z(\theta,z)^2 \leqslant 4 \iff 2z^2 \leqslant 4 \iff 0 \leqslant z \leqslant \sqrt{2}$$

In the cases of J, K, L, for $\mathbf{r}(x,y) = x(x,y)\,\hat{\boldsymbol{\imath}} + y(x,y)\,\hat{\boldsymbol{\jmath}} + z(x,y)\,\hat{\mathbf{k}}$ to lie inside S_3 we need

$$x(x,y)^2 + y(x,y)^2 + z(x,y)^2 \leqslant 4 \iff 2x^2 + 2y^2 \leqslant 4$$

So parametrizations H and L work.

S-7: (a) In the sketch below, the point (x,y,z) deviates from the centre $(2,2,4)$ by $\sin\theta$ units in the $\hat{\mathbf{k}}$ direction, and by $\cos\theta$ units in the $\sqrt{\frac{1}{\sqrt{2}}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})$ direction. So, $(x,y,z) = (2 + \frac{1}{\sqrt{2}}\cos\theta\ ,\ 2 + \frac{1}{\sqrt{2}}\cos\theta\ ,\ 4 + \sin\theta)$.



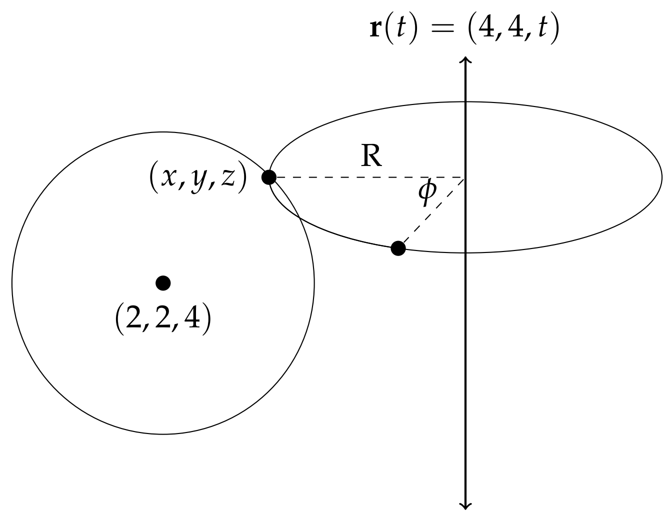
So, we can parametrize the circle as $(x,y,z) = (2 + \frac{1}{\sqrt{2}}\cos\theta\ ,\ 2 + \frac{1}{\sqrt{2}}\cos\theta\ ,\ 4 + \sin\theta)$, with $0 \leqslant \theta \leqslant 2\pi$.

Remark: it's easy to check that this equation satisfies the two properties we desire. Since the x- and y coordinates match, it's in the plane $x = y$. To check that it's a circle centred at $(2, 2, 4)$, we note the distance from (x, y, z) to $(2, 2, 4)$ is:

$$d = \sqrt{(x-2)^2 + (y-2)^2 + (z-4)^2} = \sqrt{\left(\frac{1}{\sqrt{2}}\cos\theta\right)^2 + \left(\frac{1}{\sqrt{2}}\cos\theta\right)^2 + (\sin\theta)^2}$$
$$= \sqrt{\frac{1}{2}\cos^2\theta + \frac{1}{2}\cos^2\theta + \sin^2\theta} = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

So, our points all have distance one from the same point — that is, they lie on a circle of radius 1.

(b) Consider a point $(x, y, z) = (2 + \frac{1}{\sqrt{2}}\cos\theta\ ,\ 2 + \frac{1}{\sqrt{2}}\cos\theta\ ,\ 4 + \sin\theta)$, rotating ϕ radians about the line $x = y = 4$.



The new position of the point has the same height, $z = 4 + \sin\theta$. Its distance from the line $x = y = 4$ is also preserved: $R = \sqrt{(x-4)^2 + (y-4)^2 + (z-z)^2} = \sqrt{(\frac{1}{\sqrt{2}}\cos\theta - 2)^2 + (\frac{1}{\sqrt{2}}\cos\theta - 2)^2 + 0)} = \cos\theta - 2\sqrt{2}$.

The circle traced out by a point $(x, y, z) = (2 + \frac{1}{\sqrt{2}}\cos\theta, 2 + \frac{1}{\sqrt{2}}\cos\theta, 4 + \sin\theta)$ on the circle is centred at $(4, 4, z)$ with radius $\sqrt{2}(4 - x)$, so it has equation $x = 4 + \sqrt{2}(2 - \sqrt{2}\cos\theta)\cos\phi$, $y = 4 + \sqrt{2}(2 - \sqrt{2}\cos\theta)\sin\phi$, $z = 4\sin\theta$.

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S-1: Write $F(x, y, z) = x^2 + y^2 + (z-1)^2 - 1$ and $G(x, y, z) = x^2 + y^2 + (z+1)^2 - 1$. Let S_1 denote the surface $F(x, y, z) = 0$ and S_2 denote the surface $G(x, y, z) = 0$. First note that $F(0, 0, 0) = G(0, 0, 0) = 0$ so that the point $(0, 0, 0)$ lies on both S_1 and S_2. The gradients of F and G are

$$\nabla F(x,y,z) = \left(\frac{\partial F}{\partial x}(x,y,z)\,,\ \frac{\partial F}{\partial y}(x,y,z)\,,\ \frac{\partial F}{\partial z}(x,y,z)\right) = (2x\,,\ 2y\,,\ 2(z-1))$$
$$\nabla G(x,y,z) = \left(\frac{\partial G}{\partial x}(x,y,z)\,,\ \frac{\partial G}{\partial y}(x,y,z)\,,\ \frac{\partial G}{\partial z}(x,y,z)\right) = (2x\,,\ 2y\,,\ 2(z+1))$$

In particular,

$$\nabla F(0,0,0) = (0,0,-2) \qquad \nabla G(0,0,0) = (0,0,2)$$

so that the vector $\hat{\mathbf{k}} = -\frac{1}{2}\nabla F(0,0,0) = \frac{1}{2}\nabla G(0,0,0)$ is normal to both surfaces at $(0,0,0)$. So the tangent plane to both S_1 and S_2 at $(0,0,0)$ is

$$\hat{\mathbf{k}} \cdot (x-0, y-0, z-0) = 0 \qquad \text{or} \qquad z = 0$$

Denote by P the plane $z=0$. Thus S_1 is tangent to P at $(0,0,0)$ and P is tangent to S_2 at $(0,0,0)$. So it is reasonable to say that S_1 and S_2 are tangent at $(0,0,0)$.

S-2: Denote by S the surface $G(x,y,z) = 0$ and by C the parametrized curve $\mathbf{r}(t) = \big(x(t), y(t), z(t)\big)$. To start, we'll find the tangent plane to S at $\mathbf{r}_0$ and the tangent line to C at $\mathbf{r}_0$.

- The tangent vector to C at $\mathbf{r}_0$ is $(x'(t_0)\,,\ y'(t_0)\,,\ z'(t_0))$, so the parametric equations for the tangent line to C at $\mathbf{r}_0$ are

$$x - x_0 = tx'(t_0) \qquad y - y_0 = tx'(t_0) \qquad z - z_0 = tx'(t_0) \tag{E_1}$$

- The gradient $\left(\frac{\partial G}{\partial x}(x_0\,,\ y_0\,,\ z_0)\,,\ \frac{\partial G}{\partial y}(x_0\,,\ y_0\,,\ z_0)\,,\ \frac{\partial G}{\partial z}(x_0\,,\ y_0\,,\ z_0)\right)$ is a normal vector to the surface S at (x_0, y_0, z_0). So the tangent plane to the surface S at (x_0, y_0, z_0) is

$$\left(\frac{\partial G}{\partial x}(x_0\,,\ y_0\,,\ z_0)\,,\ \frac{\partial G}{\partial y}(x_0\,,\ y_0\,,\ z_0)\,,\ \frac{\partial G}{\partial z}(x_0\,,\ y_0\,,\ z_0)\right) \cdot (x - x_0\,,\ y - y_0\,,\ z - z_0) = 0$$

  or

$$\frac{\partial G}{\partial x}(x_0\,,\ y_0\,,\ z_0)\ (x - x_0) + \frac{\partial G}{\partial y}(x_0\,,\ y_0\,,\ z_0)\ (y - y_0) + \frac{\partial G}{\partial z}(x_0\,,\ y_0\,,\ z_0)\ (z - z_0) = 0 \tag{E_2}$$

Next, we'll show that the tangent vector $(x'(t_0)\,,\ y'(t_0)\,,\ z'(t_0))$ to C at $\mathbf{r}_0$ and the normal vector $\left(\frac{\partial G}{\partial x}(x_0\,,\ y_0\,,\ z_0)\,,\ \frac{\partial G}{\partial y}(x_0\,,\ y_0\,,\ z_0)\,,\ \frac{\partial G}{\partial z}(x_0\,,\ y_0\,,\ z_0)\right)$ to S at $\mathbf{r}_0$ are perpendicular to each other. To do so, we observe that, for every t, the point $\big(x(t), y(t), z(t)\big)$ lies on the surface $G(x,y,z) = 0$ and so obeys

$$G\big(x(t), y(t), z(t)\big) = 0$$

Differentiating this equation with respect to t gives, by the chain rule,

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t} G\big(x(t), y(t), z(t)\big) \\ &= \frac{\partial G}{\partial x}\big(x(t)\,,\ y(t)\,,\ z(t)\big)\ x'(t) + \frac{\partial G}{\partial y}\big(x(t)\,,\ y(t)\,,\ z(t)\big)\ y'(t) + \frac{\partial G}{\partial z}\big(x(t)\,,\ y(t)\,,\ z(t)\big)\ z'(t) \end{aligned}$$

Then setting $t = t_0$ gives

$$\frac{\partial G}{\partial x}(x_0\,,\,y_0\,,\,z_0)\ x'(t_0) + \frac{\partial G}{\partial y}(x_0\,,\,y_0\,,\,z_0)\ y'(t_0) + \frac{\partial G}{\partial z}(x_0\,,\,y_0\,,\,z_0)\ z'(t_0) = 0 \qquad (E_3)$$

Finally, we are in a position to show that if (x, y, z) is any point on the tangent line to C at $\mathbf{r}_0$, then (x, y, z) is also on the tangent plane to S at $\mathbf{r}_0$. As (x, y, z) is on the tangent line to C at $\mathbf{r}_0$ then there is a t such that, by (E_1),

$$\begin{aligned}
&\frac{\partial G}{\partial x}(x_0\,,\,y_0\,,\,z_0)\ \{x - x_0\} + \frac{\partial G}{\partial y}(x_0\,,\,y_0\,,\,z_0)\ \{y - y_0\} + \frac{\partial G}{\partial z}(x_0\,,\,y_0\,,\,z_0)\ \{z - z_0\} \\
&= \frac{\partial G}{\partial x}(x_0\,,\,y_0\,,\,z_0)\ \{t\, x'(t_0)\} + \frac{\partial G}{\partial y}(x_0\,,\,y_0\,,\,z_0)\ \{t\, y'(t_0)\} + \frac{\partial G}{\partial z}(x_0\,,\,y_0\,,\,z_0)\ \{t\, z'(t_0)\} \\
&= t\left[\frac{\partial G}{\partial x}(x_0\,,\,y_0\,,\,z_0)\ x'(t_0) + \frac{\partial G}{\partial y}(x_0\,,\,y_0\,,\,z_0)\ y'(t_0) + \frac{\partial G}{\partial z}(x_0\,,\,y_0\,,\,z_0)\ z'(t_0)\right] = 0
\end{aligned}$$

by (E_3). That is, (x, y, z) obeys the equation, (E_2), of the tangent plane to S at $\mathbf{r}_0$ and so is on that tangent plane. So the tangent line to C at $\mathbf{r}_0$ is contained in the tangent plane to S at $\mathbf{r}_0$.

S-3: By part (b) of Theorem 3.2.1 in the CLP-4 text,

$$\mathbf{n} = -f_x(x_0, y_0)\,\hat{\imath} - f_y(x_0, y_0)\,\hat{\jmath} + \hat{\mathbf{k}}$$

is normal to the surface at (x_0, y_0, z_0). So the parametric equations of the normal line are

$$\begin{aligned}
&(x - x_0\,,\, y - y_0\,,\, z - z_0) = t\big(-f_x(x_0, y_0)\,,\, -f_y(x_0, y_0)\,,\, 1\big) \qquad \text{or} \\
&x = x_0 - t f_x(x_0, y_0) \quad y = y_0 - t f_y(x_0, y_0) \quad z = f(x_0, y_0) + t
\end{aligned}$$

S-4: Use S_1 to denote the surface $F(x, y, z) = 0$, S_2 to denote the surface $G(x, y, z) = 0$ and C to denote the curve of intersection of S_1 and S_2.

- Since C is contained in S_1, the tangent line to C at (x_0, y_0, z_0) is contained in the tangent plane to S_1 at (x_0, y_0, z_0), by Q[2]. In particular, any tangent vector, $\mathbf{t}$, to C at (x_0, y_0, z_0) must be perpendicular to $\nabla F(x_0, y_0, z_0)$, the normal vector to S_1 at (x_0, y_0, z_0).
- Since C is contained in S_2, the tangent line to C at (x_0, y_0, z_0) is contained in the tangent plane to S_2 at (x_0, y_0, z_0), by Q[2]. In particular, any tangent vector, $\mathbf{t}$, to C at (x_0, y_0, z_0) must be perpendicular to $\nabla G(x_0, y_0, z_0)$, the normal vector to S_2 at (x_0, y_0, z_0).

So any tangent vector to C at (x_0, y_0, z_0) must be perpendiular to both $\nabla F(x_0, y_0, z_0)$ and $\nabla G(x_0, y_0, z_0)$. One such tangent vector is

$$\mathbf{t} = \nabla F(x_0, y_0, z_0) \times \nabla G(x_0, y_0, z_0)$$

(Because the vectors $\nabla F(x_0,y_0,z_0)$ and $\nabla G(x_0,y_0,z_0)$ are nonzero and not parallel, $\mathbf{t}$ is nonzero.) So the normal plane in question passes through (x_0,y_0,z_0) and has normal vector $\mathbf{n}=\mathbf{t}$. Consquently, the normal plane is

$$\mathbf{n}\cdot(x-x_0\,,\,y-y_0\,,\,z-z_0)=0 \qquad \text{where } \mathbf{n}=\mathbf{t}=\nabla F(x_0,y_0,z_0)\times\nabla G(x_0,y_0,z_0)$$

S-5: Use S_1 to denote the surface $z=f(x,y)$, S_2 to denote the surface $z=g(x,y)$ and C to denote the curve of intersection of S_1 and S_2.

- Since C is contained in S_1, the tangent line to C at (x_0,y_0,z_0) is contained in the tangent plane to S_1 at (x_0,y_0,z_0), by Q[2]. In particular, any tangent vector, $\mathbf{t}$, to C at (x_0,y_0,z_0) must be perpendicular to $-f_x(x_0,y_0)\,\hat{\imath}-f_y(x_0,y_0)\,\hat{\jmath}+\hat{\mathbf{k}}$, the normal vector to S_1 at (x_0,y_0,z_0). (See part (b) of Theorem 3.2.1 in the CLP-4 text.)
- Since C is contained in S_2, the tangent line to C at (x_0,y_0,z_0) is contained in the tangent plane to S_2 at (x_0,y_0,z_0), by Q[2]. In particular, any tangent vector, $\mathbf{t}$, to C at (x_0,y_0,z_0) must be perpendicular to $-g_x(x_0,y_0)\,\hat{\imath}-g_y(x_0,y_0)\,\hat{\jmath}+\hat{\mathbf{k}}$, the normal vector to S_2 at (x_0,y_0,z_0).

So any tangent vector to C at (x_0,y_0,z_0) must be perpendicular to both of the vectors $-f_x(x_0,y_0)\,\hat{\imath}-f_y(x_0,y_0)\,\hat{\jmath}+\hat{\mathbf{k}}$ and $-g_x(x_0,y_0)\,\hat{\imath}-g_y(x_0,y_0)\,\hat{\jmath}+\hat{\mathbf{k}}$. One such tangent vector is

$$\begin{aligned}
\mathbf{t}&=\big[-f_x(x_0,y_0)\,\hat{\imath}-f_y(x_0,y_0)\,\hat{\jmath}+\hat{\mathbf{k}}\big]\times\big[-g_x(x_0,y_0)\,\hat{\imath}-g_y(x_0,y_0)\,\hat{\jmath}+\hat{\mathbf{k}}\big]\\
&=\det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ -f_x(x_0,y_0) & -f_y(x_0,y_0) & 1\\ -g_x(x_0,y_0) & -g_y(x_0,y_0) & 1\end{bmatrix}\\
&=\big(g_y(x_0,y_0)-f_y(x_0,y_0)\,,\,f_x(x_0,y_0)-g_x(x_0,y_0)\,,\,f_x(x_0,y_0)g_y(x_0,y_0)-f_y(x_0,y_0)g_x(x_0,y_0)\big)
\end{aligned}$$

So the tangent line in question passes through (x_0,y_0,z_0) and has direction vector $\mathbf{d}=\mathbf{t}$. Consquently, the tangent line is

$$(x-x_0\,,\,y-y_0\,,\,z-z_0)=t\,\mathbf{d}$$

or

$$\begin{aligned}
x&=x_0+t\big[g_y(x_0,y_0)-f_y(x_0,y_0)\big]\\
y&=y_0+t\big[f_x(x_0,y_0)-g_x(x_0,y_0)\big]\\
z&=z_0+t\big[f_x(x_0,y_0)g_y(x_0,y_0)-f_y(x_0,y_0)g_x(x_0,y_0)\big]
\end{aligned}$$

S-6: We are going to use part (b) of Theorem 3.2.1 in the CLP-4 text. To do so, we need the first order derivatives of $f(x,y)$ at $(x,y)=(-1,1)$. So we find them first.

$$\begin{aligned}
f_x(x,y)&=\frac{2xy}{x^4+2y^2}-\frac{x^2y(4x^3)}{(x^4+2y^2)^2} &\qquad f_x(-1,1)&=-\frac{2}{3}+\frac{4}{3^2}=-\frac{2}{9}\\
f_y(x,y)&=\frac{x^2}{x^4+2y^2}-\frac{x^2y(4y)}{(x^4+2y^2)^2} &\qquad f_y(-1,1)&=\frac{1}{3}-\frac{4}{3^2}=-\frac{1}{9}
\end{aligned}$$

So $(2/9\,,\,1/9\,,\,1)$ is a normal vector to the surface at $(-1,1,1/3)$ and the tangent plane is

$$\begin{aligned}\frac{2}{9}(x+1)+\frac{1}{9}(y-1)+\Big(z-\frac{1}{3}\Big)&=0\\ \frac{2}{9}x+\frac{1}{9}y+z&=-\frac{2}{9}+\frac{1}{9}+\frac{1}{3}=\frac{2}{9}\end{aligned}$$

or $2x+y+9z=2$.

S-7: The equation of the given surface is of the form $G(x,y,z)=9$ with $G(x,y,z)=\frac{27}{\sqrt{x^2+y^2+z^2+3}}$. So, by part (c) of Theorem 3.2.1 in the CLP-4 text, a normal vector to the surface at $(2,1,1)$ is

$$\begin{aligned}\nabla G(2,1,1)&=-\frac{1}{2}\,\frac{27}{(x^2+y^2+z^2+3)^{3/2}}(2x\,,\,2y\,,\,2z)\Big|_{(x,y,z)=(2,1,1)}\\ &=-(2\,,\,1\,,\,1)\end{aligned}$$

and the equation of the tangent plane is

$$-(2\,,\,1\,,\,1)\cdot(x-2\,,\,y-1\,,\,z-1)=0\qquad\text{or}\qquad 2x+y+z=6$$

S-8: We may use $G(x,y,z)=xyz^2+y^2z^3-3-x^2=0$ as an equation for the surface. Note that $(-1,1,2)$ really is on the surface since

$$G(-1,1,2)=(-1)(1)(2)^2+(1)^2(2)^3-3-(-1)^2=-4+8-3-1=0$$

By part (c) of Theorem 3.2.1 in the CLP-4 text, since

$$\begin{aligned}G_x(x,y,z)&=yz^2-2x & G_x(-1,1,2)&=6\\ G_y(x,y,z)&=xz^2+2yz^3 & G_y(-1,1,2)&=12\\ G_z(x,y,z)&=2xyz+3y^2z^2 & G_z(-1,1,2)&=8\end{aligned}$$

one normal vector to the surface at $(-1,1,2)$ is $\nabla G(-1,1,2)=(6\,,\,12\,,\,8)$ and an equation of the tangent plane to the surface at $(-1,1,2)$ is

$$(6\,,\,12\,,\,8)\cdot(x+1\,,\,y-1\,,\,z-2)=0\qquad\text{or}\qquad 6x+12y+8z=22$$

or

$$z=-\frac{3}{4}x-\frac{3}{2}y+\frac{11}{4}$$

S-9: (a) The surface is $G(x,y,z)=z-x^2+2xy-y^2=0$. When $x=a$ and $y=2a$ and (x,y,z) is on the surface, we have $z=a^2-2(a)(2a)+(2a)^2=a^2$. So, by part (c) of Theorem 3.2.1 in the CLP-4 text, a normal vector to this surface at $(a,2a,a^2)$ is

$$\nabla G(a,2a,a^2)=(-2x+2y\,,\,2x-2y\,,\,1)\Big|_{(x,y,z)=(a,2a,a^2)}=(2a\,,\,-2a\,,\,1)$$

and the equation of the tangent plane is

$$(2a\,,\,-2a\,,\,1)\cdot(x-a\,,\,y-2a\,,\,z-a^2)=0 \qquad \text{or} \qquad 2ax-2ay+z=-a^2$$

(b) The two planes are parallel when their two normal vectors, namely $(2a\,,\,-2a\,,\,1)$ and $(1\,,\,-1\,,\,1)$, are parallel. This is the case if and only if $a=\frac{1}{2}$.

S-10: A plane is determined by one point on the plane and one vector perpendicular to the plane. We are told that $(8,1,5)$ is on the plane, so it suffices to find a normal vector. The given surface is parametrized by

$$\mathbf{r}(u,v)=2u^2\,\hat{\boldsymbol{\imath}}+v^2\,\hat{\boldsymbol{\jmath}}+(u^2+v^3)\,\hat{\mathbf{k}}$$

so the vectors

$$\begin{aligned}\frac{\partial\mathbf{r}}{\partial u}(u,v)&=\big(4u\,,\,0\,,\,2u\big)\\ \frac{\partial\mathbf{r}}{\partial v}(u,v)&=\big(0\,,\,2v\,,\,3v^2\big)\end{aligned}$$

are tangent to S at $\mathbf{r}(u,v)$. Note that $\mathbf{r}(2,1)=(8,1,5)$. So

$$\begin{aligned}\frac{\partial\mathbf{r}}{\partial u}(2,1)&=\big(8\,,\,0\,,\,4\big)\\ \frac{\partial\mathbf{r}}{\partial v}(2,1)&=\big(0\,,\,2\,,\,3\big)\end{aligned}$$

are tangent to S at $\mathbf{r}(2,1)=(8,1,5)$ and

$$\begin{aligned}\frac{\partial\mathbf{r}}{\partial u}(2,1)\times\frac{\partial\mathbf{r}}{\partial v}(2,1)&=\big(8\,,\,0\,,\,4\big)\times\big(0\,,\,2\,,\,3\big)\\ &=\det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ 8 & 0 & 4\\ 0 & 2 & 3\end{bmatrix}\\ &=\big(-8\,,\,-24\,,\,16\big)\end{aligned}$$

or $\frac{1}{-8}\big(-8\,,\,-24\,,\,16\big)=\big(1\,,\,3\,,\,-2\big)$ is normal to S at $(8,1,5)$. So the tangent plane is

$$(1,3,-2)\cdot\big\{(x,y,z)-(8,1,5)\big\}=0 \qquad \text{or} \qquad x+3y-2z=1$$

S-11: To find the tangent plane we have to find a normal vector to the surface at $(2,2,0)$. Since

$$\begin{aligned}\frac{\partial\mathbf{r}}{\partial u}&=\big(1\,,\,2u\,,\,1\big)\\ \frac{\partial\mathbf{r}}{\partial v}&=\big(1\,,\,2v\,,\,-1\big)\end{aligned}$$

a normal vector to the surface at $\mathbf{r}(u,v)$ is

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v} &= \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ 1 & 2u & 1\\ 1 & 2v & -1\end{bmatrix}\\ &= \big(-2u-2v\,,\,2\,,\,2v-2u\big)\end{aligned}$$

As $\mathbf{r}(u,v)=(2,2,0)$ when (the x-coordinate) $u+v=2$ and (the z-coordinate) $u-v=0$, i.e when $u=v=1$, a normal vector to the surface at $(2,2,0)=\mathbf{r}(1,1)$ is

$$(-4,2,0) \qquad \text{or} \qquad (-2,1,0)$$

and the equation of the specified tangent plane is

$$-2(x-2)+(y-2)+0z=0 \qquad \text{or} \qquad y=2x-2$$

S-12: The first order partial derivatives of f are

$$\begin{aligned} f_x(x,y) &= -\frac{4xy}{(x^2+y^2)^2} & f_x(-1,2) &= \frac{8}{25}\\ f_y(x,y) &= \frac{2}{x^2+y^2}-\frac{4y^2}{(x^2+y^2)^2} & f_y(-1,2) &= \frac{2}{5}-\frac{16}{25}=-\frac{6}{25}\end{aligned}$$

So, by part (b) of Theorem 3.2.1 in the CLP-4 text, a normal vector to the surface at $(x,y)=(-1,2)$ is $(\frac{8}{25}\,,\,-\frac{6}{25}\,,\,-1)$. As $f(-1,2)=\frac{4}{5}$, the tangent plane is

$$\Big(\frac{8}{25}\,,\,-\frac{6}{25}\,,\,-1\Big)\cdot\Big(x+1\,,\,y-2\,,\,z-\frac{4}{5}\Big)=0 \quad\text{or}\quad \frac{8}{25}x-\frac{6}{25}y-z=-\frac{8}{5}$$

and the normal line is

$$(x,y,z)=\Big(-1,2,\frac{4}{5}\Big)+t\Big(\frac{8}{25}\,,\,-\frac{6}{25}\,,\,-1\Big)$$

S-13: A normal vector to the surface $x^2+9y^2+4z^2=17$ at the point (x,y,z) is $(2x\,,\,18y\,,\,8z)$. A normal vector to the plane $x-8z=0$ is $(1\,,\,0\,,\,-8)$. So we want $(2x\,,\,18y\,,\,8z)$ to be parallel to $(1\,,\,0\,,\,-8)$, i.e. to be a nonzero constant times $(1\,,\,0\,,\,-8)$. This is the case whenever $y=0$ and $z=-2x$ with $x\neq 0$. In addition, we want (x,y,z) to lie on the surface $x^2+9y^2+4z^2=17$. So we want $y=0$, $z=-2x$ and

$$17=x^2+9y^2+4z^2=x^2+4(-2x)^2=17x^2 \implies x=\pm 1$$

So the allowed points are $\pm(1,0,-2)$.

S-14: The equation of S is of the form $G(x,y,z) = x^2 + 2y^2 + 2y - z = 1$. So one normal vector to S at the point (x_0, y_0, z_0) is

$$\nabla G(x_0, y_0, z_0) = 2x_0\,\hat{\imath} + (4y_0 + 2)\,\hat{\jmath} - \hat{\mathbf{k}}$$

and the normal line to S at (x_0, y_0, z_0) is

$$(x, y, z) = (x_0, y_0, z_0) + t(2x_0\,,\, 4y_0 + 2\,,\, -1)$$

For this normal line to pass through the origin, there must be a t with

$$(0, 0, 0) = (x_0, y_0, z_0) + t(2x_0\,,\, 4y_0 + 2\,,\, -1)$$

or

$$x_0 + 2x_0\,t = 0 \tag{E1}$$
$$y_0 + (4y_0 + 2)t = 0 \tag{E2}$$
$$z_0 - t = 0 \tag{E3}$$

Equation (E3) forces $t = z_0$. Substituting this into equations (E1) and (E2) gives

$$x_0(1 + 2z_0) = 0 \tag{E1}$$
$$y_0 + (4y_0 + 2)z_0 = 0 \tag{E2}$$

The question specifies that $x_0 \neq 0$, so (E1) forces $z_0 = -\frac{1}{2}$. Substituting $z_0 = -\frac{1}{2}$ into (E2) gives

$$-y_0 - 1 = 0 \implies y_0 = -1$$

Finally x_0 is determined by the requirement that (x_0, y_0, z_0) must lie on S and so must obey

$$z_0 = x_0^2 + 2y_0^2 + 2y_0 - 1 \implies -\frac{1}{2} = x_0^2 + 2(-1)^2 + 2(-1) - 1 \implies x_0^2 = \frac{1}{2}$$

So the allowed points P are $\big(\frac{1}{\sqrt{2}}\,,\, -1\,,\, -\frac{1}{2}\big)$ and $\big(-\frac{1}{\sqrt{2}}\,,\, -1\,,\, -\frac{1}{2}\big)$.

S-15: Let (x_0, y_0, z_0) be a point on the hyperboloid $z^2 = 4x^2 + y^2 - 1$ where the tangent plane is parallel to the plane $2x - y + z = 0$. A normal vector to the plane $2x - y + z = 0$ is $(2, -1, 1)$. Because the hyperboloid is $G(x, y, z) = 4x^2 + y^2 - z^2 - 1$ and $\nabla G(x, y, z) = (8x, 2y, -2z)$, a normal vector to the hyperboloid at (x_0, y_0, z_0) is $\nabla G(x_0, y_0, z_0) = (8x_0, 2y_0, -2z_0)$. So (x_0, y_0, z_0) satisfies the required conditions if and only if there is a nonzero t obeying

$$\begin{aligned}
&(8x_0, 2y_0, -2z_0) = t(2, -1, 1) \text{ and } z_0^2 = 4x_0^2 + y_0^2 - 1 \\
\iff\ & x_0 = \frac{t}{4},\ y_0 = z_0 = -\frac{t}{2} \text{ and } z_0^2 = 4x_0^2 + y_0^2 - 1 \\
\iff\ & \frac{t^2}{4} = \frac{t^2}{4} + \frac{t^2}{4} - 1 \text{ and } x_0 = \frac{t}{4},\ y_0 = z_0 = -\frac{t}{2} \\
\iff\ & t = \pm 2 \qquad (x_0, y_0, z_0) = \pm\big(\tfrac{1}{2}, -1, -1\big)
\end{aligned}$$

S-16: (a) A vector perpendicular to $x^2+z^2=10$ at $(1,1,3)$ is

$$\nabla(x^2+z^2)\big|_{(1,1,3)} = (2x\hat{\imath}+2z\hat{\mathbf{k}})\big|_{(1,1,3)} = 2\hat{\imath}+6\hat{\mathbf{k}} \text{ or } \frac{1}{2}(2,0,6)=(1,0,3)$$

(b) A vector perpendicular to $y^2+z^2=10$ at $(1,1,3)$ is

$$\nabla(y^2+z^2)\big|_{(1,1,3)} = (2y\hat{\jmath}+2z\hat{\mathbf{k}})\big|_{(1,1,3)} = 2\hat{\jmath}+6\hat{\mathbf{k}} \text{ or } \frac{1}{2}(0,2,6)=(0,1,3)$$

A vector is tangent to the specified curve at the specified point if and only if it perpendicular to both $(1,0,3)$ and $(0,1,3)$. One such vector is

$$(0,1,3)\times(1,0,3) = \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ 0 & 1 & 3\\ 1 & 0 & 3\end{bmatrix} = (3,3,-1)$$

(c) The specified tangent line passes through $(1,1,3)$ and has direction vector $(1,1,3)$ and so has vector parametric equation

$$\mathbf{r}(t) = (1,1,3)+t(3,3,-1)$$

S-17: $\mathbf{r}(t) = (x(t)\,,\,y(t)\,,\,z(t))$ intersects $z^3+xyz-2=0$ when

$$z(t)^3+x(t)\,y(t)\,z(t)-2=0 \iff \left(t^2\right)^3+\left(t^3\right)(t)\left(t^2\right)-2=0 \iff 2t^6=2 \iff t=1$$

since t is required to be positive. The direction vector for the curve at $t=1$ is

$$\mathbf{r}'(1) = 3\,\hat{\imath}+\hat{\jmath}+2\,\hat{\mathbf{k}}$$

A normal vector for the surface at $\mathbf{r}(1)=(1,1,1)$ is

$$\nabla(z^3+xyz)\big|_{(1,1,1)} = [yz\hat{\imath}+xz\hat{\jmath}+(3z^2+xy)\hat{\mathbf{k}}]_{(1,1,1)} = \hat{\imath}+\hat{\jmath}+4\hat{\mathbf{k}}$$

The angle θ between the curve and the normal vector to the surface is determined by

$$\begin{aligned}\big|(3,1,2)\big|\,\big|(1,1,4)\big|\cos\theta = (3,1,2)\cdot(1,1,4) &\iff \sqrt{14}\sqrt{18}\cos\theta = 12\\ &\iff \sqrt{7\times 36}\cos\theta = 12\\ &\iff \cos\theta = \frac{2}{\sqrt{7}}\\ &\iff \theta = 40.89^\circ\end{aligned}$$

The angle between the curve and the surface is $90-40.89=49.11^\circ$ (to two decimal places).

S-18: Let (x_0, y_0, z_0) be any point on the surface. A vector normal to the surface at (x_0, y_0, z_0) is

$$\begin{aligned}\nabla\Big(xye^{-(x^2+y^2)/2} - z\Big)\Big|_{(x_0,y_0,z_0)}\\ &= \Big(y_0e^{-(x_0^2+y_0^2)/2} - x_0^2y_0e^{-(x_0^2+y_0^2)/2}, x_0e^{-(x_0^2+y_0^2)/2} - x_0y_0^2e^{-(x_0^2+y_0^2)/2}, -1\Big)\end{aligned}$$

The tangent plane to the surface at (x_0, y_0, z_0) is horizontal if and only if this vector is vertical, which is the case if and only if its x- and y-components are zero, which in turn is the case if and only if

$$\begin{aligned}&y_0(1-x_0^2) = 0 \text{ and } x_0(1-y_0^2) = 0\\ &\iff \{y_0 = 0 \text{ or } x_0 = 1 \text{ or } x_0 = -1\} \text{ and } \{x_0 = 0 \text{ or } y_0 = 1 \text{ or } y_0 = -1\}\\ &\iff (x_0, y_0) = (0,0) \text{ or } (1,1) \text{ or } (1,-1) \text{ or } (-1,1) \text{ or } (-1,-1)\end{aligned}$$

The values of z_0 at these points are 0, e^{-1}, $-e^{-1}$, $-e^{-1}$ and e^{-1}, respectively. So the horizontal tangent planes are $z = 0$, $z = e^{-1}$ and $z = -e^{-1}$. At the highest and lowest points of the surface, the tangent plane is horizontal. So the largest and smallest values of z are e^{-1} and $-e^{-1}$, respectively.

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S-1: (a) S is the part of the plane $z = y\tan\theta$ that lies above the rectangle in the xy-plane with vertices $(0,0)$, $(a,0)$, $(0,b)$, (a,b). So S is the rectangle with vertices $(0,0,0)$, $(a,0,0)$, $(0,b,b\tan\theta)$, $(a,b,b\tan\theta)$. So it has side lengths

$$\begin{aligned}|(a,0,0) - (0,0,0)| &= a\\ |(0,b,b\tan\theta) - (0,0,0)| &= \sqrt{b^2 + b^2\tan^2\theta}\end{aligned}$$

and hence area $ab\sqrt{1+\tan^2\theta} = ab\sec\theta$.

(b) S is the part of the surface $z = f(x,y)$ with $f(x,y) = y\tan\theta$ and with (x,y) running over

$$\mathcal{D} = \big\{\, (x,y) \;\big|\; 0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b \,\big\}$$

Hence by (3.3.2) in the CLP-4 text

$$\mathrm{d}S = \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\,\mathrm{d}x\,\mathrm{d}y = \sqrt{1+0^2+\tan^2\theta}\,\mathrm{d}x\,\mathrm{d}y$$

and

$$\begin{aligned}\text{Area}(S) &= \iint_S \mathrm{d}S = \iint_{\mathcal{D}} \sqrt{1+\tan^2\theta}\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_0^a \mathrm{d}x \int_0^b \mathrm{d}y\,\sqrt{1+\tan^2\theta}\\ &= ab\sqrt{1+\tan^2\theta} = ab\sec\theta\end{aligned}$$

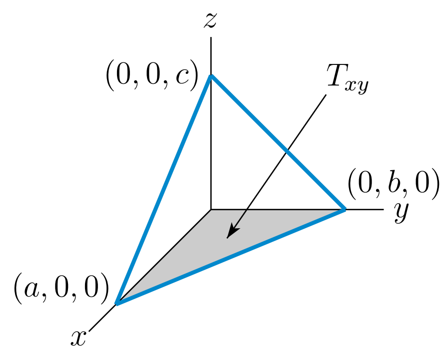
S-2: Note that all three vertices $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$ lie on the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. So the triangle is part of that plane.

Method 1. S is the part of the surface $z=f(x,y)$ with $f(x,y)=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$ and with (x,y) running over the triangle T_{xy} in the xy-plane with vertices $(0,0,0)$ $(a,0,0)$ and $(0,b,0)$. Hence by the first part of (3.3.2) in the CLP-4 text,

$$\begin{aligned}
\text{Area}(S) &= \iint_{T_{xy}} \sqrt{1+f_x(x,y)^2+f_y(x,y)^2}\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_{T_{xy}} \sqrt{1+\frac{c^2}{a^2}+\frac{c^2}{b^2}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \sqrt{1+\frac{c^2}{a^2}+\frac{c^2}{b^2}}\;A(T_{xy})
\end{aligned}$$

where $A(T_{xy})$ is the area of T_{xy}. Since the triangle T_{xy} has base a and height b (see the figure below), it has area $\frac{1}{2}ab$. So

$$\text{Area}(S) = \frac{1}{2}\sqrt{1+\frac{c^2}{a^2}+\frac{c^2}{b^2}}\;ab = \frac{1}{2}\sqrt{a^2b^2+a^2c^2+b^2c^2}$$



Method 2. S is the part of the surface $x=g(y,z)$ with $g(y,z)=a\left(1-\frac{y}{b}-\frac{z}{c}\right)$ and with (y,z) running over the triangle T_{yz} in the yz-plane with vertices $(0,0,0)$ $(0,b,0)$ and $(0,0,c)$. Hence by the second part of (3.3.2) in the CLP-4 text,

$$\begin{aligned}
\text{Area}(S) &= \iint_{T_{yz}} \sqrt{1+g_y(y,z)^2+g_z(y,z)^2}\,\mathrm{d}y\,\mathrm{d}z \\
&= \iint_{T_{yz}} \sqrt{1+\frac{a^2}{b^2}+\frac{a^2}{c^2}}\,\mathrm{d}y\,\mathrm{d}z \\
&= \sqrt{1+\frac{a^2}{b^2}+\frac{a^2}{c^2}}\;A(T_{yz})
\end{aligned}$$

where $A(T_{yz})$ is the area of T_{yz}. Since T_{yz} has base b and height c, it has area $\frac{1}{2}bc$. So

$$\text{Area}(S) = \frac{1}{2}\sqrt{1+\frac{a^2}{b^2}+\frac{a^2}{c^2}}\;bc = \frac{1}{2}\sqrt{a^2b^2+a^2c^2+b^2c^2}$$

Method 3. S is the part of the surface $y = h(x,z)$ with $h(x,z) = b\left(1 - \frac{x}{a} - \frac{z}{c}\right)$ and with (x,z) running over the triangle T_{xz} in the xz-plane with vertices $(0,0,0)$ $(a,0,0)$ and $(0,0,c)$. Hence by the third part of (3.3.2) in the CLP-4 text,

$$\begin{aligned}
\text{Area}(S) &= \iint_{T_{xz}} \sqrt{1 + h_x(x,z)^2 + h_z(x,z)^2}\,\mathrm{d}x\,\mathrm{d}z \\
&= \iint_{T_{xz}} \sqrt{1 + \frac{b^2}{a^2} + \frac{b^2}{c^2}}\,\mathrm{d}x\,\mathrm{d}z \\
&= \sqrt{1 + \frac{b^2}{a^2} + \frac{b^2}{c^2}}\,A(T_{xz})
\end{aligned}$$

where $A(T_{xz})$ is the area of T_{xz}. Since T_{xz} has base a and height c, it has area $\frac{1}{2}ac$. So

$$\text{Area}(S) = \frac{1}{2}\sqrt{1 + \frac{b^2}{a^2} + \frac{b^2}{c^2}}\,bc = \frac{1}{2}\sqrt{a^2b^2 + a^2c^2 + b^2c^2}$$

(b) We have already seen in the solution to part (a) that

$$\text{Area}(T_{xy}) = \frac{ab}{2} \qquad \text{Area}(T_{xz}) = \frac{ac}{2} \qquad \text{Area}(T_{yz}) = \frac{bc}{2}$$

Hence

$$\begin{aligned}
\text{Area}(S) &= \sqrt{\frac{a^2b^2}{4} + \frac{a^2c^2}{4} + \frac{b^2c^2}{4}} \\
&= \sqrt{\text{Area}(T_{xy})^2 + \text{Area}(T_{xz})^2 + \text{Area}(T_{yz})^2}
\end{aligned}$$

S-3: (a) Think of the cylinder as being a piece of paper that has been partially rolled up. If you flatten the piece of paper out, you get a rectangle with the length of one side being h and the length of the other side being one quarter of the circumference of a circle of radius a, i.e. $\frac{1}{4}(2\pi a) = \frac{\pi a}{2}$. So the area of S is $\frac{\pi a h}{2}$.

(b) S is parametrized by

$$x(\theta, y) = a\cos\theta \qquad y(\theta, y) = y \qquad z(\theta, z) = a\sin\theta$$

with (θ, y) running over $0 \leqslant \theta \leqslant \frac{\pi}{2}$, $0 \leqslant y \leqslant h$. Then, by (3.3.1) in the CLP-4 text,

$$\begin{aligned}
\left(\frac{\partial x}{\partial\theta}, \frac{\partial y}{\partial\theta}, \frac{\partial z}{\partial\theta}\right) &= (-a\sin\theta, 0, a\cos\theta) \\
\left(\frac{\partial x}{\partial y}, \frac{\partial y}{\partial y}, \frac{\partial z}{\partial y}\right) &= (0, 1, 0) \\
\mathrm{d}S &= \left|\left(\frac{\partial x}{\partial\theta}, \frac{\partial y}{\partial\theta}, \frac{\partial z}{\partial\theta}\right) \times \left(\frac{\partial x}{\partial y}, \frac{\partial y}{\partial y}, \frac{\partial z}{\partial y}\right)\right| \mathrm{d}\theta \mathrm{d}y \\
&= \left|(-a\cos\theta, 0, -a\sin\theta)\right|\,\mathrm{d}\theta\,\mathrm{d}y \\
&= a\,\mathrm{d}\theta\,\mathrm{d}y
\end{aligned}$$

So

$$\text{Area}(S) = \iint_S \mathrm{d}S = \int_0^{\pi/2} \mathrm{d}\theta \int_0^h \mathrm{d}y\; a = a\left(\frac{\pi}{2}\right) h$$

S-4: The surface is $z = f(x,y)$ with $f(x,y) = xy$. So, by (3.3.2) in the CLP-4 text,

$$\mathrm{d}S = \sqrt{1 + f_x^2 + f_y^2}\, \mathrm{d}x\, \mathrm{d}y = \sqrt{1 + x^2 + y^2}\, \mathrm{d}x\, \mathrm{d}y$$

and

$$\begin{aligned} I = \iint_S (x^2 + y^2)\, \mathrm{d}S &= \iint_{x^2+y^2 \leqslant 3} (x^2 + y^2)\, \sqrt{1 + x^2 + y^2}\, \mathrm{d}x\, \mathrm{d}y \\ &= \int_0^{2\pi} \mathrm{d}\theta \int_0^{\sqrt{3}} \mathrm{d}r\; r\, r^2 \sqrt{1 + r^2} \end{aligned}$$

We switched to polar coordinates in the last step. Making the change of variables $u = 1 + r^2$, $\mathrm{d}u = 2r\, \mathrm{d}r$

$$I = \pi \int_1^4 \mathrm{d}u\; (u-1)\sqrt{u} = \pi\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^4 = \pi\left[\frac{64}{5} - \frac{16}{3} - \frac{2}{5} + \frac{2}{3}\right] = \frac{116}{15}\pi$$

S-5: First observe that any point (x, y, z) on the paraboliod lies above the xy-plane if and only if

$$0 \leqslant z = a^2 - x^2 - y^2 \iff x^2 + y^2 \leqslant a^2$$

That is, if and only if (x, y) lies in the circular disk of radius a centred on the origin. The equation of the paraboloid is of the form $z = f(x, y)$ with $f(x, y) = a^2 - x^2 - y^2$. So, by (3.3.2) in the CLP-4 text,

$$\begin{aligned} \text{Surface area} &= \iint_{x^2+y^2 \leqslant a^2} \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\, \mathrm{d}x\, \mathrm{d}y \\ &= \iint_{x^2+y^2 \leqslant a^2} \sqrt{1 + 4x^2 + 4y^2}\, \mathrm{d}x\, \mathrm{d}y \end{aligned}$$

Switching to polar coordinates,

$$\begin{aligned}
\text{Surface area} &= \int_0^a \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\ r\sqrt{1+4r^2} \\
&= 2\pi \int_0^a \mathrm{d}r\ r\sqrt{1+4r^2} \\
&= 2\pi \int_1^{1+4a^2} \frac{\mathrm{d}s}{8}\sqrt{s} \qquad \text{with } s = 1+4r^2, \mathrm{d}s = 8r\,\mathrm{d}r \\
&= \frac{\pi}{4}\ \frac{2}{3}s^{3/2}\Big|_{s=1}^{s=1+4a^2} \\
&= \frac{\pi}{6}\left[\left(1+4a^2\right)^{3/2} - 1\right]
\end{aligned}$$

S-6: First observe that any point (x, y, z) on the cone lies between the planes $z = 2$ and $z = 3$ if and only if $4 \leqslant x^2 + y^2 \leqslant 9$. The equation of the cone can be rewritten in the form $z = f(x, y)$ with $f(x, y) = \sqrt{x^2 + y^2}$. Note that

$$f_x(x, y) = \frac{x}{\sqrt{x^2+y^2}} \qquad f_y(x, y) = \frac{y}{\sqrt{x^2+y^2}}$$

So, by (3.3.2) in the CLP-4 text,

$$\begin{aligned}
\text{Surface area} &= \iint\limits_{4\leqslant x^2+y^2\leqslant 9} \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint\limits_{4\leqslant x^2+y^2\leqslant 9} \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \sqrt{2} \iint\limits_{4\leqslant x^2+y^2\leqslant 9} \mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

Now the domain of integration is a circular washer with outside radius 3 and inside radius 2 and hence of area $\pi(3^2 - 2^2) = 5\pi$. So the surface area is $5\sqrt{2}\pi$.

S-7: The equation of the surface is of the form $z = f(x, y)$ with $f(x, y) = \frac{2}{3}\left(x^{3/2} + y^{3/2}\right)$. Note that

$$f_x(x, y) = \sqrt{x} \qquad f_y(x, y) = \sqrt{y}$$

So, by (3.3.2) in the CLP-4 text,

$$\begin{aligned}
\text{Surface area} &= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\ \sqrt{1+f_x(x,y)^2+f_y(x,y)^2} \\
&= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\ \sqrt{1+x+y} \\
&= \int_0^1 \mathrm{d}x\ \Big[\frac{2}{3}(1+x+y)^{3/2}\Big]_{y=0}^{y=1} \\
&= \frac{2}{3}\int_0^1 \mathrm{d}x\ \big[(2+x)^{3/2}-(1+x)^{3/2}\big] \\
&= \frac{2}{3}\,\frac{2}{5}\Big[(2+x)^{5/2}-(1+x)^{5/2}\Big]_{x=0}^{x=1} \\
&= \frac{4}{15}\big[3^{5/2}-2^{5/2}-2^{5/2}+1^{5/2}\big] \\
&= \frac{4}{15}\big[9\sqrt{3}-8\sqrt{2}+1\big]
\end{aligned}$$

S-8: (a) By (3.3.2) in the CLP-4 text, $F(x,y)=\sqrt{1+f_x(x,y)^2+f_y(x,y)^2}$.

(b) (i) The "dimple" to be painted is part of the upper sphere $x^2+y^2+\big(z-2\sqrt{3}\big)^2=4$. It is on the bottom half of the sphere and so has equation $z=f(x,y)=2\sqrt{3}-\sqrt{4-x^2-y^2}$. Note that

$$f_x(x,y)=\frac{x}{\sqrt{4-x^2-y^2}} \qquad f_y(x,y)=\frac{y}{\sqrt{4-x^2-y^2}}$$

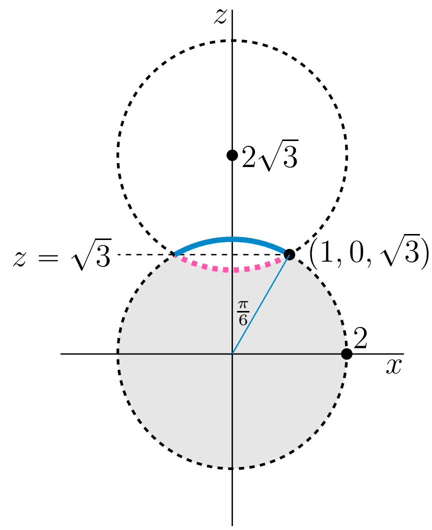
The point on the dimple with the largest value of x is $(1,0,\sqrt{3})$. (It is marked by a dot in the figure above.) The dimple is invariant under rotations around the z–axis and so has (x,y) running over $x^2+y^2\leqslant 1$. So, by (3.3.2) in the CLP-4 text,

$$\begin{aligned}
\text{Surface area} &= \iint_{x^2+y^2\leqslant 1} \sqrt{1+f_x(x,y)^2+f_y(x,y)^2}\ \mathrm{d}x\,\mathrm{d}y \\
&= \iint_{x^2+y^2\leqslant 1} \sqrt{1+\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}}\ \mathrm{d}x\,\mathrm{d}y \\
&= \iint_{x^2+y^2\leqslant 1} \frac{2}{\sqrt{4-x^2-y^2}}\ \mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

Switching to polar coordinates,

$$\text{Surface area} = \int_0^{2\pi}\mathrm{d}\theta\int_0^1 \mathrm{d}r\ \frac{2r}{\sqrt{4-r^2}}$$

(b) (ii) Observe that if we flip the dimple up by reflecting it in the plane $z=\sqrt{3}$, as in the figure below, the "Death Star" becomes a perfect ball of radius 2.



The area of the pink dimple in the figure above is identical to the area of the blue cap in that figure. So the total surface area of the Death Star is exactly the surface area of a sphere of radius 2 and so is $\frac{4}{3}\pi 2^3 = \frac{32\pi}{3}$.

S-9: On the upper half of the cone

$$z = f(x,y) = \sqrt{x^2+y^2} \qquad f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}} \qquad f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}$$

so that

$$\mathrm{d}S = \sqrt{1+f_x(x,y)^2+f_y(x,y)^2}\,\mathrm{d}x\,\mathrm{d}y = \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\,\mathrm{d}x\,\mathrm{d}y = \sqrt{2}\,\mathrm{d}x\,\mathrm{d}y$$

and

$$\begin{aligned}\text{Area} &= \iint\limits_{1\leqslant x^2+y^2\leqslant 16^2} \sqrt{2}\,\mathrm{d}x\,\mathrm{d}y \\ &= \sqrt{2}\Big[\text{area of }\{\ (x,y)\mid x^2+y^2\leqslant 16^2\ \} - \text{area of }\{\ (x,y)\mid x^2+y^2\leqslant 1\ \}\Big] \\ &= \sqrt{2}\left[\pi 16^2 - \pi 1^2\right] = 255\sqrt{2}\pi \approx 1132.9\end{aligned}$$

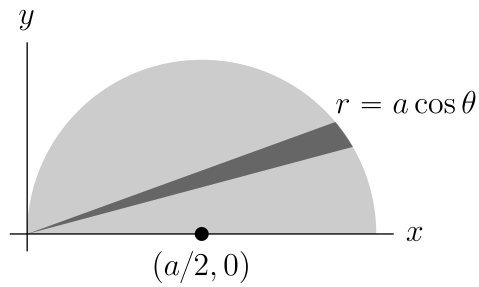
S-10: We are to find the surface area of part of a hemisphere. On the hemisphere

$$z = f(x,y) = \sqrt{a^2-x^2-y^2} \qquad f_x(x,y) = -\frac{x}{\sqrt{a^2-x^2-y^2}} \qquad f_y(x,y) = -\frac{y}{\sqrt{a^2-x^2-y^2}}$$

so that

$$\begin{aligned}\mathrm{d}S &= \sqrt{1+f_x(x,y)^2+f_y(x,y)^2}\,\mathrm{d}x\,\mathrm{d}y = \sqrt{1+\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}}\,\mathrm{d}x\,\mathrm{d}y \\ &= \sqrt{\frac{a^2}{a^2-x^2-y^2}}\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

In polar coordinates, this is $\mathrm{d}S = \frac{a}{\sqrt{a^2-r^2}}\, r\,\mathrm{d}r\,\mathrm{d}\theta$. We are to find the surface area of the part of the hemisphere that is inside the cylinder, $x^2 - ax + y^2 = 0$, which is polar coordinates is becomes $r^2 - ar\cos\theta = 0$ or $r = a\cos\theta$. The top half of the domain of integration is sketched below.



So the

$$\begin{aligned}
\text{Surface Area} &= 2\int_0^{\pi/2} \mathrm{d}\theta \int_0^{a\cos\theta} \mathrm{d}r\; r\frac{a}{\sqrt{a^2-r^2}} = 2a\int_0^{\pi/2}\mathrm{d}\theta\,\Big[-\sqrt{a^2-r^2}\Big]_0^{a\cos\theta} \\
&= 2a\int_0^{\pi/2}\mathrm{d}\theta\,\big[a - a\sin\theta\big] \\
&= 2a^2\Big[\theta + \cos\theta\Big]_0^{\pi/2} = a^2[\pi - 2]
\end{aligned}$$

S-11: The upper half cone obeys $f(x,y,z) = x^2+y^2-z^2 = 0$. So, by (3.3.3) in the CLP-4 text,

$$\mathrm{d}S = \left|\frac{\nabla f}{\nabla f\cdot\hat{\mathbf{k}}}\right|\mathrm{d}x\,\mathrm{d}y = \left|\frac{2x\hat{\imath}+2y\hat{\jmath}-2z\hat{\mathbf{k}}}{-2z}\right|\mathrm{d}x\,\mathrm{d}y = \frac{\sqrt{x^2+y^2+z^2}}{z}\mathrm{d}x\,\mathrm{d}y$$

But on the cone $x^2+y^2 = z^2$, and $z > 0$, so that

$$\mathrm{d}S = \frac{\sqrt{x^2+y^2+z^2}}{z}\mathrm{d}x\,\mathrm{d}y = \frac{\sqrt{2z^2}}{z}\mathrm{d}x\,\mathrm{d}y = \sqrt{2}\,\mathrm{d}x\,\mathrm{d}y$$

and

$$x^4 - y^4 + y^2z^2 - z^2x^2 + 1 = x^4 - y^4 + y^2\big(x^2+y^2\big) - \big(x^2+y^2\big)x^2 + 1 = 1$$

We have to integrate (x,y) over the interior of $x^2+y^2 = 2x$, or equivalently, the interior of $(x-1)^2+y^2 = 1$, which is the disk

$$D = \big\{\ (x,y)\ \big|\ (x-1)^2 + y^2 \leqslant 1\ \big\}$$

So

$$\iint_S \overbrace{\big(x^4 - y^4 + y^2z^2 - z^2x^2 + 1\big)}^{=1}\,\mathrm{d}S = \sqrt{2}\iint_D \mathrm{d}x\,\mathrm{d}y = \sqrt{2}\ \text{Area}(D) = \sqrt{2}\,\pi$$

S-12: As we saw in Example 3.1.5 of the CLP-4 text, the torus may be parametrized by

$$\mathbf{r}(\theta,\psi) = (R+r\cos\theta)\cos\psi\,\hat{\boldsymbol{\imath}} + (R+r\cos\theta)\sin\psi\,\hat{\boldsymbol{\jmath}} + r\sin\theta\,\hat{\mathbf{k}} \qquad 0\leqslant\theta,\psi\leqslant 2\pi$$

Then

$$\frac{\partial\mathbf{r}}{\partial\psi} = (R+r\cos\theta)\big[-\sin\psi\hat{\boldsymbol{\imath}} + \cos\psi\hat{\boldsymbol{\jmath}}\big] \qquad \frac{\partial\mathbf{r}}{\partial\theta} = r\big[-\sin\theta\cos\psi\,\hat{\boldsymbol{\imath}} - \sin\theta\sin\psi\,\hat{\boldsymbol{\jmath}} + \cos\theta\,\hat{\mathbf{k}}\big]$$

and

$$\begin{aligned}\frac{\partial\mathbf{r}}{\partial\psi}\times\frac{\partial\mathbf{r}}{\partial\theta} &= r(R+r\cos\theta)\ \big[-\sin\psi\hat{\boldsymbol{\imath}} + \cos\psi\hat{\boldsymbol{\jmath}}\big] \times \big[-\sin\theta\cos\psi\,\hat{\boldsymbol{\imath}} - \sin\theta\sin\psi\,\hat{\boldsymbol{\jmath}} + \cos\theta\,\hat{\mathbf{k}}\big] \\ &= r(R+r\cos\theta)\det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ -\sin\psi & \cos\psi & 0 \\ -\sin\theta\cos\psi & -\sin\theta\sin\psi & \cos\theta\end{bmatrix} \\ &= r(R+r\cos\theta)\big[\cos\psi\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\psi\cos\theta\,\hat{\boldsymbol{\jmath}} + \sin\theta\,\hat{\mathbf{k}}\big]\end{aligned}$$

As $\big[\cos\psi\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\psi\cos\theta\,\hat{\boldsymbol{\jmath}} + \sin\theta\,\hat{\mathbf{k}}\big]$ is a unit vector, (we could have shortened this computation by observing that $-\sin\psi\,\hat{\boldsymbol{\imath}} + \cos\psi\,\hat{\boldsymbol{\jmath}}$ and $-\sin\theta\cos\psi\,\hat{\boldsymbol{\imath}} - \sin\theta\sin\psi\,\hat{\boldsymbol{\jmath}} + \cos\theta\,\hat{\mathbf{k}}$ are mutually perpendicular unit vectors, so that their cross product is automatically a unit vector) and

$$\left|\frac{\partial\mathbf{r}}{\partial\psi}\times\frac{\partial\mathbf{r}}{\partial\theta}\right| = r(R+r\cos\theta) \implies \mathrm{d}S = r(R+r\cos\theta)\,d\psi\,\mathrm{d}\theta$$

The total surface area of the torus is

$$r\int_0^{2\pi}\mathrm{d}\theta\int_0^{2\pi}d\psi\,(R+r\cos\theta) = 2\pi r\int_0^{2\pi}\mathrm{d}\theta\,(R+r\cos\theta) = (2\pi)^2Rr$$

S-13: By symmetry, the centroid $(\bar{x},\bar{y},\bar{z})$ obeys $\bar{x}=\bar{y}=\bar{z}$. Parametrize the sphere using spherical coordinates.

$$\mathbf{r}(\theta,\varphi) = a\sin\varphi\cos\theta\,\hat{\boldsymbol{\imath}} + a\sin\varphi\sin\theta\,\hat{\boldsymbol{\jmath}} + a\cos\varphi\,\hat{\mathbf{k}}$$

Then

$$\frac{\partial\mathbf{r}}{\partial\theta} = -a\sin\varphi\sin\theta\,\hat{\boldsymbol{\imath}} + a\sin\varphi\cos\theta\,\hat{\boldsymbol{\jmath}} \qquad \frac{\partial\mathbf{r}}{\partial\varphi} = a\cos\varphi\cos\theta\,\hat{\boldsymbol{\imath}} + a\cos\varphi\sin\theta\,\hat{\boldsymbol{\jmath}} - a\sin\varphi\,\hat{\mathbf{k}}$$

so that

$$\begin{aligned}\frac{\partial\mathbf{r}}{\partial\theta}\times\frac{\partial\mathbf{r}}{\partial\varphi} &= \det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ -a\sin\varphi\sin\theta & a\sin\varphi\cos\theta & 0 \\ a\cos\varphi\cos\theta & a\cos\varphi\sin\theta & -a\sin\varphi\end{bmatrix} \\ &= -a^2\sin^2\varphi\cos\theta\,\hat{\boldsymbol{\imath}} - a^2\sin^2\varphi\sin\theta\,\hat{\boldsymbol{\jmath}} - a^2\sin\varphi\cos\varphi\,\hat{\mathbf{k}} \\ \implies \mathrm{d}S &= \left|\frac{\partial\mathbf{r}}{\partial\theta}\times\frac{\partial\mathbf{r}}{\partial\varphi}\right|\mathrm{d}\theta\,\mathrm{d}\varphi = a^2\sin\varphi\,\mathrm{d}\theta\,\mathrm{d}\varphi\end{aligned}$$

As the surface area of the part of the sphere in the first octant is $\frac{1}{8}4\pi a^2 = \frac{\pi a^2}{2}$

$$\begin{aligned}\bar{x} = \bar{y} = \bar{z} = \frac{\iint_S z\,\mathrm{d}S}{\iint_S \mathrm{d}S} &= \frac{1}{\pi a^2/2}\int_0^{\pi/2} \mathrm{d}\theta \int_0^{\pi/2} \mathrm{d}\varphi\ (a^2 \sin\varphi)(\overbrace{a\cos\varphi}^{z}) \\ &= \frac{2a}{\pi}\frac{\pi}{2}\int_0^{\pi/2} \mathrm{d}\varphi\ \sin\varphi\cos\varphi = a\Big[\frac{1}{2}\sin^2\varphi\Big]_0^{\pi/2} = \frac{a}{2}\end{aligned}$$

S-14: In cylindrical coordinates

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

In these coordinates the equation, $x^2 + y^2 = 2ay$, of the cylinder becomes

$$r^2 = 2ar\sin\theta \qquad \text{or} \qquad r = 2a\sin\theta$$

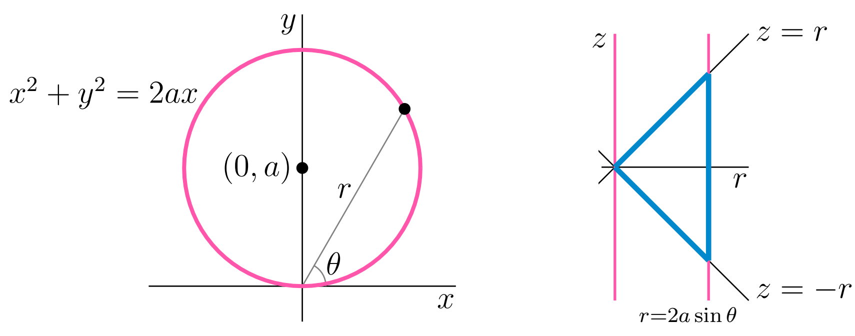
That is, $r = f(\theta)$ with $f(\theta) = 2a\sin\theta$. Parametrize the cylinder by $\mathbf{r}(\theta, z) = x(\theta, z)\,\hat{\boldsymbol{\imath}} + y(\theta, z)\,\hat{\boldsymbol{\jmath}} + z(\theta, z)\,\hat{\mathbf{k}}$ with

$$\begin{aligned} x(\theta, z) &= f(\theta)\cos\theta = 2a\sin\theta\cos\theta = a\sin 2\theta \\ y(\theta, z) &= f(\theta)\sin\theta = 2a\sin\theta\sin\theta = a(1 - \cos 2\theta) \\ z(\theta, z) &= z \end{aligned}$$

Under this parametrization,

$$\begin{aligned} \frac{\partial\mathbf{r}}{\partial\theta} = 2a\cos 2\theta\,\hat{\boldsymbol{\imath}} + 2a\sin 2\theta\,\hat{\boldsymbol{\jmath}} \qquad \frac{\partial\mathbf{r}}{\partial z} = \hat{\mathbf{k}} \implies \frac{\partial\mathbf{r}}{\partial\theta} \times \frac{\partial\mathbf{r}}{\partial z} &= -2a\cos 2\theta\,\hat{\boldsymbol{\jmath}} + 2a\sin 2\theta\,\hat{\boldsymbol{\imath}} \\ \implies \mathrm{d}S = \left|\frac{\partial\mathbf{r}}{\partial\theta} \times \frac{\partial\mathbf{r}}{\partial z}\right| \mathrm{d}\theta\,\mathrm{d}z &= 2a\,\mathrm{d}\theta\,\mathrm{d}z \end{aligned}$$

We still have to determine the limits of integration. The figure on the left below provides a top view of the cylinder.



From it we see that $0 \leqslant \theta \leqslant \pi$. The cone $z^2 = x^2 + y^2 = r^2$ (i.e. $z = \pm r$) and the cylinder $r = 2a\sin\theta$ intersect at $z^2 = r^2 = 4a^2\sin^2\theta$. So, for each fixed θ, z runs from $-2a\sin\theta$ to $z = +2a\sin\theta$. (See the figure on the right above. It shows a constant θ cross-section.) Finally,

$$\begin{aligned}\text{Area} &= \int_{|z|\leqslant 2a\sin\theta} 2a\,\mathrm{d}\theta\,\mathrm{d}z = 2a\int_0^{\pi}\mathrm{d}\theta\int_{-2a\sin\theta}^{2a\sin\theta}\mathrm{d}z = 8a^2\int_0^{\pi}\mathrm{d}\theta\ \sin\theta = 8a^2\Big[-\cos\theta\Big]_0^{\pi} \\ &= 16a^2\end{aligned}$$

S-15: (a) This right circular cone symmetric about the z-axis projects down onto a disk $\mathcal{D}$ in the plane $z = 0$. Setting $z = b$ gives

$$\mathcal{D} = \{ (x, y, z) \mid x^2 + y^2 \leqslant a^2,\ z = 0 \}$$

Since $G(x, y, z) = b^2(x^2 + y^2) - a^2z^2$ is constant on $\mathcal{S}$, the area elements $\mathrm{d}S$ on $\mathcal{S}$ are related to area elements $\mathrm{d}x\mathrm{d}y$ on $\mathcal{D}$ as follows:

$$\mathrm{d}S = \frac{|\nabla G(x, y, z)|}{|\nabla G(x, y, z) \cdot \hat{\mathbf{k}}|}\,\mathrm{d}x\mathrm{d}y = \frac{2|(b^2x, b^2y, -a^2z)|}{2|a^2z|}\,\mathrm{d}x\mathrm{d}y = \frac{\sqrt{b^4(x^2 + y^2) + a^4z^2}}{a^2z}\,\mathrm{d}x\mathrm{d}y$$

by (3.3.3) in the CLP-4 text. The defining equation for $\mathcal{S}$ gives $z = \frac{b}{a}\sqrt{x^2 + y^2}$, so

$$\mathrm{d}S = \frac{\sqrt{b^4(x^2 + y^2) + a^2b^2(x^2 + y^2)}}{a\sqrt{b^2(x^2 + y^2)}}\,\mathrm{d}x\mathrm{d}y = \frac{1}{a}\sqrt{a^2 + b^2}\,\mathrm{d}x\mathrm{d}y.$$

Hence $I = \frac{\sqrt{a^2+b^2}}{a}\iint_{\mathcal{D}}(x^2 + y^2)\,\mathrm{d}x\mathrm{d}y$.

(b) Or, parametrize the surface $\mathcal{S}$ using θ and t as follows:

$$x = t\cos\theta,\ y = t\sin\theta,\ z = \frac{b}{a}\sqrt{x^2 + y^2} = \frac{b}{a}t, \qquad 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant t \leqslant a. \tag{$*$}$$

Then we have, by (3.3.1) in the CLP-4 text,

$$\frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial \theta} = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \cos\theta & \sin\theta & b/a \\ -t\sin\theta & t\cos\theta & 0 \end{bmatrix} = \Big(-\frac{b}{a}t\cos\theta, -\frac{b}{a}t\sin\theta, t\Big),$$

$$\text{so}\quad \mathrm{d}S = \left|\frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial \theta}\right|\mathrm{d}t\,\mathrm{d}\theta = t\sqrt{1 + b^2/a^2}\,\mathrm{d}t\,\mathrm{d}\theta.$$

It follows that for the rectangular region $\mathcal{R}$ of the $t\theta$-plane described in $(*)$,

$$I = \iint_{\mathcal{R}} (t^2)t\sqrt{1 + b^2/a^2}\,\mathrm{d}t\,\mathrm{d}\theta.$$

(c) Using polar coordinates in (a) would give

$$I = \frac{\sqrt{a^2 + b^2}}{a}\int_{\theta=0}^{2\pi}\int_{r=0}^{a} r^2\,r\,\mathrm{d}r\,\mathrm{d}\theta = \frac{\pi}{2}a^3\sqrt{a^2 + b^2}.$$

Direct integration in (b) gives the same thing, because

$$I = \iint_{\mathcal{D}} (t^2)t\sqrt{1 + b^2/a^2}\,\mathrm{d}t\,\mathrm{d}\theta = \frac{\sqrt{a^2 + b^2}}{a}\int_{\theta=0}^{2\pi}\int_{t=0}^{a} t^3\,\mathrm{d}t\,\mathrm{d}\theta.$$

S-16: (a) The surface is $g(x,y,z) = x^2 + y^2 + z^2 - a^2 = 0$. So, on the surface of the sphere,

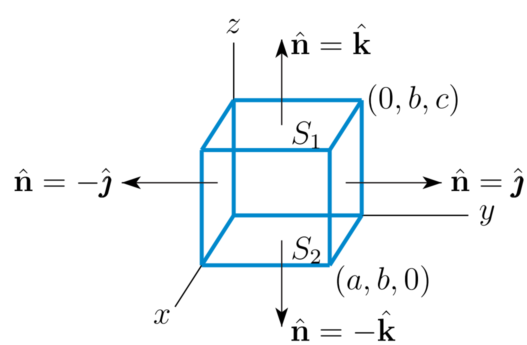
$$\hat{\mathbf{n}} = \frac{\nabla g}{|\nabla g|} = \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{\mathbf{k}}}{\sqrt{x^2+y^2+z^2}} \Rightarrow \mathbf{F}\cdot\hat{\mathbf{n}} = \left(x^2+y^2+z^2\right)^{n++1-1/2} = \left(a^2\right)^{n+1/2} = a^{2n+1}$$

$$\Rightarrow \iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = a^{2n+1}\iint_S \mathrm{d}S = a^{2n+1}\mathrm{Area}(S) = 4\pi a^{2n+3}$$

since the surface area of a sphere of radius a is $4\pi a^2$.

(b) The box has six faces.

$$\begin{aligned}
S_1 &= \left\{\, (x,y,z) \,\middle|\, 0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b,\ z = c \,\right\} && \text{with outward normal } \hat{\mathbf{n}} = \hat{\mathbf{k}}\\
S_2 &= \left\{\, (x,y,z) \,\middle|\, 0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b,\ z = 0 \,\right\} && \text{with outward normal } \hat{\mathbf{n}} = -\hat{\mathbf{k}}\\
S_3 &= \left\{\, (x,y,z) \,\middle|\, 0 \leqslant x \leqslant a,\ 0 \leqslant z \leqslant c,\ y = b \,\right\} && \text{with outward normal } \hat{\mathbf{n}} = \hat{\jmath}\\
S_4 &= \left\{\, (x,y,z) \,\middle|\, 0 \leqslant x \leqslant a,\ 0 \leqslant z \leqslant c,\ y = 0 \,\right\} && \text{with outward normal } \hat{\mathbf{n}} = -\hat{\jmath}\\
S_5 &= \left\{\, (x,y,z) \,\middle|\, 0 \leqslant y \leqslant b,\ 0 \leqslant z \leqslant c,\ x = a \,\right\} && \text{with outward normal } \hat{\mathbf{n}} = \hat{\imath}\\
S_6 &= \left\{\, (x,y,z) \,\middle|\, 0 \leqslant y \leqslant b,\ 0 \leqslant z \leqslant c,\ x = 0 \,\right\} && \text{with outward normal } \hat{\mathbf{n}} = -\hat{\imath}
\end{aligned}$$



For S_1, i.e. the $z = c$ face, and S_2, i.e. the $z = 0$ face,

$$\begin{aligned}
\int_{\substack{z=c\\ \text{face}}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \int_{\substack{z=c\\ \text{face}}} \left(x\,\hat{\imath} + y\,\hat{\jmath} + c\,\hat{\mathbf{k}}\right)\cdot\hat{\mathbf{k}}\,\mathrm{d}x\,\mathrm{d}y \qquad = c\int_{\substack{z=c\\ \text{face}}} \mathrm{d}x\,\mathrm{d}y = abc\\
\int_{\substack{z=0\\ \text{face}}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \int_{\substack{z=0\\ \text{face}}} \left(x\,\hat{\imath} + y\,\hat{\jmath} + 0\,\hat{\mathbf{k}}\right)\cdot\left(-\hat{\mathbf{k}}\right)\mathrm{d}x\,\mathrm{d}y = 0
\end{aligned}$$

because the $z = c$ face has area ab. Similarly,

$$\int_{\substack{x=0\\ \text{face}}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_{\substack{y=0\\ \text{face}}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 0 \qquad \int_{\substack{x=a\\ \text{face}}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_{\substack{y=b\\ \text{face}}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = abc$$

The total flux is $3abc$.

(c) The base of the cone is $\left\{\, (x,y,z) \,\middle|\, x^2 + y^2 \leqslant 1,\ z = 0 \,\right\}$ and has (outward) normal $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$. So The flux through the base is

$$\int_{\text{base}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_{x^2+y^2\leqslant 1} (y\hat{\imath})\cdot\left(-\hat{\mathbf{k}}\right)\mathrm{d}x\,\mathrm{d}y = 0$$

In cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ and the equation $z = 1-\sqrt{x^2+y^2}$ of the top part of the cone becomes $z = 1-r$. So we may parametrize the top part of the cone by

$$\mathbf{r}(r,\theta) = r\cos\theta\,\hat{\boldsymbol{\imath}} + r\sin\theta\,\hat{\boldsymbol{\jmath}} + (1-r)\,\hat{\mathbf{k}} \qquad \text{with } 0\leqslant\theta\leqslant 2\pi,\ 0\leqslant r\leqslant 1$$

Then

$$\begin{aligned}
\frac{\partial\mathbf{r}}{\partial r} &= \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}} - \hat{\mathbf{k}}\\
\frac{\partial\mathbf{r}}{\partial\theta} &= -r\sin\theta\,\hat{\boldsymbol{\imath}} + r\cos\theta\,\hat{\boldsymbol{\jmath}}\\
\frac{\partial\mathbf{r}}{\partial r}\times\frac{\partial\mathbf{r}}{\partial\theta} &= \det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ \cos\theta & \sin\theta & -1\\ -r\sin\theta & r\cos\theta & 0\end{bmatrix}\\
&= -r\cos\theta\,\hat{\boldsymbol{\imath}} - r\sin\theta\,\hat{\boldsymbol{\jmath}} + r\,\hat{\mathbf{k}}\\
\implies\ \hat{\mathbf{n}}\,\mathrm{d}S &= \frac{\partial\mathbf{r}}{\partial r}\times\frac{\partial\mathbf{r}}{\partial\theta}\,\mathrm{d}r\,\mathrm{d}\theta\\
&= \big(-r\cos\theta\,\hat{\boldsymbol{\imath}} - r\sin\theta\,\hat{\boldsymbol{\jmath}} + r\hat{\mathbf{k}}\big)\,\mathrm{d}r\,\mathrm{d}\theta
\end{aligned}$$

by (3.1.1) in the CLP-4 text. We have the orientation correct because the $\hat{\mathbf{k}}$ component of $\hat{\mathbf{n}}$ is positive. The flux through the top, as well as the total flux, is

$$\begin{aligned}
\int_{\text{top}}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \int_0^1\mathrm{d}r\int_0^{2\pi}\mathrm{d}\theta\ \big(\overbrace{r\sin\theta}^{y}\ \hat{\boldsymbol{\imath}} + \overbrace{(1-r)}^{z}\ \hat{\mathbf{k}}\big)\cdot\big(-r\cos\theta\,\hat{\boldsymbol{\imath}} - r\sin\theta\,\hat{\boldsymbol{\jmath}} + r\,\hat{\mathbf{k}}\big)\\
&= \int_0^1\mathrm{d}r\int_0^{2\pi}\mathrm{d}\theta\ \big(-r^2\sin\theta\cos\theta + r(1-r)\big)\\
&= -\left[\int_0^1\mathrm{d}r\ r^2\right]\left[\int_0^{2\pi}\mathrm{d}\theta\ \frac{1}{2}\sin(2\theta)\right] + 2\pi\int_0^1\mathrm{d}r\big[r-r^2\big]\\
&= -\frac{1}{3}\times 0 + 2\pi\left[\frac{1}{2}-\frac{1}{3}\right] = \frac{\pi}{3}
\end{aligned}$$

S-17: Let $G(x,y,z) = x^2+y^2+2z$. Then, by (3.3.3) of the CLP-4 text,

$$\begin{aligned}
\hat{\mathbf{n}}\,\mathrm{d}S &= \frac{\boldsymbol{\nabla}G}{\boldsymbol{\nabla}G\cdot\hat{\mathbf{k}}}\,\mathrm{d}x\mathrm{d}y = \frac{2x\hat{\boldsymbol{\imath}} + 2y\hat{\boldsymbol{\jmath}} + 2\hat{\mathbf{k}}}{2}\,\mathrm{d}x\mathrm{d}y = (x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}})\,\mathrm{d}x\mathrm{d}y\\
\frac{x^2+y^2}{\sqrt{1+x^2+y^2}}\,\mathrm{d}S &= \frac{x^2+y^2}{\sqrt{1+x^2+y^2}}\,\sqrt{1+x^2+y^2}\,\mathrm{d}x\mathrm{d}y = (x^2+y^2)\,\mathrm{d}x\mathrm{d}y\\
\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \big[x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\mathbf{k}}\big]\cdot\big[x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\big]\,\mathrm{d}x\mathrm{d}y = \big[x^2+y^2+z\big]\,\mathrm{d}x\mathrm{d}y\\
&= \left[1+\frac{1}{2}(x^2+y^2)\right]\mathrm{d}x\mathrm{d}y
\end{aligned}$$

since $z = 1-\frac{1}{2}(x^2+y^2)$ on $\mathcal{S}$.

(a)

$$\iint_{\mathcal{S}} \frac{x^2+y^2}{\sqrt{1+x^2+y^2}}\,\mathrm{d}S = \iint_{\mathcal{S}} (x^2+y^2)\,\mathrm{d}x\mathrm{d}y = 4\int_0^1 \mathrm{d}x\int_0^1 \mathrm{d}y\ (x^2+y^2)$$
$$= 4\int_0^1 \mathrm{d}x\ \big(x^2+\frac{1}{3}\big) = 4\Big(\frac{1}{3}+\frac{1}{3}\Big) = \frac{8}{3}$$

(b)

$$\iint_{\mathcal{S}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_{-1}^1 \mathrm{d}x\int_{-1}^1 \mathrm{d}y\ \Big[1+\frac{1}{2}(x^2+y^2)\Big] = 2\times 2+\frac{1}{2}\times\frac{8}{3} = \frac{16}{3}$$

S-18: Let $G(x,y,z) = z - xy$. Then, using (3.3.3) in the CLP-4 text,

$$\hat{\mathbf{n}}\,\mathrm{d}S = \frac{\nabla G}{\nabla G\cdot\hat{\mathbf{k}}}\,\mathrm{d}x\mathrm{d}y = \frac{-y\hat{\boldsymbol{\imath}} - x\hat{\boldsymbol{\jmath}}+\hat{\mathbf{k}}}{1}\,\mathrm{d}x\mathrm{d}y = (-y\hat{\boldsymbol{\imath}} - x\hat{\boldsymbol{\jmath}}+\hat{\mathbf{k}})\,\mathrm{d}x\mathrm{d}y$$
$$\frac{x^2y}{\sqrt{1+x^2+y^2}}\,\mathrm{d}S = \frac{x^2y}{\sqrt{1+x^2+y^2}}\,\sqrt{y^2+x^2+1}\,\mathrm{d}x\mathrm{d}y = x^2y\ \mathrm{d}x\mathrm{d}y$$
$$\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \big[x\hat{\boldsymbol{\imath}}+y\hat{\boldsymbol{\jmath}}+\hat{\mathbf{k}}\big]\cdot\big[-y\hat{\boldsymbol{\imath}} - x\hat{\boldsymbol{\jmath}}+\hat{\mathbf{k}}\big]\,\mathrm{d}x\mathrm{d}y = \big[1-2xy\big]\,\mathrm{d}x\mathrm{d}y$$

(a)

$$\iint_{\mathcal{S}} \frac{x^2y}{\sqrt{1+x^2+y^2}}\,\mathrm{d}S = \iint_{\mathcal{S}} x^2y\ \mathrm{d}x\mathrm{d}y = \int_0^1 \mathrm{d}x\int_0^1 \mathrm{d}y\ x^2y = \int_0^1 \mathrm{d}x\ \tfrac{1}{2}x^2 = \frac{1}{6}$$

(b)

$$\iint_{\mathcal{S}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_0^1 \mathrm{d}x\int_0^1 \mathrm{d}y\ [1-2xy] = \int_0^1 \mathrm{d}x\ [1-x] = 1-\frac{1}{2} = \frac{1}{2}$$

S-19: For the surface $z = f(x,y) = y^{3/2}$,

$$\mathrm{d}S = \sqrt{1+f_x^2+f_y^2}\,\mathrm{d}x\mathrm{d}y = \sqrt{1+\Big(\frac{3}{2}\sqrt{y}\Big)^2}\,\mathrm{d}x\mathrm{d}y = \sqrt{1+\frac{9}{4}y}\,\mathrm{d}x\mathrm{d}y$$

by (3.3.2) in the CLP-4 text. So the area is

$$\int_0^1 \mathrm{d}x\int_0^1 \mathrm{d}y\ \sqrt{1+\frac{9}{4}y} = \int_0^1 \mathrm{d}x\ \frac{8}{27}\Big[\Big(1+\frac{9}{4}y\Big)^{3/2}\Big]_0^1 = \int_0^1 \mathrm{d}x\ \frac{8}{27}\Big[\Big(\frac{13}{4}\Big)^{3/2}-1\Big]$$
$$= \frac{8}{27}\left[\left(\frac{13}{4}\right)^{3/2}-1\right]$$

S-20: The surface is a sphere of radius 2 centered on $(0,0,2)$, The plane $z=1$ intersects the sphere on the circle $x^2+y^2=3$. Let $F(x,y,z)=x^2+y^2+(z-2)^2$. Then, by (3.3.3) in the CLP-4 text,

$$\begin{aligned}
\mathrm{d}S &= \Big|\frac{\nabla F}{\nabla F\cdot\hat{\mathbf{k}}}\Big|\,\mathrm{d}x\mathrm{d}y = \Big|\frac{2x\,\hat{\imath}+2y\,\hat{\jmath}+2(z-2)\,\hat{\mathbf{k}}}{2(z-2)}\Big|\,\mathrm{d}x\mathrm{d}y = \Big|\frac{x\,\hat{\imath}+y\,\hat{\jmath}+(z-2)\,\hat{\mathbf{k}}}{(z-2)}\Big|\,\mathrm{d}x\mathrm{d}y\\
&= \frac{\sqrt{x^2+y^2+(z-2)^2}}{|z-2|}\,\mathrm{d}x\mathrm{d}y = \frac{2}{|z-2|}\,\mathrm{d}x\mathrm{d}y
\end{aligned}$$

since $x^2+y^2+(z-2)^2=4$ on $\mathcal{S}$. On $\mathcal{S}$, $z\leqslant 2$, so $|z-2|=2-z$ and

$$\iint_{\mathcal{S}} f(x,y,z)\,\mathrm{d}S = \iint_{x^2+y^2\leqslant 3}(2-z)(x^2+y^2)\,\frac{2}{|z-2|}\,\mathrm{d}x\mathrm{d}y = 2\iint_{x^2+y^2\leqslant 3}(x^2+y^2)\,\mathrm{d}x\mathrm{d}y$$

Switching to polar coordinates

$$\iint_{\mathcal{S}} f(x,y,z)\,\mathrm{d}S = 2\int_0^{\sqrt{3}}\mathrm{d}r\,r\int_0^{2\pi}\mathrm{d}\theta\,r^2 = 2(2\pi)\frac{r^4}{4}\Big|_0^{\sqrt{3}} = 9\pi$$

S-21: (a) Each (horizontal) constant z cross-section is a circle centred on the z-axis. The radius varies linearly from 2, when $z=0$ to 0, when $z=3$. So the radius at height z is $\frac{2}{3}(3-z)$ and we can use

$$\mathbf{r}(\theta,z) = \frac{2}{3}(3-z)\cos\theta\,\hat{\imath}+\frac{2}{3}(3-z)\sin\theta\,\hat{\jmath}+z\,\hat{\mathbf{k}} \qquad 0\leqslant\theta<2\pi,\quad 0\leqslant z\leqslant 3$$

as the parametrization.

(b) By symmetry the centre of mass will lie on the z-axis. We are only asked for the z-coordinate anyway. The z-coordinate of the centre of mass is the weighted average of z over the cone. Since a density has not been specified, we assume that it is a constant. We may take the density to be 1, so the z-coordinate of the centre of mass is $\iint_S z\,\mathrm{d}S/\iint_S \mathrm{d}S$.

Since

$$\begin{aligned}
\tfrac{\partial\mathbf{r}}{\partial\theta} &= \big(-\tfrac{2}{3}(3-z)\sin\theta\,,\ \tfrac{2}{3}(3-z)\cos\theta\,,\ 0\big)\\
\tfrac{\partial\mathbf{r}}{\partial z} &= \big(-\tfrac{2}{3}\cos\theta\,,\ -\tfrac{2}{3}\sin\theta\,,\ 1\big)\\
\tfrac{\partial\mathbf{r}}{\partial\theta}\times\tfrac{\partial\mathbf{r}}{\partial z} &= \big(\tfrac{2}{3}(3-z)\cos\theta\,,\ \tfrac{2}{3}(3-z)\sin\theta\,,\ \tfrac{4}{9}(3-z)\big)
\end{aligned}$$

the element of surface area for this parametrization is

$$\begin{aligned}
\mathrm{d}S &= \big|\tfrac{\partial\mathbf{r}}{\partial\theta}\times\tfrac{\partial\mathbf{r}}{\partial z}\big|\mathrm{d}\theta\mathrm{d}z = \tfrac{2}{3}(3-z)\big|\big(\cos\theta\,,\ \sin\theta\,,\ \tfrac{2}{3}\big)\big|\mathrm{d}\theta\mathrm{d}z\\
&= \tfrac{2\sqrt{13}}{9}(3-z)\mathrm{d}\theta\mathrm{d}z
\end{aligned}$$

So the surface area, $\iint_S \mathrm{d}S$, of the cone is

$$\begin{aligned}
\int_0^3\mathrm{d}z\int_0^{2\pi}\mathrm{d}\theta\ \tfrac{2\sqrt{13}}{9}(3-z) &= \tfrac{4\sqrt{13}}{9}\pi\int_0^3\mathrm{d}z\ (3-z) = -\tfrac{2\sqrt{13}}{9}\pi(3-z)^2\Big|_0^3\\
&= 2\sqrt{13}\pi
\end{aligned}$$

and the z-coordinate of the centre of mass is

$$\begin{aligned}\bar{z} &= \frac{1}{2\sqrt{13}\pi}\int_0^3 \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta\ \tfrac{2\sqrt{13}}{9}(3-z)z = \frac{2}{9}\int_0^3 \mathrm{d}z\ (3z-z^2) = \frac{2}{9}\Big[\frac{3z^2}{2}-\frac{z^3}{3}\Big]_0^3 \\ &= \frac{2}{9}\frac{27}{6} = 1\end{aligned}$$

This is a little less than half way up the cone, which is reasonable since the cone is "bottom heavy".

S-22: Each constant z cross-section of the cone is a circle. When $z=0$, that circle has radius a. When $z=a$ that circle has radius 0. Thus the radius decreases linearly from a to 0 as z increases from 0 to a. So the radius at height z is $a-z$ and we can parametrize the cone by

$$\mathbf{r}(\theta,z) = (a-z)\cos\theta\,\hat{\boldsymbol{\imath}} + (a-z)\sin\theta\,\hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}} \qquad 0\leqslant\theta<2\pi,\ 0\leqslant z\leqslant a$$

Since

$$\begin{aligned}\tfrac{\partial\mathbf{r}}{\partial\theta} &= \big(-(a-z)\sin\theta\,,\ (a-z)\cos\theta\,,\ 0\big) \\ \tfrac{\partial\mathbf{r}}{\partial z} &= \big(-\cos\theta\,,\ -\sin\theta\,,\ 1\big) \\ \tfrac{\partial\mathbf{r}}{\partial\theta}\times\tfrac{\partial\mathbf{r}}{\partial z} &= \big((a-z)\cos\theta\,,\ (a-z)\sin\theta\,,\ a-z\big)\end{aligned}$$

the element of surface area for this parametrization is

$$\begin{aligned}\mathrm{d}S &= \big|\tfrac{\partial\mathbf{r}}{\partial\theta}\times\tfrac{\partial\mathbf{r}}{\partial z}\big|\mathrm{d}\theta\mathrm{d}z = (a-z)\big|(\cos\theta\,,\ \sin\theta\,,\ 1)\big|\mathrm{d}\theta\mathrm{d}z \\ &= \sqrt{2}(a-z)\,\mathrm{d}\theta\mathrm{d}z\end{aligned}$$

by (3.3.1) in the CLP-4 text. So the surface area of the cone is

$$\begin{aligned}\iint_S \mathrm{d}S &= \int_0^a \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta\ \sqrt{2}(a-z) \\ &= 2\sqrt{2}\,\pi\int_0^a \mathrm{d}z\ (a-z) = -\sqrt{2}\,\pi(a-z)^2\Big|_0^a \\ &= \sqrt{2}\,\pi a^2\end{aligned}$$

and the z-coordinate of the centre of mass is

$$\begin{aligned}\bar{z} &= \frac{\iint_S z\,\mathrm{d}S}{\iint_S \mathrm{d}S} = \frac{1}{\sqrt{2}\,\pi a^2}\int_0^a \mathrm{d}z\int_0^{2\pi}\mathrm{d}\theta\ \sqrt{2}(a-z)z = \frac{2}{a^2}\int_0^a \mathrm{d}z\ (az-z^2) \\ &= \frac{2}{a^2}\Big[\frac{az^2}{2}-\frac{z^3}{3}\Big]_0^a = \frac{2}{a^2}\frac{a^3}{6} = \frac{a}{3}\end{aligned}$$

This is a little less than half way up the cone, which is reasonable since the cone is "bottom heavy".

S-23: Parametrize the surface by

$$x(\theta,z) = \cos\theta \qquad y(\theta,z) = 2\sin\theta \qquad z(\theta,z) = z$$

with (θ, z) running over $0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant z \leqslant 1$. Then, by (3.3.1) in the CLP-4 text,

$$\begin{aligned}
\Big(\frac{\partial x}{\partial\theta},\frac{\partial y}{\partial\theta},\frac{\partial z}{\partial\theta}\Big) &= (-\sin\theta, 2\cos\theta, 0)\\
\Big(\frac{\partial x}{\partial z},\frac{\partial y}{\partial z},\frac{\partial z}{\partial z}\Big) &= (0,0,1)\\
\hat{\mathbf{n}}\,\mathrm{d}S &= \pm\Big(\frac{\partial x}{\partial\theta},\frac{\partial y}{\partial\theta},\frac{\partial z}{\partial\theta}\Big)\times\Big(\frac{\partial x}{\partial z},\frac{\partial y}{\partial z},\frac{\partial z}{\partial z}\Big)\mathrm{d}\theta\mathrm{d}z\\
&= (2\cos\theta, \sin\theta, 0)\,\mathrm{d}\theta\,\mathrm{d}z \qquad (+\text{ for outward normal})\\
\mathbf{F}\big(x(\theta,z), y(\theta,z), z(\theta,z)\big) &= \cos\theta\,\hat{\imath} + 2z\sin\theta\cos\theta\,\hat{\jmath} + 16z\sin^4\theta\,\hat{\mathbf{k}}
\end{aligned}$$

So

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \int_0^1 \mathrm{d}z \int_0^{2\pi}\mathrm{d}\theta\ \big[2\cos^2\theta + 2z\sin^2\theta\cos\theta\big]\\
&= \int_0^1 \mathrm{d}z \int_0^{2\pi}\mathrm{d}\theta\ \big[1+\cos(2\theta) + 2z\sin^2\theta\cos\theta\big]\\
&= \int_0^1 \mathrm{d}z\ \Big[\theta + \frac{1}{2}\sin(2\theta) + \frac{2}{3}z\sin^3\theta\Big]_0^{2\pi} = 2\pi
\end{aligned}$$

For an efficient, sneaky, way to evaluate $\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta$, see Example 2.4.4 in the CLP-4 text.

S-24: By (3.3.2) of the CLP-4 text, with $f(x,y) = 4 - x^2 - y^2$,

$$\begin{aligned}
\hat{\mathbf{n}}\,\mathrm{d}S &= \pm\big(-f_x\,,\,-f_y\,,\,1\big)\mathrm{d}x\mathrm{d}y\\
&= \pm\big(2x\,,\,2y\,,\,1\big)\,\mathrm{d}x\mathrm{d}y
\end{aligned}$$

To get the downward pointing normal, we want the minus sign. Set

$$T = \big\{\ (x,y)\ \big|\ 0\leqslant x\leqslant 1,\ 0\leqslant y\leqslant 1-x\ \big\}$$

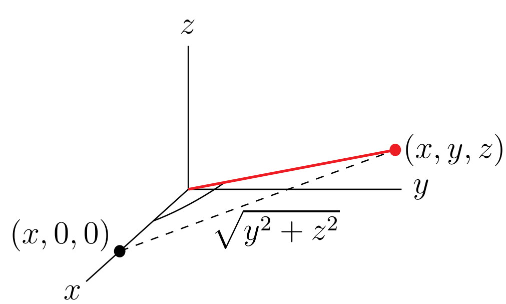
Then

$$
\begin{aligned}
\iint\limits_{S} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= -\iint\limits_{T} \big(x+1\,,\,y+1\,,\,2\overbrace{(4-x^2-y^2)}^{z}\,\big)\cdot\big(2x\,,\,2y\,,\,1\big)\,\mathrm{d}x\mathrm{d}y \\
&= -\iint\limits_{T} \big(8+2x+2y\big)\,\mathrm{d}x\mathrm{d}y \\
&= -\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\ \big(8+2x+2y\big) \\
&= -\int_0^1 \mathrm{d}x\ \big(8(1-x)+2x(1-x)+(1-x)^2\big) \\
&= -\int_0^1 \mathrm{d}x\ \big(9-8x-x^2\big) \\
&= -\Big(9-4-\frac{1}{3}\Big) = -\frac{14}{3}
\end{aligned}
$$

S-25: First we have to parametrize S. It is natural to use spherical coordinates with $\rho=\sqrt{2}$. However if we use the standard spherical coordinates

$$x=\sqrt{2}\,\sin\varphi\cos\theta \qquad y=\sqrt{2}\,\sin\varphi\sin\theta \qquad z=\sqrt{2}\,\cos\varphi$$

the condition $x\geqslant\sqrt{y^2+z^2}$, i.e. $\frac{\sqrt{y^2+z^2}}{x}\leqslant 1$, becomes $\frac{\sqrt{\sin^2\varphi\sin^2\theta+\cos^2\varphi}}{\sin\varphi\cos\theta}\leqslant 1$, which is very complicated. So let's back up and think a bit before we compute. From the sketch below



we see that $\frac{\sqrt{y^2+z^2}}{x}$ is the tangent of the angle between the radius vector (x,y,z) and the x-axis. The angle between the radius vector (x,y,z) and the z-axis (not the x-axis) is exactly spherical coordinate φ. So let's modify spherical coordinates to make the x-axis play the role of the z-axis. The easy way to do is to just rename $x=Z$, $y=X$, $z=Y$. Then the integral we are to compute becomes $\iint_S ZX^2\,\mathrm{d}S$, and the condition $x\geqslant\sqrt{y^2+z^2}$ becomes $Z\geqslant\sqrt{X^2+Y^2}$. Under the parametrization

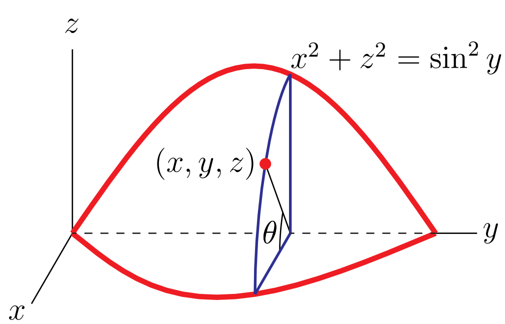
$$X=\sqrt{2}\,\sin\varphi\cos\theta \qquad Y=\sqrt{2}\,\sin\varphi\sin\theta \qquad Z=\sqrt{2}\,\cos\varphi$$

the condition $Z\geqslant\sqrt{X^2+Y^2}$ is $\frac{\sqrt{X^2+Y^2}}{Z}=\frac{\sin\varphi}{\cos\varphi}\leqslant 1$, which is turn is $0\leqslant\varphi\leqslant\frac{\pi}{4}$. As $\mathrm{d}S=2\sin\varphi\,\mathrm{d}\theta\mathrm{d}\varphi$ (see Appendix F.3 in the CLP-4 text and recall that $\rho=\sqrt{2}$) the specified integral is

$$\begin{aligned}
\iint\limits_S xy^2\,\mathrm{d}S = \iint\limits_S ZX^2\,\mathrm{d}S &= 2\int_0^{\pi/4}\mathrm{d}\varphi\ \sin\varphi\int_0^{2\pi}\mathrm{d}\theta\ \big(\sqrt{2}\,\cos\varphi\big)\big(\sqrt{2}\,\sin\varphi\cos\theta\big)^2\\
&= 4\sqrt{2}\left\{\int_0^{\pi/4}\mathrm{d}\varphi\ \cos\varphi\sin^3\varphi\right\}\left\{\int_0^{2\pi}\mathrm{d}\theta\ \cos^2\theta\right\}\\
&= 4\sqrt{2}\left\{\int_0^{\pi/4}\mathrm{d}\varphi\ \cos\varphi\sin^3\varphi\right\}\left\{\int_0^{2\pi}\mathrm{d}\theta\ \frac{\cos(2\theta)+1}{2}\right\}\\
&= 4\sqrt{2}\left[\frac{\sin^4\varphi}{4}\right]_0^{\pi/4}\left[\frac{\sin(2\theta)}{4}+\frac{\theta}{2}\right]_0^{2\pi}\\
&= \frac{\sqrt{2}\,\pi}{4}
\end{aligned}$$

For an efficient, sneaky, way to evaluate $\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta$, see Example 2.4.4 in the CLP-4 text.

S-26: Here is a sketch of the part of S that is in the first octant.



For each fixed y, $x^2+z^2=\sin^2 y$ is a circle of radius $\sin y$. (It's the blue circle in the sketch above.) So we may parametrize the surface by

$$\mathbf{r}(\theta,y) = \big(\sin y\,\cos\theta\,,\,y\,,\,\sin y\,\sin\theta\big)\qquad 0\leqslant\theta<2\pi,\ 0\leqslant y\leqslant\pi$$

Then, by (3.3.1) in the CLP-4 text,

$$\begin{aligned}
\frac{\partial\mathbf{r}}{\partial\theta} &= \big(-\sin y\,\sin\theta\,,\,0\,,\,\sin y\,\cos\theta\big)\\
\frac{\partial\mathbf{r}}{\partial y} &= \big(\cos y\,\cos\theta\,,\,1\,,\,\cos y\,\sin\theta\big)\\
\frac{\partial\mathbf{r}}{\partial\theta}\times\frac{\partial\mathbf{r}}{\partial y} &= \big(-\sin y\,\cos\theta\,,\,\sin y\,\cos y\,,\,-\sin y\,\sin\theta\big)\\
\mathrm{d}S &= \left|\frac{\partial\mathbf{r}}{\partial\theta}\times\frac{\partial\mathbf{r}}{\partial y}\right|\mathrm{d}\theta\,\mathrm{d}y = \sin y\sqrt{1+\cos^2 y}\,\mathrm{d}\theta\,\mathrm{d}y
\end{aligned}$$

So the specified integral is

$$\begin{aligned}
\iint_S \sqrt{1+\cos^2 y}\,\mathrm{d}S &= \int_0^\pi \mathrm{d}y \int_0^{2\pi} \mathrm{d}\theta\ \sin y\{1+\cos^2 y\} \\
&= 2\pi \int_0^\pi \mathrm{d}y\ \sin y\{1+\cos^2 y\} \\
&= -2\pi \int_1^{-1} \mathrm{d}u\ \{1+u^2\} \quad \text{with } u=\cos y,\ \mathrm{d}u = -\sin y\,\mathrm{d}y \\
&= 4\pi \int_0^1 \mathrm{d}u\ \{1+u^2\} \\
&= 4\pi \left[u + \frac{u^3}{3}\right]_0^1 \\
&= \frac{16}{3}\pi
\end{aligned}$$

S-27: The paraboloid is

$$S = \left\{\ (x,y,z)\ \middle|\ z = 1-x^2-y^2,\ z \geqslant 0\ \right\} = \left\{\ (x,y,z)\ \middle|\ z = 1-x^2-y^2,\ x^2+y^2 \leqslant 1\ \right\}$$

By (3.3.2) in the CLP-4 text, the paraboloid has

$$\begin{aligned}
\mathrm{d}S &= \sqrt{1+f_x(x,y)^2+f_y(x,y)^2}\,\mathrm{d}x\mathrm{d}y \qquad \text{with } z = f(x,y) = 1-x^2-y^2 \\
&= \sqrt{1+4x^2+4y^2}\,\mathrm{d}x\mathrm{d}y
\end{aligned}$$

By symmetry, the centre of mass will lie on the z-axis. By definition, the z-coordinate of the centre of mass is the weighted average of z over S, which is

$$\bar{z} = \frac{\iint_S z\,\rho(x,y,z)\ \mathrm{d}S}{\iint_S \rho(x,y,z)\ \mathrm{d}S}$$

On S,

$$\rho(x,y,z) = \frac{z}{\sqrt{5-4z}} = \frac{1-x^2-y^2}{\sqrt{1+4x^2+4y^2}}$$

so that

$$\rho(x,y,z)\ \mathrm{d}S = (1-x^2-y^2)\,\mathrm{d}x\mathrm{d}y$$

So, using polar coordinates, the denominator of $\bar{z}$ is

$$\begin{aligned}
\iint_S \rho(x,y,z)\ \mathrm{d}S = \iint_{x^2+y^2\leqslant 1} (1-x^2-y^2)\,\mathrm{d}x\mathrm{d}y &= \int_0^1 \mathrm{d}r\, r \int_0^{2\pi} \mathrm{d}\theta\,(1-r^2) \\
&= 2\pi \int_0^1 r(1-r^2)\ \mathrm{d}r \\
&= 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 \\
&= \frac{\pi}{2}
\end{aligned}$$

and the numerator of $\bar{z}$ is

$$\begin{aligned}
\iint_S z\rho(x,y,z)\,\mathrm{d}S &= \iint_{x^2+y^2\leqslant 1} \left(1-x^2-y^2\right)^2 \mathrm{d}x\mathrm{d}y = \int_0^1 \mathrm{d}r\, r \int_0^{2\pi} \mathrm{d}\theta\, \left(1-r^2\right)^2 \\
&= 2\pi \int_0^1 r\left(1-r^2\right)^2\,\mathrm{d}r \\
&= 2\pi \Big[\frac{r^2}{2} - 2\frac{r^4}{4} + \frac{r^6}{6}\Big]_0^1 \\
&= \frac{\pi}{3}
\end{aligned}$$

and

$$\bar{z} = \frac{\pi/3}{\pi/2} = \frac{2}{3}$$

S-28: The equation of the plane is $z = f(x,y) = 2 - x - y$. So by (3.3.2) in the CLP-4 text,

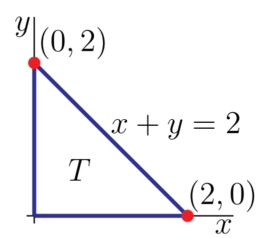
$$\hat{\mathbf{n}}\,\mathrm{d}S = \big[-f_x(x,y)\,\hat{\imath} - f_y(x,y)\,\hat{\jmath} + \hat{\mathbf{k}}\big]\,\mathrm{d}x\mathrm{d}y = \big[\hat{\imath} + \hat{\jmath} + \hat{\mathbf{k}}\big]\,\mathrm{d}x\mathrm{d}y$$

A point (x,y,z) on the plane lies in the first octant if and only if

$$x \geqslant 0 \quad \text{and} \quad y \geqslant 0 \quad \text{and} \quad z = 2 - x - y \geqslant 0$$

So the domain of integration is the triangle

$$T = \big\{\, (x,y) \bigm| x \geqslant 0,\ y \geqslant 0,\ x + y \leqslant 2 \,\big\}$$



and

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iint_T \big[x\,\hat{\imath} + y\,\hat{\jmath} + (\overbrace{2-x-y}^{z})\,\hat{\mathbf{k}}\big]\cdot\big[\hat{\imath} + \hat{\jmath} + \hat{\mathbf{k}}\big]\,\mathrm{d}x\mathrm{d}y \\
&= 2\iint_T \mathrm{d}x\mathrm{d}y \\
&= 2\frac{1}{2}(2)(2) = 4
\end{aligned}$$

S-29: Since

$$\begin{aligned}
\tfrac{\partial \mathbf{r}}{\partial u} &= \left(v^2\,,\, 2uv\,,\, v\right) \\
\tfrac{\partial \mathbf{r}}{\partial v} &= \left(2uv\,,\, u^2\,,\, u\right) \\
\tfrac{\partial \mathbf{r}}{\partial u} \times \tfrac{\partial \mathbf{r}}{\partial v} &= \left(u^2v\,,\, uv^2\,,\, -3u^2v^2\right)
\end{aligned}$$

(3.3.1) in the CLP-4 text gives

$$\hat{\mathbf{n}}\,\mathrm{d}S = \pm\left(u^2v\,,\, uv^2\,,\, -3u^2v^2\right)\mathrm{d}u\mathrm{d}v$$

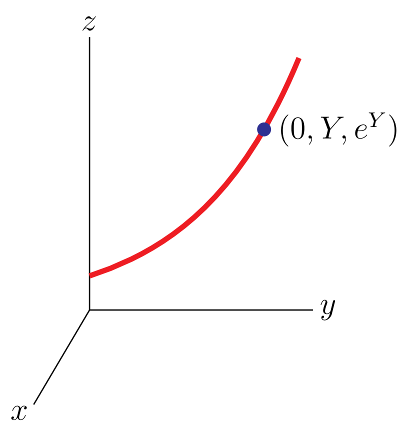
We are told that $\hat{\mathbf{n}}$ should have a positive z-component, so

$$\hat{\mathbf{n}}\,\mathrm{d}S = -\big(u^2v\,,\,uv^2\,,\,-3u^2v^2\big)\,\mathrm{d}u\mathrm{d}v = \big(-u^2v\,,\,-uv^2\,,\,3u^2v^2\big)\,\mathrm{d}u\mathrm{d}v$$

and

$$\begin{aligned}\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iint_S \overbrace{(uv^2\,,\,u^2v\,,\,uv)}^{\mathbf{F}}\cdot\big(-u^2v\,,\,-uv^2\,,\,3u^2v^2\big)\,\mathrm{d}u\mathrm{d}v\\ &= \int_0^1 \mathrm{d}u\int_0^3 \mathrm{d}v\;u^3v^3 = \left[\int_0^1 \mathrm{d}u\;u^3\right]\left[\int_0^3 \mathrm{d}v\;v^3\right]\\ &= \frac{1}{4}\,\frac{3^4}{4} = \frac{81}{16}\end{aligned}$$

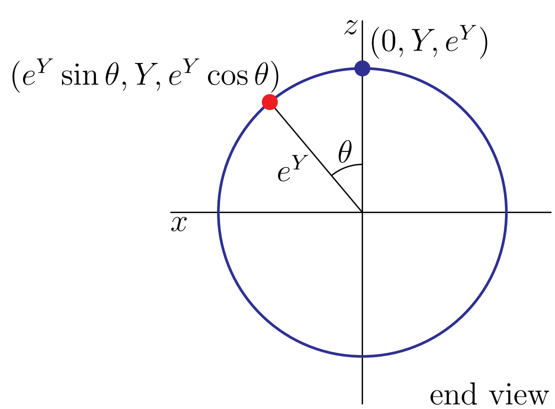
S-30: (a) We start by just sketching the curve $z = e^y$, considering the yz-plane as the plane $x = 0$ in $\mathbb{R}^3$. This curve is the red curve in the figure below. Concentrate on any one point on that curve. It is the blue dot at $(0, Y, e^Y)$ in the figure. When our curve is rotated



about the y-axis, the blue dot sweeps out a circle. The circle that the blue dot sweeps out

- lies in the vertical plane $y = Y$ and
- is centred on the y-axis and
- has radius e^Y.

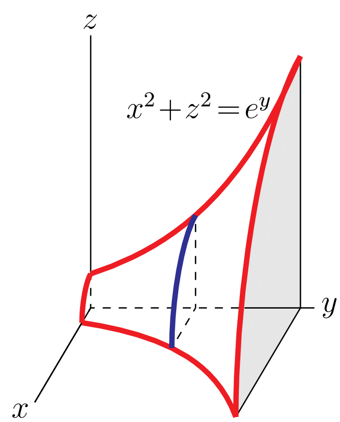
We can parametrize the circle swept out in the usual way. Here is an end view of the circle (looking down the y-axis), with the parameter, named θ, indicated.



The coordinates of the red dot are $\big(e^Y \sin\theta\,,\ Y\,,\ e^Y\cos\theta\big)$. This also gives a parametrization of the surface of revolution

$$\begin{aligned} x(Y,\theta) &= e^Y \sin\theta \\ y(Y,\theta) &= Y \\ z(Y,\theta) &= e^Y\cos\theta \\ &\qquad 0\leqslant Y\leqslant 1, \qquad 0\leqslant\theta<2\pi \end{aligned}$$

Finally here is a sketch of the part of the surface in the first octant, $x,y,z\geqslant 0$.



(b) We are using the parametrization

$$\mathbf{r}(Y,\theta) = e^Y\sin\theta\,\hat{\boldsymbol{\imath}} + Y\,\hat{\boldsymbol{\jmath}} + e^Y\cos\theta\,\hat{\mathbf{k}} \qquad 0\leqslant Y\leqslant 1,\ 0\leqslant\theta\leqslant 2\pi$$

so that

$$\frac{\partial\mathbf{r}}{\partial Y}\times\frac{\partial\mathbf{r}}{\partial\theta} = \det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ e^Y\sin\theta & 1 & e^Y\cos\theta \\ e^Y\cos\theta & 0 & -e^Y\sin\theta\end{bmatrix} = \big(-e^Y\sin\theta, e^{2Y}, -e^Y\cos\theta\big),$$

and, by (3.3.1) in the CLP-4 text,

$$\mathrm{d}S = \Big|\frac{\partial\mathbf{r}}{\partial Y}\times\frac{\partial\mathbf{r}}{\partial\theta}\Big|\,\mathrm{d}Y\mathrm{d}\theta = \sqrt{e^{2Y}+e^{4Y}}\,\mathrm{d}Y\,\mathrm{d}\theta = e^Y\sqrt{1+e^{2Y}}\,\mathrm{d}Y\mathrm{d}\theta$$

So the integral is

$$\begin{aligned}\iint_S e^y\,\mathrm{d}S &= \int_0^1 \mathrm{d}Y\int_0^{2\pi}\mathrm{d}\theta\ e^{2Y}\sqrt{1+e^{2Y}} = 2\pi\int_0^1\mathrm{d}Y\ e^{2Y}\sqrt{1+e^{2Y}} = \frac{2\pi}{3}\Big[1+e^{2Y}\Big]^{3/2}\Big|_0^1 \\ &= \frac{2\pi}{3}\Big[(1+e^2)^{3/2}-2^{3/2}\Big]\end{aligned}$$

(c) Again, we are using the parametrization

$$\mathbf{r}(Y,\theta) = e^Y\sin\theta\,\hat{\boldsymbol{\imath}} + Y\,\hat{\boldsymbol{\jmath}} + e^Y\cos\theta\,\hat{\mathbf{k}} \qquad 0\leqslant Y\leqslant 1,\ 0\leqslant\theta\leqslant 2\pi$$

so that

$$\frac{\partial\mathbf{r}}{\partial Y}\times\frac{\partial\mathbf{r}}{\partial\theta} = \big(-e^Y\sin\theta, e^{2Y}, -e^Y\cos\theta\big),$$

and, by (3.3.1) in the CLP-4 text,

$$\hat{\mathbf{n}}\,\mathrm{d}S = \pm\frac{\partial \mathbf{r}}{\partial Y}\times\frac{\partial \mathbf{r}}{\partial \theta}\,\mathrm{d}Y\mathrm{d}\theta = \pm\big(-e^Y\sin\theta, e^{2Y}, -e^Y\cos\theta\big)\,\mathrm{d}Y\,\mathrm{d}\theta$$

We choose the "+" sign so that $\hat{\mathbf{n}}$ points towards the y-axis. As an example, when $0 \leqslant \theta \leqslant \frac{\pi}{2}$, then $z = e^Y\cos\theta > 0$ while the z-coordinate of $\hat{\mathbf{n}}$ is $-e^y\cos\theta < 0$. So the integral is

$$\begin{aligned}\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \int_0^1 \mathrm{d}Y\int_0^{2\pi}\mathrm{d}\theta\,\big(\overbrace{e^Y\sin\theta}^{x}, 0, \overbrace{e^Y\cos\theta}^{z}\big)\cdot\big(-e^Y\sin\theta, e^{2Y}, -e^Y\cos\theta\big)\\ &= -\int_0^1\mathrm{d}Y\int_0^{2\pi}\mathrm{d}\theta\, e^{2Y} = -2\pi\int_0^1\mathrm{d}Y\, e^{2Y}\\ &= -\pi\big(e^2-1\big) = \pi\big(1-e^2\big)\end{aligned}$$

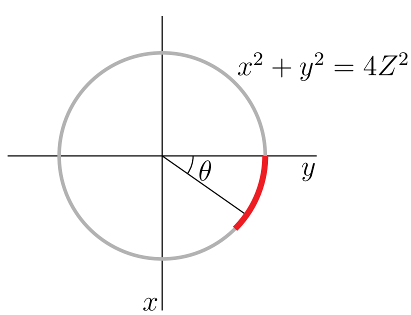
S-31: Write

$$V = \big\{\,(x,y,z)\,\big|\,1\leqslant x^2+y^2+z^2\leqslant 4\,\big\}$$

The boundary of V consists of two parts — the sphere, S_2, of radius 2, centred on the origin, with (outward) normal $\hat{\mathbf{n}} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\mathbf{r}}{2}$, and the sphere S_1 of radius 1, centred on the origin, with (inward) normal $\hat{\mathbf{n}} = -\mathbf{r}$, So,

$$\begin{aligned}\iint_{\partial V}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iint_{S_2}\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\frac{\mathbf{r}}{2}\,\mathrm{d}S - \iint_{S_1}\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{r}\,\mathrm{d}S\\ &= \iint_{S_2}\mathrm{d}S - \iint_{S_1}\mathrm{d}S\\ &= 4\pi(2)^2 - 4\pi(1)^2\\ &= 12\pi\end{aligned}$$

S-32: The part of the cone that has some fixed value, Z, of z with $0\leqslant Z\leqslant 1$ is the part of the circle $\big\{\,(x,y,z)\,\big|\,x^2+y^2=4Z^2, z=Z\,\big\}$ of radius 2Z that has $0\leqslant x\leqslant y$. Here is a sketch of the top view of that part of that circle.



So we can parametrize S by

$$\mathbf{r}(\theta, Z) = 2Z\sin\theta\,\hat{\boldsymbol{\imath}} + 2Z\cos\theta\,\hat{\boldsymbol{\jmath}} + Z\,\hat{\mathbf{k}} \qquad 0 \leqslant \theta \leqslant \frac{\pi}{4},\ 0 \leqslant Z \leqslant 1$$

So

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial \theta} &= 2Z\cos\theta\,\hat{\boldsymbol{\imath}} - 2Z\sin\theta\,\hat{\boldsymbol{\jmath}} \\ \frac{\partial \mathbf{r}}{\partial Z} &= 2\sin\theta\,\hat{\boldsymbol{\imath}} + 2\cos\theta\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\end{aligned}$$

so that

$$\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial Z} = \det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 2Z\cos\theta & -2Z\sin\theta & 0 \\ 2\sin\theta & 2\cos\theta & 1 \end{bmatrix} = \big(-2Z\sin\theta, -2Z\cos\theta, 4Z\big),$$

and, by (3.3.1) in the CLP-4 text,

$$\mathrm{d}S = \left|\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial Z}\right| \mathrm{d}\theta\mathrm{d}Z = \sqrt{20}\,Z\,\mathrm{d}\theta\mathrm{d}Z$$

and

$$\iint_S z^2\,\mathrm{d}S = \sqrt{20}\int_0^1 \mathrm{d}Z \int_0^{\pi/4} \mathrm{d}\theta\ Z^3 = \sqrt{20}\,\frac{\pi}{4}\,\frac{1}{4} = \frac{\sqrt{5}\,\pi}{8}$$

S-33: We'll start by parametrizing S. Note that as $x^2 + y^2$ runs from 0 to 4, z runs from 5 to 1, and that, for each fixed $1 \leqslant Z \leqslant 5$, the cross-section of S with $z = Z$ is the circle $x^2 + y^2 = 5 - Z$, $z = Z$. So we may parametrize S by

$$\mathbf{r}(\theta, Z) = \sqrt{5 - Z}\cos\theta\,\hat{\boldsymbol{\imath}} + \sqrt{5 - Z}\sin\theta\,\hat{\boldsymbol{\jmath}} + Z\,\hat{\mathbf{k}} \qquad 0 \leqslant \theta \leqslant 2\pi,\ 1 \leqslant Z \leqslant 5$$

Since

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial \theta} &= -\sqrt{5 - Z}\sin\theta\,\hat{\boldsymbol{\imath}} + \sqrt{5 - Z}\cos\theta\,\hat{\boldsymbol{\jmath}} \\ \frac{\partial \mathbf{r}}{\partial Z} &= -\frac{1}{2\sqrt{5 - Z}}\cos\theta\,\hat{\boldsymbol{\imath}} - \frac{1}{2\sqrt{5 - Z}}\sin\theta\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\end{aligned}$$

so that

$$\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial Z} = \det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ -\sqrt{5 - Z}\sin\theta & \sqrt{5 - Z}\cos\theta & 0 \\ -\frac{1}{2\sqrt{5-Z}}\cos\theta & -\frac{1}{2\sqrt{5-Z}}\sin\theta & 1 \end{bmatrix} = \big(\sqrt{5 - Z}\cos\theta\,,\ \sqrt{5 - Z}\sin\theta\,,\ 1/2\big),$$

(3.3.1) in the CLP-4 text gives

$$\hat{\mathbf{n}}\,\mathrm{d}S = \pm\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial Z}\,\mathrm{d}\theta\mathrm{d}Z = \pm\big(\sqrt{5 - Z}\cos\theta\,,\ \sqrt{5 - Z}\sin\theta\,,\ 1/2\big)\,\mathrm{d}\theta\mathrm{d}Z$$

Choosing the minus sign to give the downward pointing normal

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$$

$$= -\int_1^5 \mathrm{d}Z\int_0^{2\pi}\mathrm{d}\theta\ \Big(-\frac{1}{2}\overbrace{[5-Z]^{3/2}\cos^3\theta}^{x^3}-\overbrace{[5-Z]^{3/2}\cos\theta\sin^2\theta}^{xy^2}\,,\ -\frac{1}{2}\overbrace{[5-Z]^{3/2}\sin^3\theta}^{y^3}\,,\ \overbrace{Z^2}^{z^2}\Big)$$
$$\cdot\big(\sqrt{5-Z}\cos\theta\,,\ \sqrt{5-Z}\sin\theta\,,\ 1/2\big)$$
$$= -\int_1^5 \mathrm{d}Z\int_0^{2\pi}\mathrm{d}\theta\ \Big(-\frac{1}{2}[5-Z]^2\cos^4\theta-[5-Z]^2\cos^2\theta\sin^2\theta-\frac{1}{2}[5-Z]^2\sin^4\theta+\frac{1}{2}Z^2\Big)$$

Since

$$\frac{1}{2}\cos^4\theta+\cos^2\theta\sin^2\theta+\frac{1}{2}\sin^4\theta=\frac{1}{2}\big(\cos^2\theta+\sin^2\theta\big)^2=\frac{1}{2}$$

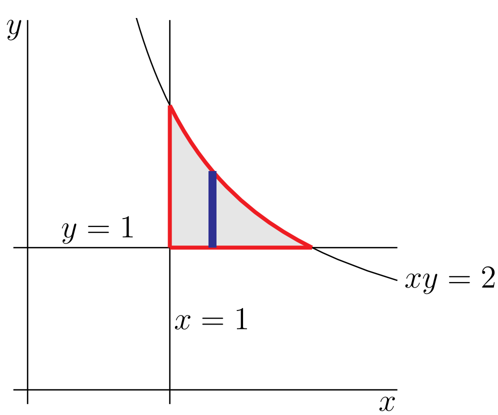
the flux

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S=\int_1^5\mathrm{d}Z\int_0^{2\pi}\mathrm{d}\theta\ \Big(\frac{1}{2}[5-Z]^2-\frac{1}{2}Z^2\Big)=\pi\int_1^5\mathrm{d}Z\ \Big([5-Z]^2-Z^2\Big)$$
$$=\pi\left[-\frac{1}{3}[5-Z]^3-\frac{Z^3}{3}\right]_1^5=\pi\left[\frac{4^3}{3}-\frac{5^3}{3}+\frac{1}{3}\right]=-20\,\pi$$

S-34: The surface is $z=f(x,y)$ with $f(x,y)=\sqrt{2xy}$. Since $f_x=\sqrt{\frac{y}{2x}}$ and $f_y=\sqrt{\frac{x}{2y}}$, (3.3.2) in the CLP-4 text gives

$$\mathrm{d}S=\sqrt{1+f_x^2+f_y^2}\,\mathrm{d}x\mathrm{d}y=\sqrt{1+\frac{y}{2x}+\frac{x}{2y}}\,\mathrm{d}x\mathrm{d}y=\sqrt{\frac{2xy+y^2+x^2}{2xy}}\,\mathrm{d}x\mathrm{d}y$$
$$=\frac{x+y}{\sqrt{2xy}}\,\mathrm{d}x\mathrm{d}y$$

On the shell, $z^2=2xy\leqslant 4$. So the x and y components of points (x,y,z) on the shell run over the region $x\geqslant 1$, $y\geqslant 1$, $xy\leqslant 2$, which is sketched below



So the mass is

$$\iint_S \rho(x,y,z)\,dS = \int_1^2 \mathrm{d}x \int_1^{2/x} \mathrm{d}y\; 3f(x,y)\frac{x+y}{\sqrt{2xy}} = \int_1^2 \mathrm{d}x \int_1^{2/x} \mathrm{d}y\; 3(x+y)$$
$$= 3\int_1^2 \mathrm{d}x\ \Big[xy+\frac{1}{2}y^2\Big]_1^{2/x} = 3\int_1^2 \mathrm{d}x\ \Big[2+\frac{2}{x^2}-x-\frac{1}{2}\Big]$$
$$= 3\left\{\frac{3}{2}-\frac{2}{x}\Big|_1^2-\frac{x^2}{2}\Big|_1^2\right\} = 3\left\{\frac{3}{2}-1+2-2+\frac{1}{2}\right\}$$
$$= 3$$

S-35: Since $x = g(y,z)$ with $g(x,y) = y^2+z^2$, (3.3.2) in the CLP-4 text gives

$$\hat{\mathbf{n}}\,\mathrm{d}S = \pm(1,-g_y,-g_z)\,\mathrm{d}y\mathrm{d}z = \pm(1,-2y,-2z)\,\mathrm{d}y\mathrm{d}z$$

We choose the $+$ sign so that $\hat{\mathbf{n}}\cdot\hat{\boldsymbol{\imath}} > 0$. Furthermore

$$\begin{aligned} S &= \big\{\ (x,y,z)\ \big|\ x=y^2+z^2,\ x\leqslant 2y\ \big\} \\ &= \big\{\ (x,y,z)\ \big|\ x=y^2+z^2,\ y^2+z^2\leqslant 2y\ \big\} \\ &= \big\{\ (x,y,z)\ \big|\ x=y^2+z^2,\ (y-1)^2+z^2\leqslant 1\ \big\} \\ &= \big\{\ (x,y,z)\ \big|\ x=y^2+z^2,\ (y,z)\text{ in }D\ \big\} \end{aligned}$$

where $D = \big\{\ (x,y)\ \big|\ (y-1)^2+z^2\leqslant 1\ \big\}$ is a disk with radius 1. Hence

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_D (2,z,y)\cdot(1,-2y,-2z)\ \mathrm{d}y\mathrm{d}z = \iint_D (2-4yz)\ \mathrm{d}y\mathrm{d}z$$

Since $-4yz$ is odd under $z\to -z$ the integral of $-4yz$ is zero and

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 2\,\mathrm{Area}(D) = 2\pi$$

S-36: For the specified $\mathbf{F}$ and the surface $x = f(x,y) = 1-\frac{1}{4}x^2-y^2$, by (3.3.2) in the CLP-4 text,

$$\begin{aligned} \hat{\mathbf{n}}\,\mathrm{d}S &= \big(-f_x\,\hat{\boldsymbol{\imath}}-f_y\,\hat{\boldsymbol{\jmath}}+\hat{\mathbf{k}}\big)\,\mathrm{d}x\mathrm{d}y = \Big(\frac{x}{2}\,\hat{\boldsymbol{\imath}}+2y\,\hat{\boldsymbol{\jmath}}+\hat{\mathbf{k}}\Big)\,\mathrm{d}x\mathrm{d}y \\ \nabla\times\mathbf{F} &= \det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y^2+z & x-x^2 & 1 \end{bmatrix} \\ &= \hat{\boldsymbol{\jmath}}+(1-2x-6y)\,\hat{\mathbf{k}} \\ \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= (2y+1-2x-6y)\,\mathrm{d}x\mathrm{d}y = (1-2x-4y)\,\mathrm{d}x\mathrm{d}y \end{aligned}$$

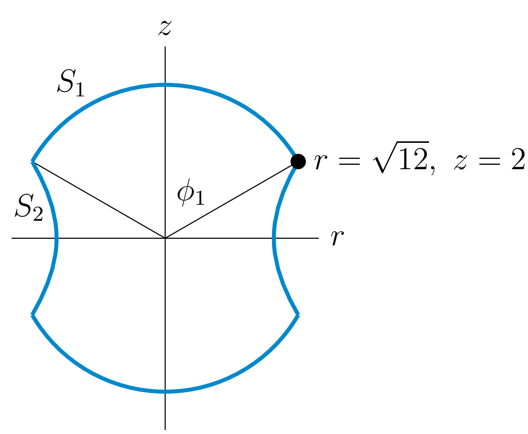
The domain of integration is $1 - \frac{1}{4}x^2 - y^2 \geqslant 0$ or $\frac{1}{4}x^2 + y^2 \leqslant 1$. This is an ellipse. Call it D. So

$$\iint\limits_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iint\limits_D (1 - 2x - 4y)\, \mathrm{d}x\mathrm{d}y$$

The integrals over D of x, which is odd under $x \to -x$, and of y, which is odd under $y \to -y$, are both zero. As the ellipse D has area $A = \pi \times 2 \times 1 = 2\pi$

$$\iint\limits_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iint\limits_D (1 - 2x - 4y)\, \mathrm{d}x\mathrm{d}y = A = 2\pi$$

S-37: Due to the symmetry of the surface and the vector field under reflection in the xy-plane, i.e. under $z \to -z$, it is sufficient to compute the integral over the upper half of the surface, where $z \geqslant 0$, and then multiply the result by 2. The upper half of the surface consists of two pieces, S_1 and S_2, where S_1 is the part on the sphere and S_2 is the part on the hyperboloid. S_1 and S_2 intersect on a circle. The circle is obtained by imposing the two equations $x^2 + y^2 + z^2 = 16$ and $x^2 + y^2 - z^2 = 8$ simultaneously. Thus we have $x^2 + y^2 = 12$ and $z = 2$, or in cylindrical coordinates $r = \sqrt{12}$, $z = 1$, on the circle. Here is a sketch of a cross-section of the apple core.



Let ϕ_1 be the angle between z-axis and the cone formed by connecting the circle to the origin. We have $\tan\phi_1 = \sqrt{12}/2 = \sqrt{3}$. Thus $\phi_1 = \pi/3$.

We'll use spherical coordinates to compute the flux integral $\iint_{S_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$. As the spherical coordinate $\rho = 4$ on all of S_1, we can paramerize S_1 by

$$\mathbf{r}(\theta, \varphi) = 4\cos\theta\sin\varphi\,\hat{\imath} + 4\sin\theta\sin\varphi\,\hat{\jmath} + 4\cos\varphi\,\hat{\mathbf{k}} \qquad 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \phi \leqslant \pi/3$$

So

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial \theta} &= -4\sin\theta\sin\varphi\,\hat{\boldsymbol{\imath}} + 4\cos\theta\sin\varphi\,\hat{\boldsymbol{\jmath}} \\
\frac{\partial \mathbf{r}}{\partial \varphi} &= 4\cos\theta\cos\varphi\,\hat{\boldsymbol{\imath}} + 4\sin\theta\cos\varphi\,\hat{\boldsymbol{\jmath}} - 4\sin\varphi\,\hat{\mathbf{k}} \\
\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi} &= \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ -4\sin\theta\sin\varphi & 4\cos\theta\sin\varphi & 0 \\ 4\cos\theta\cos\varphi & 4\sin\theta\cos\varphi & -4\sin\varphi \end{bmatrix} \\
&= -16\big(\cos\theta\sin^2\varphi\,,\ \sin\theta\sin^2\varphi\,,\ \sin\varphi\cos\varphi\big) \\
&= -4\,(\sin\varphi)\,\mathbf{r}(\theta,\varphi)
\end{aligned}$$

and, by (3.3.1) in the CLP-4 text,

$$\hat{\mathbf{n}}\,\mathrm{d}S = \pm\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi}\,\mathrm{d}\theta\mathrm{d}\varphi = \mp 4\,(\sin\varphi)\,\mathbf{r}(\theta,\varphi)\,\mathrm{d}\theta\mathrm{d}\varphi$$

To get the outward pointing normal, i.e. the normal point in the same direction as $\mathbf{r}(\theta,\varphi)$, we take the plus sign. As $\mathbf{F} = \mathbf{r}(\theta,\varphi)$,

$$\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 4\,\overbrace{|\mathbf{r}(\theta,\varphi)|^2}^{4^2}\ \sin\varphi\,\mathrm{d}\theta\,\mathrm{d}\varphi = 64\sin\varphi\,\mathrm{d}\theta\,\mathrm{d}\varphi$$

and

$$\iint_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = 64\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi/3}\mathrm{d}\varphi\ \sin\varphi = 64\cdot 2\pi\Big[-\cos\varphi\Big]_0^{\pi/3} = 64\pi$$

The surface S_2 can be parametrized using the cylindrical coordinates θ and z. Indeed, we have

$$r = \sqrt{x^2+y^2} = (8+z^2)^{1/2}$$

for the hyperboloid and we always have $x = r\cos\theta$ and $y = r\sin\theta$. Thus the hyperboloid has the following parametrization:

$$\mathbf{R}(\theta,z) = (8+z^2)^{1/2}\cos\theta\,\hat{\boldsymbol{\imath}} + (8+z^2)^{1/2}\sin\theta\,\hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}}$$

The range for the parameters of S_2 is $0 \leqslant \theta \leqslant 2\pi$ and $0 \leqslant z \leqslant 2$. We have

$$\begin{aligned}
\frac{\partial \mathbf{R}}{\partial \theta} &= -(8+z^2)^{1/2}\sin\theta\,\hat{\boldsymbol{\imath}} + (8+z^2)^{1/2}\cos\theta\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\mathbf{k}} \\
\frac{\partial \mathbf{R}}{\partial z} &= z(8+z^2)^{-1/2}\cos\theta\,\hat{\boldsymbol{\imath}} + z(8+z^2)^{-1/2}\sin\theta\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \mathbf{R}}{\partial \theta} \times \frac{\partial \mathbf{R}}{\partial z} &= \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ -(8+z^2)^{1/2}\sin\theta & (8+z^2)^{1/2}\cos\theta & 0 \\ z(8+z^2)^{-1/2}\cos\theta & z(8+z^2)^{-1/2}\sin\theta & 1 \end{bmatrix} \\
&= (8+z^2)^{1/2}\cos\theta\,\hat{\boldsymbol{\imath}} + (8+z^2)^{1/2}\sin\theta\,\hat{\boldsymbol{\jmath}} - z\,\hat{\mathbf{k}} \\
&= x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} - z\,\hat{\mathbf{k}}
\end{aligned}$$

Note that $\frac{\partial \mathbf{R}}{\partial \theta} \times \frac{\partial \mathbf{R}}{\partial z}$ is pointing downward (since $z > 0$) and hence outward. Since $F \cdot \left(\frac{\partial \mathbf{R}}{\partial \theta} \times \frac{\partial \mathbf{R}}{\partial z}\right) = (x, y, z) \cdot (x, y, -z) = x^2 + y^2 - z^2 = 8$ on S_2, we have

$$\iint_{S_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iint_{S_2} \mathbf{F} \cdot (\mathbf{R}_\theta \times \mathbf{R}_z)\, d\theta\, \mathrm{d}z = \int_0^2 \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta\, 8 = 32\pi$$

Finally, the flux integral over the whole apple core surface is

$$2\left(\iint_{S_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS + \iint_{S_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS\right) = 2\big(64\pi + 32\pi\big) = 192\pi$$

S-38: (a) The specified surface is of the form

$$G(x, y, z) = x^2 + z^2 - \cos^2 y = 0$$

So one normal vector at the point $\left(\frac{1}{2}, \frac{\pi}{4}, \frac{1}{2}\right)$ is

$$\nabla G\left(\frac{1}{2}, \frac{\pi}{4}, \frac{1}{2}\right) = \big(2x\,,\, 2\sin y \cos y\,,\, 2z\big)\Big|_{\left(\frac{1}{2}, \frac{\pi}{4}, \frac{1}{2}\right)} = (1, 1, 1)$$

and an equation for the tangent plane at $\left(\frac{1}{2}, \frac{\pi}{4}, \frac{1}{2}\right)$ is

$$(1, 1, 1) \cdot \big(x - 1/2\,,\, y - \pi/4\,,\, z - 1/2\big) = 0 \quad \text{or} \quad x + y + z = 1 + \pi/4$$

(b) For each fixed y, $x^2 + z^2 = \cos^2 y$ is a circle of radius $|\cos y|$. So we may parametrize the surface by

$$\mathbf{r}(\theta, y) = \big(\cos y\, \cos\theta\,,\, y\,,\, \cos y\, \sin\theta\big) \qquad 0 \leqslant \theta < 2\pi,\ 0 \leqslant y \leqslant \pi/2$$

Then

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial \theta} &= \big(-\cos y\, \sin\theta\,,\, 0\,,\, \cos y\, \cos\theta\big) \\
\frac{\partial \mathbf{r}}{\partial y} &= \big(-\sin y\, \cos\theta\,,\, 1\,,\, -\sin y\, \sin\theta\big) \\
\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial y} &= \big(-\cos y\, \cos\theta\,,\, -\sin y\, \cos y\,,\, -\cos y\, \sin\theta\big) \\
\mathrm{d}S &= \left|\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial y}\right| \mathrm{d}\theta\, \mathrm{d}y = \cos y \sqrt{1 + \sin^2 y}\, \mathrm{d}\theta\, \mathrm{d}y
\end{aligned}$$

So the specified integral is

$$\begin{aligned}
\iint\limits_S \sin y \,\mathrm{d}S &= \int_0^{\pi/2} \mathrm{d}y \int_0^{2\pi} \mathrm{d}\theta\ \cos y\sqrt{1+\sin^2 y}\ \sin y \\
&= 2\pi \int_0^{\pi/2} \mathrm{d}y\ \sqrt{1+\sin^2 y}\ \sin y \cos y \\
&= \pi \int_1^2 \mathrm{d}u\ \sqrt{u} \quad \text{with } u = 1+\sin^2 y,\ \mathrm{d}u = 2\sin y\cos y\,\mathrm{d}y \\
&= \pi \left[\frac{u^{3/2}}{3/2}\right]_1^2 \\
&= \frac{2\pi}{3}\left[2\sqrt{2}-1\right]
\end{aligned}$$

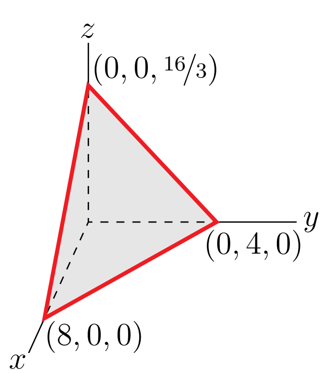
S-39: (a) By definition $\mathbf{F}$ is a conservative vector field with potential f. Suppose that the curve C starts at P_1, on S, and ends at P_2, on S. Then $f(P_1) = f(P_2) = c$ and, by Theorem 2.4.2 of the CLP-4 text,

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = f(P_2) - f(P_1) = c - c = 0$$

(b) Since $\mathbf{F} = \boldsymbol{\nabla} f$, $\mathbf{F}$ is normal to the level surfaces of f by Lemma 2.3.6 of the CLP-4 text. So, at any point of S, $\mathbf{F}$ is a scalar multiple of $\hat{\mathbf{n}}$ and $\mathbf{F}\times\mathbf{G}$ is perpendicular to $\hat{\mathbf{n}}$. Thus $(\mathbf{F}\times\mathbf{G})\cdot\hat{\mathbf{n}} = 0$ and

$$\iint\limits_S (\mathbf{F}\times\mathbf{G})\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 0.$$

S-40: (a) (i) Here is a sketch of the part of the plane in question.



We can use x and y as parameters. As we can rewrite the equation of the plane as $z = \frac{1}{3}(16-2x-4y)$, we have the parametrization

$$\mathbf{r}(x,y) = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + \frac{1}{3}(16-2x-4y)\,\hat{\mathbf{k}}$$

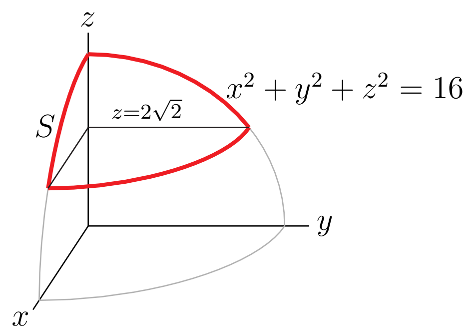
In terms of x and y, the condition $z = \frac{1}{3}(16-2x-4y) \geqslant 0$ is $16-2x-4y \geqslant 0$ or $x+2y \leqslant 8$. So the domain is

$$\{\ (x,y)\ \big|\ x \geqslant 0,\ y \geqslant 0,\ x+2y \leqslant 8\ \}$$

Renaming x to u and y to v, the parametrization is also

$$\mathbf{r}(u,v) = \Big(u\,,\,v\,,\,\frac{1}{3}(16-2u-4v)\Big)\,\hat{\mathbf{k}} \qquad u \geqslant 0,\ v \geqslant 0,\ u+2v \leqslant 8$$

(a) (ii) Here is a sketch of the part of the cap in the first octant.



The full sphere can be parametrized (using spherical coordinates with $\rho = 4$) by

$$\mathbf{r}(\theta,\varphi) = 4\cos\theta\sin\varphi\,\hat{\boldsymbol{\imath}} + 4\sin\theta\sin\varphi\,\hat{\boldsymbol{\jmath}} + 4\cos\varphi\,\hat{\mathbf{k}} \qquad 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \varphi \leqslant \pi$$

In these coordinates, the condition $4/\sqrt{2} \leqslant z \leqslant 4$ is

$$\begin{aligned} \frac{4}{\sqrt{2}} \leqslant 4\cos\varphi \leqslant 4 &\iff \frac{1}{\sqrt{2}} \leqslant \cos\varphi \leqslant 1 \\ &\iff 0 \leqslant \varphi \leqslant \frac{\pi}{4} \end{aligned}$$

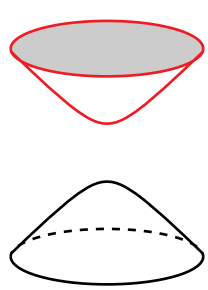
So our parametrization is

$$\mathbf{r}(\theta,\varphi) = 4\cos\theta\sin\varphi\,\hat{\boldsymbol{\imath}} + 4\sin\theta\sin\varphi\,\hat{\boldsymbol{\jmath}} + 4\cos\varphi\,\hat{\mathbf{k}} \qquad 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \varphi \leqslant \frac{\pi}{4}$$

Renaming θ to u and φ to v, the parametrization is also

$$\mathbf{r}(u,v) = \big(4\cos u\sin v\,,\,4\sin u\sin v\,,\,4\cos v\big) \qquad 0 \leqslant u \leqslant 2\pi,\ 0 \leqslant v \leqslant \frac{\pi}{4}$$

(a) (iii) Here is a sketch of the hyperboloid.



If we use x and y as parameters, then, since $z = \sqrt{1 + x^2 + y^2}$, we have the parametrization

$$\mathbf{r}(x, y) = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + \sqrt{1 + x^2 + y^2}\,\hat{\mathbf{k}}$$

In terms of x and y, the condition $1 \leqslant z \leqslant 10$ is

$$1 \leqslant \sqrt{1 + x^2 + y^2} \leqslant 10 \qquad \text{or} \qquad 0 \leqslant x^2 + y^2 \leqslant 99$$

So the domain is

$$\left\{ (x, y) \mid x^2 + y^2 \leqslant 99 \right\}$$

Renaming x to u and y to v, the parametrization is also

$$\mathbf{r}(u, v) = \big(u\,,\, v\,,\, \sqrt{1 + u^2 + v^2}\big) \qquad u^2 + v^2 \leqslant 99$$

Alternatively, if we replace x and y with the polar coordinates r and θ, we get the parametrization

$$\mathbf{r}(r, \theta) = r\cos\theta\,\hat{\boldsymbol{\imath}} + r\sin\theta\,\hat{\boldsymbol{\jmath}} + \sqrt{1 + r^2}\,\hat{\mathbf{k}} \qquad 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant r \leqslant \sqrt{99}$$

Renaming r to u and θ to v, the parametrization is also

$$\mathbf{r}(u, v) = \big(u\cos v\,,\, u\sin v\,,\, \sqrt{1 + u^2}\big) \qquad 0 \leqslant v \leqslant 2\pi,\ 0 \leqslant u \leqslant \sqrt{99}$$

(b) Let's use the parametrization

$$\mathbf{r}(\theta, \varphi) = 4\cos\theta\sin\varphi\,\hat{\boldsymbol{\imath}} + 4\sin\theta\sin\varphi\,\hat{\boldsymbol{\jmath}} + 4\cos\varphi\,\hat{\mathbf{k}} \qquad 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \varphi \leqslant \frac{\pi}{4}$$

from part (a) (ii), so that

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi} &= \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ -4\sin\theta\sin\varphi & 4\cos\theta\sin\varphi & 0 \\ 4\cos\theta\cos\varphi & 4\sin\theta\cos\varphi & -4\sin\varphi \end{bmatrix} \\
&= -16\big(\cos\theta\sin^2\varphi\,,\, \sin\theta\sin^2\varphi\,,\, \sin\varphi\cos\varphi\big)
\end{aligned}$$

and, by (3.3.1) in the CLP-4 text,

$$\begin{aligned}
\mathrm{d}S &= \Big|\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi}\Big|\,\mathrm{d}\theta\mathrm{d}\varphi = 16\sin\varphi\sqrt{\cos^2\theta\sin^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\varphi}\,\mathrm{d}\theta\mathrm{d}\varphi \\
&= 16\sin\varphi\,\mathrm{d}\theta\mathrm{d}\varphi
\end{aligned}$$

So the area is

$$\begin{aligned}
\text{Area} = \iint_S \mathrm{d}S &= \int_0^{\pi/4} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta\ 16\sin\varphi = 32\pi \int_0^{\pi/4} \mathrm{d}\varphi\ \sin\varphi = 32\pi\Big[-\cos\varphi\Big]_0^{\pi/4} \\
&= 32\pi\Big[1 - \frac{1}{\sqrt{2}}\Big]
\end{aligned}$$

S-41:

Solution 1 — using tweaked spherical coordinates.

First we have to parametrize S. It is natural to use spherical coordinates with $\rho = \sqrt{2}$. However if we use the standard spherical coordinates

$$x = \sqrt{2}\,\sin\varphi\cos\theta \qquad y = \sqrt{2}\,\sin\varphi\sin\theta \qquad z = \sqrt{2}\,\cos\varphi$$

the condition $y \geqslant 1$ becomes $\sin\varphi\sin\theta \geqslant \frac{1}{\sqrt{2}}$, which is very complicated. So let's back up and think a bit before we compute. The condition $z \geqslant 1$, as opposed to $y \geqslant 1$, is easy to implement in spherical coordinates. It is $\cos\varphi \geqslant \frac{1}{\sqrt{2}}$ or $0 \leqslant \varphi \leqslant \frac{\pi}{4}$. So let's modify spherical coordinates to make the y-axis play the role of the z-axis, by just exchanging y and z in the parametrization.

$$x = \sqrt{2}\,\sin\varphi\cos\theta \qquad y = \sqrt{2}\,\cos\varphi \qquad z = \sqrt{2}\,\sin\varphi\sin\theta$$

The condition $y \geqslant 1$ is then $\sqrt{2}\cos\varphi \geqslant 1$, which is turn is $0 \leqslant \varphi \leqslant \frac{\pi}{4}$. Since we have just exchanged y and z we could probably just guess $\hat{\mathbf{n}}\,\mathrm{d}S$ and $\mathrm{d}S$ from standard spherical coordinates. (See Appendix F.3 in the CLP-4 text and recall that $\rho = \sqrt{2}$.) But to be on the safe side, let's derive them. We are using the parametrization

$$\mathbf{r}(\theta,\varphi) = \sqrt{2}\,\sin\varphi\cos\theta\,\hat{\imath} + \sqrt{2}\,\cos\varphi\,\hat{\jmath} + \sqrt{2}\,\sin\varphi\sin\theta\,\hat{\mathbf{k}} \qquad 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \varphi \leqslant \frac{\pi}{4}$$

Since

$$\begin{aligned}
\frac{\partial\mathbf{r}}{\partial\theta} &= -\sqrt{2}\,\sin\varphi\sin\theta\,\hat{\imath} + \sqrt{2}\,\sin\varphi\cos\theta\,\hat{\mathbf{k}} \\
\frac{\partial\mathbf{r}}{\partial\varphi} &= \sqrt{2}\,\cos\varphi\cos\theta\,\hat{\imath} - \sqrt{2}\,\sin\varphi\,\hat{\jmath} + \sqrt{2}\,\cos\varphi\sin\theta\,\hat{\mathbf{k}}
\end{aligned}$$

so that

$$\begin{aligned}
\frac{\partial\mathbf{r}}{\partial\theta} \times \frac{\partial\mathbf{r}}{\partial\varphi} &= \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ -\sqrt{2}\,\sin\varphi\sin\theta & 0 & \sqrt{2}\,\sin\varphi\cos\theta \\ \sqrt{2}\,\cos\varphi\cos\theta & -\sqrt{2}\,\sin\varphi & \sqrt{2}\,\cos\varphi\sin\theta \end{bmatrix} \\
&= 2\sin^2\varphi\cos\theta\,\hat{\imath} + 2\sin\varphi\cos\varphi\,\hat{\jmath} + 2\sin^2\varphi\sin\theta\,\hat{\mathbf{k}}
\end{aligned}$$

(3.3.1) in the CLP-4 text gives

$$\begin{aligned}
\hat{\mathbf{n}}\,\mathrm{d}S &= \pm\frac{\partial\mathbf{r}}{\partial\theta} \times \frac{\partial\mathbf{r}}{\partial\varphi}\,\mathrm{d}\theta\mathrm{d}\varphi = \pm 2\sin\varphi\big(\sin\varphi\cos\theta\,,\,\cos\varphi\,,\,\sin\varphi\sin\theta\big)\,\mathrm{d}\theta\mathrm{d}\varphi \\
\mathrm{d}S &= \left|\frac{\partial\mathbf{r}}{\partial\theta} \times \frac{\partial\mathbf{r}}{\partial\varphi}\right|\,\mathrm{d}\theta\mathrm{d}\varphi = 2\sin\varphi\,\mathrm{d}\theta\mathrm{d}\varphi
\end{aligned}$$

Choose the plus sign to give the outward pointing normal.

(a) The specified integral is

$$\iint_S y^3\,\mathrm{d}S = 2\int_0^{\pi/4}\mathrm{d}\varphi\int_0^{2\pi}\mathrm{d}\theta\ \sin\varphi\ \overbrace{\left(\sqrt{2}\,\cos\varphi\right)^3}^{y^3}$$

$$= 8\sqrt{2}\,\pi\int_0^{\pi/4}\mathrm{d}\varphi\ \sin\varphi\cos^3\varphi$$

$$= 8\sqrt{2}\,\pi\left[-\frac{\cos^4\varphi}{4}\right]_0^{\pi/4}$$

$$= 2\sqrt{2}\,\pi\left[1-\frac{1}{4}\right] = \frac{3}{\sqrt{2}}\pi$$

(b) The specified integral is

$$\iint_S \left(xy\,\hat{\imath} + xz\,\hat{\jmath} + zy\,\hat{\mathbf{k}}\right)\cdot\hat{\mathbf{n}}\,\mathrm{d}S$$

$$= 2\int_0^{\pi/4}\mathrm{d}\varphi\int_0^{2\pi}\mathrm{d}\theta\ \sin\varphi\ \left(2\sin\varphi\cos\varphi\cos\theta\,,\ 2\sin^2\varphi\sin\theta\cos\theta\,,\ 2\sin\varphi\cos\varphi\sin\theta\right)\cdot \left(\sin\varphi\cos\theta\,,\ \cos\varphi\,,\ \sin\varphi\sin\theta\right)$$

$$= 4\int_0^{\pi/4}\mathrm{d}\varphi\int_0^{2\pi}\mathrm{d}\theta\ \Big\{\sin^3\varphi\cos\varphi\cos^2\theta + \sin^3\varphi\cos\varphi\sin\theta\cos\theta + \sin^3\varphi\cos\varphi\sin^2\theta\Big\}$$

$$= 4\left[\int_0^{\pi/4}\mathrm{d}\varphi\ \sin^3\varphi\cos\varphi\right]\left[\int_0^{2\pi}\mathrm{d}\theta\ \left(1+\sin\theta\cos\theta\right)\right]$$

$$= 4\left[\frac{\sin^4\varphi}{4}\right]_0^{\pi/4}\left[\theta+\frac{\sin^2\theta}{2}\right]_0^{2\pi}$$

$$= 4\times\frac{1}{16}\times(2\pi) = \frac{\pi}{2}$$

Solution 2 — parametrizing by x and z.

We can also parametrize S by using x and z as parameters. On S,

- $y = \sqrt{2-x^2-z^2}$ and
- y runs over the range $1 \leqslant y \leqslant \sqrt{2}$. Correspondingly, $x^2+z^2 = 2-y^2$ runs over $0 \leqslant x^2+z^2 \leqslant 1$

So we can use the parametrization

$$\mathbf{r}(x,z) = x\,\hat{\imath} + \sqrt{2-x^2-z^2}\,\hat{\jmath} + z\,\hat{\mathbf{k}} \qquad 0 \leqslant x^2+z^2 \leqslant 1$$

Since

$$\frac{\partial\mathbf{r}}{\partial x} = \hat{\imath} - \frac{x}{\sqrt{2-x^2-z^2}}\,\hat{\jmath}$$

$$\frac{\partial\mathbf{r}}{\partial z} = -\frac{z}{\sqrt{2-x^2-z^2}}\,\hat{\jmath} + \hat{\mathbf{k}}$$

so that

$$\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial z} = \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 1 & -\frac{x}{\sqrt{2-x^2-z^2}} & 0 \\ 0 & -\frac{z}{\sqrt{2-x^2-z^2}} & 1 \end{bmatrix}$$
$$= -\frac{x}{\sqrt{2-x^2-z^2}}\,\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} - \frac{z}{\sqrt{2-x^2-z^2}}\,\hat{\mathbf{k}}$$

(3.3.1) in the CLP-4 text gives

$$\hat{\mathbf{n}}\,\mathrm{d}S = \pm\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial z}\,\mathrm{d}x\mathrm{d}z = \mp\left(\frac{x}{\sqrt{2-x^2-z^2}}\,,1\,,\,\frac{z}{\sqrt{2-x^2-z^2}}\right)\,\mathrm{d}x\mathrm{d}z$$
$$\mathrm{d}S = \left|\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial z}\right|\,\mathrm{d}x\mathrm{d}z = \sqrt{1+\frac{x^2+z^2}{2-x^2-z^2}}\,\mathrm{d}x\mathrm{d}z = \frac{\sqrt{2}}{\sqrt{2-x^2-z^2}}\,\mathrm{d}x\mathrm{d}z$$

Choose the plus sign to give the outward pointing normal.

(a) The specified integral is

$$\iint_S y^3\,\mathrm{d}S = \iint_{x^2+z^2\leqslant 1}\mathrm{d}x\mathrm{d}z\,\frac{\sqrt{2}}{\sqrt{2-x^2-z^2}}\,\overbrace{\left(2-x^2-z^2\right)^{3/2}}^{y^3}$$
$$= \sqrt{2}\iint_{x^2+z^2\leqslant 1}\mathrm{d}x\mathrm{d}z\,\left(2-x^2-z^2\right)$$

Switching to polar coordinates with $x = r\cos\theta$ and $z = r\sin\theta$,

$$\iint_S y^3\,\mathrm{d}S = \sqrt{2}\int_0^{2\pi}\mathrm{d}\theta\int_0^1\mathrm{d}r\;r(2-r^2)$$
$$= \sqrt{2}\,(2\pi)\;\left[r^2-\frac{r^4}{4}\right]_0^1 = \sqrt{2}\,(2\pi)\;\left[\frac{3}{4}\right] = \frac{3}{\sqrt{2}}\pi$$

(b) The specified integral is

$$\iint_S \left(xy\,\hat{\boldsymbol{\imath}} + xz\,\hat{\boldsymbol{\jmath}} + zy\,\hat{\mathbf{k}}\right)\cdot\hat{\mathbf{n}}\,\mathrm{d}S$$
$$= \iint_{x^2+z^2\leqslant 1}\mathrm{d}x\mathrm{d}z\,\left(x\sqrt{2-x^2-z^2}\,,\,xz\,,\,z\sqrt{2-x^2-z^2}\right)\cdot$$
$$\left(\frac{x}{\sqrt{2-x^2-z^2}}\,,1\,,\,\frac{z}{\sqrt{2-x^2-z^2}}\right)$$
$$= \iint_{x^2+z^2\leqslant 1}\mathrm{d}x\mathrm{d}z\,\left\{x^2+xz+z^2\right\}$$

Switching to polar coordinates with $x = r\cos\theta$ and $z = r\sin\theta$,

$$
\begin{aligned}
\iint_S \big(xy\,\hat{\boldsymbol\imath} + xz\,\hat{\boldsymbol\jmath} + zy\,\hat{\mathbf{k}}\big)\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \int_0^{2\pi}\mathrm{d}\theta\int_0^1 \mathrm{d}r\; r(r^2\cos^2\theta + r^2\sin\theta\cos\theta + r^2\sin^2\theta)\\
&= \left[\int_0^1 \mathrm{d}r\; r^3\right]\left[\int_0^{2\pi}\mathrm{d}\theta\;\big(1+\sin\theta\cos\theta\big)\right]\\
&= \left[\frac{r^4}{4}\right]_0^1\left[\theta + \frac{\sin^2\theta}{2}\right]_0^{2\pi}\\
&= \frac{1}{4}\times(2\pi) = \frac{\pi}{2}
\end{aligned}
$$

S-42: First observe that,

- because $(x+y+1)^2 \geqslant 0$, all points on $(x+y+1)^2 + z^2 = 4$ have $|z| \leqslant 2$ and that,
- for $|z_0| \leqslant 2$, the surface $(x+y+1)^2 + z^2 = 4$ intersects the horizontal plane $z = z_0$ on $(x+y+1)^2 = 4 - z_0^2$, i.e. on the two lines $x + y = \pm\sqrt{4-z_0^2} - 1$, $z = z_0$.
- The line $x + y = \pm\sqrt{4-z_0^2} - 1$, $z = z_0$ intersects the first octant if and only if $z_0 \geqslant 0$ and $\pm\sqrt{4-z_0^2} - 1 \geqslant 0$.
- Thus $x + y = -\sqrt{4-z_0^2} - 1$, $z = z_0$ never intersects the first octant and
- $x + y = \sqrt{4-z_0^2} - 1$, $z = z_0$ intersects the first octant if and only if $0 \leqslant z_0 \leqslant \sqrt{3}$.
- When $z_0 = 0$, the line $x + y = \sqrt{4-z_0^2} - 1$, $z = z_0$ is $x + y = 1$, $z = 0$.
- When $z_0 = \sqrt{3}$, the line $x + y = \sqrt{4-z_0^2} - 1$, $z = z_0$ is $x + y = 0$, $z = 2$.
- So as (x, y, z) runs over $\mathcal{S}$, (x, y) runs over the triangle $x \geqslant 0$, $y \geqslant 0$, $x + y \leqslant 1$.

Let $G(x, y, z) = (x+y+1)^2 + z^2$. Then

$$
\hat{\mathbf{n}}\,\mathrm{d}S = \pm\frac{\boldsymbol\nabla G}{\boldsymbol\nabla G\cdot\hat{\mathbf{k}}}\,\mathrm{d}x\mathrm{d}y = \pm\frac{2(x+y+1)\hat{\boldsymbol\imath} + 2(x+y+1)\hat{\boldsymbol\jmath} + 2z\hat{\mathbf{k}}}{2z}\,\mathrm{d}x\mathrm{d}y
$$

For the downward normal, we need the minus sign, so

$$
\begin{aligned}
\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= -\big[xy\,\hat{\boldsymbol\imath} + (z-xy)\,\hat{\boldsymbol\jmath}\big]\cdot\left[\frac{(x+y+1)\hat{\boldsymbol\imath} + (x+y+1)\hat{\boldsymbol\jmath} + z\hat{\mathbf{k}}}{z}\right]\mathrm{d}x\mathrm{d}y\\
&= -\frac{1}{z}\big[xy(x+y+1) + (z-xy)(x+y+1)\big]\,\mathrm{d}x\mathrm{d}y\\
&= -\frac{1}{z}\big[z(x+y+1)\big]\,\mathrm{d}x\mathrm{d}y\\
&= -(x+y+1)\,\mathrm{d}x\mathrm{d}y
\end{aligned}
$$

The domain of integration is $x \geqslant 0,\ y \geqslant 0,\ x + y \leqslant 1$, so

$$\iint_{\mathcal{S}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = -\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\ (x+y+1) = -\int_0^1 \mathrm{d}x\ \Big[(1+x)(1-x) + \frac{1}{2}(1-x)^2\Big]$$
$$= -\int_0^1 \mathrm{d}x\ \Big[\frac{3}{2} - x - \frac{1}{2}x^2\Big] = -\Big[\frac{3}{2} - \frac{1}{2} - \frac{1}{6}\Big] = -\frac{5}{6}$$

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S-1: (a) A. The angle between $\mathbf{F}$ and $\mathrm{d}\mathbf{r}$ is less than $90°$ along the entire path. So $\mathbf{F}\cdot\mathrm{d}\mathbf{r} > 0$ along the entire path and the work is positive.

(b) B. $\mathbf{F}$ is perpendicular to $\mathrm{d}\mathbf{r}$ along all of C_2. So $\int_{C_2} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = 0$.

(c) C. It looks like $P_x = Q_y = 0$ at N. So $\boldsymbol{\nabla}\cdot\mathbf{F} = 0$ at N.

(d) A. At Q, the vertical component of $\mathbf{F}$ is increasing from left to right (so that $Q_x > 0$) and the horizontal component of $\mathbf{F}$ is decreasing from bottom to top (so that $P_y < 0$). So $Q_x - P_y > 0$ at N.

(e) B. At D, the horizontal component of $\mathbf{F}$ is increasing from left to right, so that $P_x > 0$.

S-2: No. The vector field $\mathbf{F}(x,y,z) = \hat{\imath} + y\,\hat{\mathbf{k}}$ has

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{F} &= \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 1 & 0 & y\end{bmatrix}\\ &= \hat{\imath}\end{aligned}$$

has dot product 1 with $\mathbf{F}(x,y,z)$ (for all x, y, z) and so is not perpendicular to it.

S-3: (a) By the product rule

$$\begin{aligned}\boldsymbol{\nabla}\cdot(f\mathbf{F}) &= \frac{\partial}{\partial x}(fF_1) + \frac{\partial}{\partial y}(fF_2) + \frac{\partial}{\partial z}(fF_3)\\ &= \quad f\frac{\partial F_1}{\partial x} + f\frac{\partial F_2}{\partial y} + f\frac{\partial F_3}{\partial z}\\ &\quad + F_1\frac{\partial f}{\partial x} + F_2\frac{\partial f}{\partial y} + F_3\frac{\partial f}{\partial z}\\ &= f\,\boldsymbol{\nabla}\cdot\mathbf{F} + \mathbf{F}\cdot\boldsymbol{\nabla} f\end{aligned}$$

(b) Again by the product rule

$$\begin{aligned}
\nabla \cdot (\mathbf{F} \times \mathbf{G}) &= \frac{\partial}{\partial x}(F_2G_3 - F_3G_2) + \frac{\partial}{\partial y}(F_3G_1 - F_1G_3) + \frac{\partial}{\partial z}(F_1G_2 - F_2G_1) \\
&= \quad \frac{\partial F_2}{\partial x}G_3 - \frac{\partial F_3}{\partial x}G_2 + \frac{\partial F_3}{\partial y}G_1 - \frac{\partial F_1}{\partial y}G_3 + \frac{\partial F_1}{\partial z}G_2 - \frac{\partial F_2}{\partial z}G_1 \\
&\quad + F_2\frac{\partial G_3}{\partial x} - F_3\frac{\partial G_2}{\partial x} + F_3\frac{\partial G_1}{\partial y} - F_1\frac{\partial G_3}{\partial y} + F_1\frac{\partial G_2}{\partial z} - F_2\frac{\partial G_1}{\partial z} \\
&= \quad \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)G_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)G_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)G_3 \\
&\quad - F_1\left(\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}\right) - F_2\left(\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}\right) - F_3\left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right) \\
&= \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})
\end{aligned}$$

(c) Recall that $\nabla^2(fg) = \nabla \cdot \big[\nabla(fg)\big]$. First

$$\begin{aligned}
\nabla(fg) &= \hat{\imath}\frac{\partial}{\partial x}(fg) + \hat{\jmath}\frac{\partial}{\partial y}(fg) + \hat{\mathbf{k}}\frac{\partial}{\partial z}(fg) \\
&= \quad \hat{\imath}g\frac{\partial f}{\partial x} + \hat{\jmath}g\frac{\partial f}{\partial y} + \hat{\mathbf{k}}g\frac{\partial f}{\partial z} \\
&\quad + \hat{\imath}f\frac{\partial g}{\partial x} + \hat{\jmath}f\frac{\partial g}{\partial y} + \hat{\mathbf{k}}f\frac{\partial g}{\partial z} \\
&= g\nabla f + f\nabla g
\end{aligned}$$

So by part (a), twice,

$$\begin{aligned}
\nabla^2(fg) &= \nabla \cdot (g\nabla f) + \nabla \cdot (f\nabla g) \\
&= g(\nabla \cdot \nabla f) + (\nabla g) \cdot (\nabla f) + f(\nabla \cdot \nabla g) + (\nabla f) \cdot (\nabla g) \\
&= f\,\nabla^2 g + 2\nabla f \cdot \nabla g + g\,\nabla^2 f
\end{aligned}$$

S-4: (a) By definition

$$\nabla \cdot (x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{\mathbf{k}}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

$$\nabla \times (x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{\mathbf{k}}) = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{bmatrix} = \mathbf{0}$$

(b) By definition

$$\nabla \cdot (xy^2\hat{\boldsymbol{\imath}} - yz^2\hat{\boldsymbol{\jmath}} + zx^2\hat{\mathbf{k}}) = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(-yz^2) + \frac{\partial}{\partial z}(zx^2) = y^2 - z^2 + x^2$$

$$\nabla \times (xy^2\hat{\boldsymbol{\imath}} - yz^2\hat{\boldsymbol{\jmath}} + zx^2\hat{\mathbf{k}}) = \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & -yz^2 & zx^2 \end{bmatrix} = 2yz\,\hat{\boldsymbol{\imath}} - 2xz\,\hat{\boldsymbol{\jmath}} - 2xy\,\hat{\mathbf{k}}$$

(c) By definition

$$\begin{aligned} \nabla \cdot \left(\frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\imath}} + \frac{y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\jmath}}\right) &= \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}}\right) \\ &= \frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{[x^2+y^2]^{3/2}} + \frac{1}{\sqrt{x^2+y^2}} - \frac{y^2}{[x^2+y^2]^{3/2}} \\ &= \frac{x^2+y^2 \ - x^2 \ + x^2+y^2 \ - \ y^2}{[x^2+y^2]^{3/2}} \\ &= \frac{1}{\sqrt{x^2+y^2}} \end{aligned}$$

$$\begin{aligned} \nabla \times \left(\frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\imath}} + \frac{y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\jmath}}\right) &= \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \end{bmatrix} \\ &= \left(-\frac{xy}{[x^2+y^2]^{3/2}} + \frac{xy}{[x^2+y^2]^{3/2}}\right)\hat{\mathbf{k}} = \mathbf{0} \end{aligned}$$

(d) By definition

$$\begin{aligned} \nabla \cdot \left(-\frac{y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\imath}} + \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\jmath}}\right) &= \frac{\partial}{\partial x}\left(-\frac{y}{\sqrt{x^2+y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right) \\ &= \frac{xy}{[x^2+y^2]^{3/2}} - \frac{xy}{[x^2+y^2]^{3/2}} = 0 \end{aligned}$$

$$\begin{aligned} \nabla \times \left(-\frac{y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\imath}} + \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\jmath}}\right) &= \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \end{bmatrix} \\ &= \left(\frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{[x^2+y^2]^{3/2}} + \frac{1}{\sqrt{x^2+y^2}} - \frac{y^2}{[x^2+y^2]^{3/2}}\right)\hat{\mathbf{k}} \\ &= \frac{x^2+y^2 \ - x^2 \ + x^2+y^2 \ - \ y^2}{[x^2+y^2]^{3/2}}\,\hat{\mathbf{k}} = \frac{\hat{\mathbf{k}}}{\sqrt{x^2+y^2}} \end{aligned}$$

S-5: (a) We are to compute the divergence of $\frac{\mathbf{r}}{r} = \frac{x\,\hat{\boldsymbol{\imath}}+y\,\hat{\boldsymbol{\jmath}}+z\,\hat{\mathbf{k}}}{[x^2+y^2+z^2]^{1/2}}$. Since

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{x}{[x^2+y^2+z^2]^{1/2}} &= \frac{1}{[x^2+y^2+z^2]^{1/2}} - \frac{1}{2}\frac{x(2x)}{[x^2+y^2+z^2]^{3/2}} = \frac{y^2+z^2}{[x^2+y^2+z^2]^{3/2}}\\
\frac{\partial}{\partial y}\frac{y}{[x^2+y^2+z^2]^{1/2}} &= \frac{1}{[x^2+y^2+z^2]^{1/2}} - \frac{1}{2}\frac{y(2y)}{[x^2+y^2+z^2]^{3/2}} = \frac{x^2+z^2}{[x^2+y^2+z^2]^{3/2}}\\
\frac{\partial}{\partial z}\frac{z}{[x^2+y^2+z^2]^{1/2}} &= \frac{1}{[x^2+y^2+z^2]^{1/2}} - \frac{1}{2}\frac{z(2z)}{[x^2+y^2+z^2]^{3/2}} = \frac{x^2+y^2}{[x^2+y^2+z^2]^{3/2}}
\end{aligned}$$

the specified divergence is

$$\boldsymbol{\nabla}\left(\frac{\mathbf{r}}{r}\right) = \frac{2x^2+2y^2+2z^2}{[x^2+y^2+z^2]^{3/2}} = \frac{2r^2}{r^3} = \frac{2}{r}$$

(b)

$$\boldsymbol{\nabla}\times\left(yz\,\hat{\boldsymbol{\imath}}+2xz\,\hat{\boldsymbol{\jmath}}+e^{xy}\,\hat{\mathbf{k}}\right) = \det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ yz & 2xz & e^{xy}\end{bmatrix} = \left(xe^{xy}-2x\right)\hat{\boldsymbol{\imath}} - \left(ye^{xy}-y\right)\hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}}$$

S-6: (a) Since $r^k = \left(x^2+y^2+z^2\right)^{k/2}$,

$$\begin{aligned}
\frac{\partial}{\partial x}r^k &= 2x\,\frac{k}{2}\left(x^2+y^2+z^2\right)^{\frac{k}{2}-1} = k\,(\mathbf{r}\cdot\hat{\boldsymbol{\imath}})\,r^{k-2}\\
\frac{\partial}{\partial y}r^k &= 2y\,\frac{k}{2}\left(x^2+y^2+z^2\right)^{\frac{k}{2}-1} = k\,(\mathbf{r}\cdot\hat{\boldsymbol{\jmath}})\,r^{k-2}\\
\frac{\partial}{\partial z}r^k &= 2z\,\frac{k}{2}\left(x^2+y^2+z^2\right)^{\frac{k}{2}-1} = k\,(\mathbf{r}\cdot\hat{\mathbf{k}})\,r^{k-2}
\end{aligned}$$

We want $k=-3$.

(b) Using the computation in part (a)

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(r^k\mathbf{r}) &= \frac{\partial}{\partial x}(xr^k) + \frac{\partial}{\partial y}(yr^k) + \frac{\partial}{\partial z}(zr^k)\\
&= 3r^k + x\frac{\partial}{\partial x}r^k + y\frac{\partial}{\partial y}r^k + z\frac{\partial}{\partial z}r^k\\
&= 3r^k + x\left(kx\,r^{k-2}\right) + y\left(ky\,r^{k-2}\right) + z\left(kz\,r^{k-2}\right)\\
&= \left(3+k\right)r^k
\end{aligned}$$

We want $k=2$.

(c) Recalling that $\boldsymbol{\nabla}^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$,

$$\begin{aligned}
\boldsymbol{\nabla}^2(r^k) &= \boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}(r^k)\right)\\
&= \boldsymbol{\nabla}\cdot(kr^{k-2}\,\mathbf{r}) && \text{by part (a)}\\
&= k(3+k-2)r^{k-2} && \text{by part (b), but with } k \text{ replaced by } k-2
\end{aligned}$$

We want $k = -2$.

S-7: (a)

$$\nabla \cdot \mathbf{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

(b)

$$\nabla(r^2) = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\left(x^2+y^2+z^2\right) = 2x\,\hat{\imath} + 2y\,\hat{\jmath} + 2\,\hat{\mathbf{k}} = 2\mathbf{r}$$

(c) Since

$$\mathbf{r}\times\mathbf{a} = \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ x & y & z\\ a_1 & a_2 & a_3\end{bmatrix} = \hat{\imath}\big(a_3y - a_2z\big) + \hat{\jmath}\big(a_1z - a_3x\big) + \hat{\mathbf{k}}\big(a_2x - a_1y\big)$$

we have

$$\begin{aligned}\nabla\times(\mathbf{r}\times\mathbf{a}) &= \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ a_3y - a_2z & a_1z - a_3x & a_2x - a_1y\end{bmatrix} = -2a_1\,\hat{\imath} - 2a_2\,\hat{\jmath} - 2a_3\,\hat{\mathbf{k}}\\ &= -2\mathbf{a}\end{aligned}$$

(d) Since

$$\begin{aligned}\nabla(r) &= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\left(x^2+y^2+z^2\right)^{1/2}\\ &= \hat{\imath}\frac{x}{\left(x^2+y^2+z^2\right)^{1/2}} + \hat{\jmath}\frac{y}{\left(x^2+y^2+z^2\right)^{1/2}} + \hat{\mathbf{k}}\frac{x}{\left(x^2+y^2+z^2\right)^{1/2}}\end{aligned}$$

we have

$$\begin{aligned}\nabla\cdot(\nabla(r)) &= \frac{\partial}{\partial x}\frac{x}{\left(x^2+y^2+z^2\right)^{1/2}} + \frac{\partial}{\partial y}\frac{y}{\left(x^2+y^2+z^2\right)^{1/2}} + \frac{\partial}{\partial z}\frac{z}{\left(x^2+y^2+z^2\right)^{1/2}}\\ &= \frac{3}{\left(x^2+y^2+z^2\right)^{1/2}} - \frac{1}{2}\frac{2x^2+2y^2+2z^2}{\left(x^2+y^2+z^2\right)^{3/2}} = \frac{2}{\left(x^2+y^2+z^2\right)^{1/2}} = \frac{2}{r}\end{aligned}$$

S-8: (a) Since

$$\begin{aligned}\nabla\left(\frac{1}{r}\right) &= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\left(x^2+y^2+z^2\right)^{-1/2}\\ &= -\hat{\imath}\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}} - \hat{\jmath}\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}} - \hat{\mathbf{k}}\frac{z}{\left(x^2+y^2+z^2\right)^{3/2}}\\ &= -\hat{\imath}\frac{x}{r^3} - \hat{\jmath}\frac{y}{r^3} - \hat{\mathbf{k}}\frac{x}{r^3}\end{aligned}$$

we have $a = -3$.

(b) Since

$$\begin{aligned}\nabla\cdot(r\,\mathbf{r}) &= \frac{\partial}{\partial x}\Big[\big(x^2+y^2+z^2\big)^{1/2}x\Big] + \frac{\partial}{\partial y}\Big[\big(x^2+y^2+z^2\big)^{1/2}y\Big] + \frac{\partial}{\partial z}\Big[\big(x^2+y^2+z^2\big)^{1/2}z\Big]\\ &= 3\big(x^2+y^2+z^2\big)^{1/2} + \frac{1}{2}\frac{2x^2+2y^2+2z^2}{\big(x^2+y^2+z^2\big)^{1/2}} = 4\big(x^2+y^2+z^2\big)^{1/2} = 4r\end{aligned}$$

we have $a = 4$.

(c) Since

$$\begin{aligned}\nabla(r^3) &= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\big(x^2+y^2+z^2\big)^{3/2}\\ &= \hat{\imath}\,3x\big(x^2+y^2+z^2\big)^{1/2} + \hat{\jmath}\,3y\big(x^2+y^2+z^2\big)^{1/2} + \hat{\mathbf{k}}\,3z\big(x^2+y^2+z^2\big)^{1/2}\\ &= 3r\mathbf{r}\end{aligned}$$

we have

$$\begin{aligned}\nabla\cdot\big(\nabla(r^3)\big) &= \nabla\cdot(3r\mathbf{r}) = 3\,\nabla\cdot(r\mathbf{r}) = 3\,(4r) \qquad \text{by part (b)}\\ &= 12r\end{aligned}$$

so that $a = 12$.

S-9: (a) Since $\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(1+yz) + \frac{\partial}{\partial y}(2y+zx) + \frac{\partial}{\partial z}(3z^2+xy) = 2+6z \neq 0$, $\mathbf{F}$ fails the screening test and cannot have a vector potential.

(b) The vector field $\mathbf{A} = A_1\hat{\imath} + A_2\hat{\jmath}$ is a vector potential for $\mathbf{G}$ if and only if $\mathbf{G} = \nabla\times\mathbf{A}$, which is the case if and only if

$$\begin{aligned}-\frac{\partial A_2}{\partial z} &= yz &&\iff & A_2 &= -\frac{1}{2}yz^2 + B_2(x,y)\\ \frac{\partial A_1}{\partial z} &= zx &&\iff & A_1 &= \frac{1}{2}xz^2 + B_1(x,y)\\ \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} &= xy &&\iff & \frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y} &= xy\end{aligned}$$

There are infinitely many solutions to $\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y} = xy$. In fact B_2 is completely arbitrary. If one chooses $B_2 = 0$, then $B_1 = -\frac{1}{2}xy^2$ does the job. If one chooses $B_1 = 0$, then $B_2 = \frac{1}{2}x^2y$ does the job. Thus two solutions are $\mathbf{A} = \frac{1}{2}(z^2-y^2)x\hat{\imath} - \frac{1}{2}yz^2\hat{\jmath}$ and $\mathbf{A} = \frac{1}{2}xz^2\hat{\imath} + \frac{1}{2}(x^2-z^2)y\hat{\jmath}$.

S-10: (a) $\mathbf{F}$ is well-defined wherever the denominator x^2+z^2 is nonzero. So the (largest possible) domain is

$$D = \big\{\ (x,y,z)\ \big|\ x^2+z^2 \neq 0\ \big\}$$

(b) As preliminary computations, let's find

$$\frac{\partial}{\partial z}\left(\frac{-z}{x^2+z^2}\right) = \frac{-1}{x^2+z^2} - \frac{2z(-z)}{(x^2+z^2)^2} = \frac{-x^2+z^2}{(x^2+z^2)^2}$$
$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+z^2}\right) = \frac{1}{x^2+z^2} - \frac{2x(x)}{(x^2+z^2)^2} = \frac{-x^2+z^2}{(x^2+y^2)^2}$$

So the curl of **F** is

$$\boldsymbol{\nabla}\times\mathbf{F} = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-z}{x^2+z^2} & y & \frac{x}{x^2+z^2}\end{bmatrix} = -\left(\frac{-x^2+z^2}{(x^2+y^2)^2} - \frac{-x^2+z^2}{(x^2+y^2)^2}\right)\hat{\jmath} = \mathbf{0}$$

on the domain of **F**.

(c) As preliminary computations, let's find

$$\frac{\partial}{\partial x}\left(\frac{-z}{x^2+z^2}\right) = -\frac{2x(-z)}{(x^2+z^2)^2} = \frac{2xz}{(x^2+z^2)^2}$$
$$\frac{\partial}{\partial z}\left(\frac{x}{x^2+z^2}\right) = -\frac{2z(x)}{(x^2+z^2)^2} = \frac{-2xz}{(x^2+y^2)^2}$$

So the divergence of **F** is

$$\boldsymbol{\nabla}\cdot\mathbf{F} = \frac{\partial}{\partial x}\left(\frac{-z}{x^2+z^2}\right) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}\left(\frac{x}{x^2+z^2}\right) = 1$$

(d) By part (b), the vector field passes the conservative field screening test $\boldsymbol{\nabla}\times\mathbf{F} = \mathbf{0}$. But we should still be suspicious because of the similarity of **F** to the vector field of Examples 2.3.14 and 4.3.8 in the CLP-4 text.

So let's compute the line integral of **F** around the (closed) circle $y = 0$, $x^2+z^2 = 1$, parametrized by

$$\mathbf{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\mathbf{k}} \qquad \mathbf{r}'(t) = -\sin t\,\hat{\imath} + \cos t\,\hat{\mathbf{k}}$$

The line integral is

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \int_0^{2\pi}\Big\{\overbrace{-\sin t}^{\frac{-z}{x^2+z^2}}\,\hat{\imath} + \overbrace{\cos t}^{\frac{x}{x^2+y^2}}\,\hat{\mathbf{k}}\Big\}\cdot\Big\{\overbrace{-\sin t\,\hat{\imath} + \cos t\,\hat{\mathbf{k}}}^{\mathbf{r}'(t)}\Big\}\mathrm{d}t \\ &= \int_0^{2\pi}\mathrm{d}t = 2\pi\end{aligned}$$

As the integral of **F** around the simple closed curve C is not zero, **F** cannot be conservative on D. See Theorem 2.4.6 and Examples 2.3.14 and 4.3.8 in the CLP-4 text.

S-11: (a) By the vector identity of Theorem 4.1.7.a in the CLP-4 text,

$$\nabla \cdot \mathbf{F} = \nabla \cdot \nabla \times \mathbf{G} = 0$$

So we must have

$$0 = \nabla \cdot \mathbf{F} = \nabla \cdot \big((xz+xy)\hat{\boldsymbol{\imath}} + \alpha(yz-xy)\hat{\boldsymbol{\jmath}} + \beta(yz+xz)\hat{\mathbf{k}}\big) = (z+y) + \alpha(z-x) + \beta(y+x)$$

This is true for all (x,y,z) if and only if $\alpha = \beta = -1$.

(b) Since

$$\nabla \times \mathbf{G} = \nabla \times \big(xyz\hat{\boldsymbol{\imath}} - xyz\hat{\boldsymbol{\jmath}} + g(x,y,z)\hat{\mathbf{k}}\big) = (g_y + xy)\,\hat{\boldsymbol{\imath}} - (g_x - xy)\,hj + (-yz - xz)\,\hat{\mathbf{k}}$$

we will have that $\nabla \times \mathbf{G} = \mathbf{F}$ if and only if

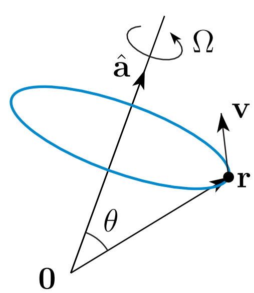
$$(g_y + xy)\,\hat{\boldsymbol{\imath}} - (g_x - xy)\,\hat{\boldsymbol{\jmath}} + (-yz - xz)\,\hat{\mathbf{k}} = (xz+xy)\,\hat{\boldsymbol{\imath}} - (yz-xy)\,\hat{\boldsymbol{\jmath}} - (yz+xz)\,\hat{\mathbf{k}}$$

which is the case if and only if

$$g_y = xz, \quad g_x = yz$$

The first equation, $g_y = xz$, is satisfied if and only if $g = xyz + h(x,z)$. The second equation is also satisfied if and only if $g_x = yz + h_x(x,z) = yz$. This is the case if and only if $h_x(x,z) = 0$. That is, if and only if h is independent of x. Equivalently, if and only if $h(x,z) = w(z)$ for some function $w(z)$. So, in fact, any function of the form $g(x,y,z) = xyz + w(z)$ will work.

S-12: (a) Denote by θ the angle between $\hat{\mathbf{a}}$ and $\mathbf{r}$. The point $\mathbf{r}$ is a distance $\ell = |\mathbf{r}|\,\sin\theta$ from the axis of rotation. So as the body rotates, the point sweeps out a circle of radius ℓ centred on the axis of rotation. In one second the point sweeps out an arc of this circle



that subtends an angle of Ω radians. This arc is the fraction $\frac{\Omega}{2\pi}$ of a full circle and so has length $\frac{\Omega}{2\pi}2\pi\ell = \Omega\ell = \Omega|\mathbf{r}|\,\sin\theta$. Thus the point is moving with speed $\Omega|\mathbf{r}|\,\sin\theta$. The velocity vector of the point must have length $\Omega|\mathbf{r}|\,\sin\theta$ and direction perpendicular to both $\hat{\mathbf{a}}$ and $\mathbf{r}$. The vector $\boldsymbol{\Omega} \times \mathbf{r}$ is perpendicular to both $\mathbf{r}$ and $\boldsymbol{\Omega} = \Omega\hat{\mathbf{a}}$ and has length $|\boldsymbol{\Omega}|\,|\mathbf{r}|\,\sin\theta = \Omega|\mathbf{r}|\,\sin\theta$ as desired. So the velocity vector is either $\boldsymbol{\Omega} \times \mathbf{r}$ or its negative. By the right hand rule it is $\boldsymbol{\Omega} \times \mathbf{r}$.

(b) By vector identities

$$\begin{aligned}
\boldsymbol{\nabla} \cdot (\mathbf{F} \times \mathbf{G}) &= \mathbf{G} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) - \mathbf{F} \cdot (\boldsymbol{\nabla} \times \mathbf{G}) \\
\boldsymbol{\nabla} \times (\mathbf{F} \times \mathbf{G}) &= \mathbf{F}(\boldsymbol{\nabla} \cdot \mathbf{G}) - (\boldsymbol{\nabla} \cdot \mathbf{F})\mathbf{G} + (\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F} - (\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{G}
\end{aligned}$$

(which are Theorems 4.1.4(d) and 4.1.5(d) in the CLP-4 text) and the assumption that $\mathbf{\Omega}$ is constant

$$\begin{aligned}
\nabla\times(\mathbf{\Omega}\times\mathbf{r}) &= \mathbf{\Omega}(\nabla\cdot\mathbf{r}) - (\nabla\cdot\mathbf{\Omega})\mathbf{r} + (\mathbf{r}\cdot\nabla)\mathbf{\Omega} - (\mathbf{\Omega}\cdot\nabla)\mathbf{r} = \mathbf{\Omega}(\nabla\cdot\mathbf{r}) - (\mathbf{\Omega}\cdot\nabla)\mathbf{r}\\
\nabla\cdot(\mathbf{\Omega}\times\mathbf{r}) &= \mathbf{r}\cdot(\nabla\times\mathbf{\Omega}) - \mathbf{\Omega}\cdot(\nabla\times\mathbf{r}) = -\mathbf{\Omega}\cdot(\nabla\times\mathbf{r})
\end{aligned}$$

Substituting in

$$\begin{aligned}
\nabla\cdot\mathbf{r} &= \frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z} = 3\\
\nabla\times\mathbf{r} &= (\frac{\partial z}{\partial y}-\frac{\partial y}{\partial z})\hat{\imath} + (\frac{\partial x}{\partial z}-\frac{\partial z}{\partial x})\hat{\jmath} + (\frac{\partial y}{\partial x}-\frac{\partial x}{\partial y})\hat{\mathbf{k}} = \mathbf{0}\\
(\mathbf{\Omega}\cdot\nabla)\mathbf{r} &= \big(\Omega_1\frac{\partial}{\partial x}+\Omega_2\frac{\partial}{\partial y}+\Omega_3\frac{\partial}{\partial z}\big)\big(x\hat{\imath}+y\hat{\jmath}+z\hat{\mathbf{k}}\big) = \Omega_1\hat{\imath}+\Omega_2\hat{\jmath}+\Omega_3\hat{\mathbf{k}} = \mathbf{\Omega}
\end{aligned}$$

gives

$$\nabla\times(\mathbf{\Omega}\times\mathbf{r}) = 2\mathbf{\Omega} \qquad \nabla\cdot(\mathbf{\Omega}\times\mathbf{r}) = 0$$

(c) The students are a distance $6378\sin(90^\circ-49^\circ) = 6378\cos(49^\circ) = 4184$ km from the axis of rotation. The rate of rotation is $\Omega = \frac{2\pi}{24}$ radians per hour. In each hour the students sweep out an arc of $\frac{2\pi}{24}$ radians from a circle of radius 4184 km. Their speed is $\frac{2\pi}{24}\times 4184 = 1095$km/hr.

S-13: We shall show that $\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} = F_1$. The other components are similar. First we have

$$t\,\mathbf{F}\big(\mathbf{r}(t)\big)\times\frac{d\mathbf{r}}{dt}(t) = t\,\mathbf{F}\big(tx,ty,tz\big)\times\big(x\,\hat{\imath}+y\,\hat{\jmath}+z\,\hat{\mathbf{k}}\big) = t\det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ F_1 & F_2 & F_3\\ x & y & z\end{bmatrix}$$

Reading off the $\hat{\mathbf{k}}$ and $\hat{\jmath}$ components of the determinant gives

$$\begin{aligned}
G_3(x,y,z) &= \int_0^1 t\big[F_1\big(tx,ty,tz\big)\,y - F_2\big(tx,ty,tz\big)\,x\big]\,dt\\
G_2(x,y,z) &= \int_0^1 t\big[F_3\big(tx,ty,tz\big)\,x - F_1\big(tx,ty,tz\big)\,z\big]\,dt
\end{aligned}$$

So

$$\begin{aligned}
\frac{\partial G_3}{\partial y} &= \int_0^1 t\Big[F_1\big(tx,ty,tz\big) + \frac{\partial F_1}{\partial y}\big(tx,ty,tz\big)\,ty - \frac{\partial F_2}{\partial y}\big(tx,ty,tz\big)\,tx\Big]\,dt\\
\frac{\partial G_2}{\partial z} &= \int_0^1 t\Big[\frac{\partial F_3}{\partial z}\big(tx,ty,tz\big)\,tx - \frac{\partial F_1}{\partial z}\big(tx,ty,tz\big)\,tz - F_1\big(tx,ty,tz\big)\Big]\,dt\\
\Rightarrow\ \frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} &= \int_0^1 \Big[2t\,F_1\big(tx,ty,tz\big) + t^2y\,\frac{\partial F_1}{\partial y}\big(tx,ty,tz\big) + t^2z\frac{\partial F_1}{\partial z}\big(tx,ty,tz\big)\\
&\qquad - t^2x\,\frac{\partial F_2}{\partial y}\big(tx,ty,tz\big) - t^2x\,\frac{\partial F_3}{\partial z}\big(tx,ty,tz\big)\Big]\,dt
\end{aligned}$$

Since, by hypothesis, $\nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0$, the last two terms

$$-t^2 x\Big\{\frac{\partial F_2}{\partial y}(tx,ty,tz) + \frac{\partial F_3}{\partial z}(tx,ty,tz)\Big\} = -t^2 x\Big\{-\frac{\partial F_1}{\partial x}(tx,ty,tz)\Big\}$$

so that

$$\begin{aligned}
&\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} \\
&\quad = \int_0^1 \Big[2tF_1(tx,ty,tz) + t^2 x\frac{\partial F_1}{\partial x}(tx,ty,tz) + t^2 y\frac{\partial F_1}{\partial y}(tx,ty,tz) + t^2 z\frac{\partial F_1}{\partial z}(tx,ty,tz)\Big]\,dt \\
&\quad = \int_0^1 \frac{d}{dt}\Big[t^2 F_1(tx,ty,tz)\Big]\,dt = \Big[t^2 F_1(tx,ty,tz)\Big]_{t=0}^{t=1} = F_1(x,y,z)
\end{aligned}$$

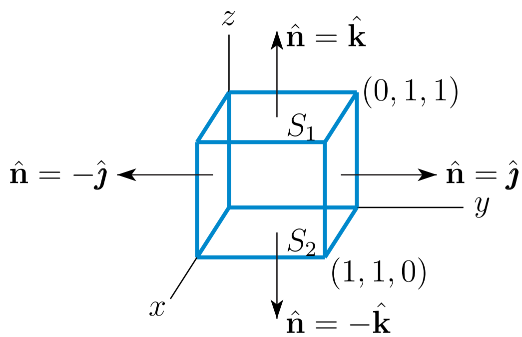
Solutions to Exercises 4.2 — Jump to TABLE OF CONTENTS

S-1: (a) Expressing the left hand side as an iterated integral, with z as the innermost integration variable, we have

$$\begin{aligned}
\iiint_V \frac{\partial f}{\partial z}(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\,\Big[\int_0^1 \mathrm{d}z\,\frac{\partial f}{\partial z}(x,y,z)\Big] \\
&= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\,\big[f(x,y,1) - f(x,y,0)\big] \\
&\qquad\qquad \text{by the fundamental theorem of calculus} \\
&= \iint_R \big[f(x,y,1) - f(x,y,0)\big]\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_R f(x,y,1)\,\mathrm{d}x\,\mathrm{d}y - \iint_R f(x,y,0)\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

(b) Define the vector field $\mathbf{F}(x,y,z) = f(x,y,z)\,\hat{\mathbf{k}}$. Then the divergence of $\mathbf{F}$ is $\nabla \cdot \mathbf{F}(x,y,z) = \frac{\partial f}{\partial z}(x,y,z)$. The boundary of the cube V is the union of six faces

$$\begin{aligned}
S_1 &= \big\{\,(x,y,z)\,\big|\,0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1,\ z = 1\,\big\} && \text{with outward normal } \hat{\mathbf{n}} = \hat{\mathbf{k}} \\
S_2 &= \big\{\,(x,y,z)\,\big|\,0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1,\ z = 0\,\big\} && \text{with outward normal } \hat{\mathbf{n}} = -\hat{\mathbf{k}} \\
S_3 &= \big\{\,(x,y,z)\,\big|\,0 \leqslant x \leqslant 1,\ 0 \leqslant z \leqslant 1,\ y = 1\,\big\} && \text{with outward normal } \hat{\mathbf{n}} = \hat{\boldsymbol{\jmath}} \\
S_4 &= \big\{\,(x,y,z)\,\big|\,0 \leqslant x \leqslant 1,\ 0 \leqslant z \leqslant 1,\ y = 0\,\big\} && \text{with outward normal } \hat{\mathbf{n}} = -\hat{\boldsymbol{\jmath}} \\
S_5 &= \big\{\,(x,y,z)\,\big|\,0 \leqslant y \leqslant 1,\ 0 \leqslant z \leqslant 1,\ x = 1\,\big\} && \text{with outward normal } \hat{\mathbf{n}} = \hat{\boldsymbol{\imath}} \\
S_6 &= \big\{\,(x,y,z)\,\big|\,0 \leqslant y \leqslant 1,\ 0 \leqslant z \leqslant 1,\ x = 0\,\big\} && \text{with outward normal } \hat{\mathbf{n}} = -\hat{\boldsymbol{\imath}}
\end{aligned}$$



Observe that

$$\mathbf{F}\cdot\hat{\mathbf{n}} = f\,\hat{\mathbf{k}}\cdot\hat{\mathbf{n}} = \begin{cases} +f & \text{on } S_1 \\ -f & \text{on } S_2 \\ 0 & \text{on } S_3,\ S_4,\ S_5,\ S_6 \end{cases}$$

So the divergence theorem gives

$$\begin{aligned}\iiint_V \frac{\partial f}{\partial z}(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \iint_{\partial V} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \sum_{j=1}^{6}\iint_{S_j} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S \\ &= \iint_{S_1} f\,\mathrm{d}S - \iint_{S_2} f\,\mathrm{d}S \\ &= \iint_R f(x,y,1)\,\mathrm{d}x\,\mathrm{d}y - \iint_R f(x,y,0)\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

S-2: (a) The divergence of $\phi\,\mathbf{a}$ is $\boldsymbol{\nabla}\cdot(\phi\,\mathbf{a}) = \boldsymbol{\nabla}\phi\cdot\mathbf{a} + \phi\boldsymbol{\nabla}\cdot\mathbf{a} = \boldsymbol{\nabla}\phi\cdot\mathbf{a}$, since $\mathbf{a}$ is constant. So, by the divergence theorem,

$$\iint_{\partial V} \phi\,\mathbf{a}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint_V \boldsymbol{\nabla}\cdot(\phi\,\mathbf{a})\,\mathrm{d}V = \iiint_V \boldsymbol{\nabla}\phi\cdot\mathbf{a}\,\mathrm{d}V \implies \left[\iint_{\partial V}\phi\hat{\mathbf{n}}\,\mathrm{d}S - \iiint_V \boldsymbol{\nabla}\phi\,\mathrm{d}V\right]\cdot\mathbf{a} = 0$$

This is true for all vectors $\mathbf{a}$. In particular, applying this with $\mathbf{a} = \hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\mathbf{k}}$, we have that all three components of $\left[\iint_{\partial V}\phi\hat{\mathbf{n}}\,\mathrm{d}S - \iiint_V \boldsymbol{\nabla}\phi\,\mathrm{d}V\right]$ are zero. So

$$\iint_{\partial V}\phi\,\hat{\mathbf{n}}\,\mathrm{d}S - \iiint_V \boldsymbol{\nabla}\phi\,\mathrm{d}V = 0$$

(b) By part (a), with $\phi = x^2+y^2+z^2$ and $\boldsymbol{\nabla}\phi = 2x\,\hat{\boldsymbol{\imath}} + 2y\,\hat{\boldsymbol{\jmath}} + 2x\,\hat{\mathbf{k}}$,

$$\frac{1}{2|V|}\iint_{\partial V}(x^2+y^2+z^2)\,\hat{\mathbf{n}}\,\mathrm{d}S = \frac{1}{2|V|}\iiint_V (2x\,\hat{\boldsymbol{\imath}} + 2y\,\hat{\boldsymbol{\jmath}} + 2z\,\hat{\mathbf{k}})\,\mathrm{d}V = (\bar{x},\bar{y},\bar{z})$$

S-3: (a) We'll parametrize the sphere using the spherical coordinates θ and φ.

$$\begin{aligned} x &= \sin\varphi\cos\theta \\ y &= \sin\varphi\sin\theta \\ z &= \cos\varphi \end{aligned}$$

with $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant \varphi \leqslant \pi$. Since

$$\begin{aligned} \Big(\frac{\partial x}{\partial\theta}, \frac{\partial y}{\partial\theta}, \frac{\partial z}{\partial\theta}\Big) &= \big(-\sin\varphi\sin\theta\,,\ \sin\varphi\cos\theta\,,\ 0\big) \\ \Big(\frac{\partial x}{\partial\varphi}, \frac{\partial y}{\partial\varphi}, \frac{\partial z}{\partial\varphi}\Big) &= (\cos\varphi\cos\theta\,,\ \cos\varphi\sin\theta\,,\ -\sin\varphi) \end{aligned}$$

(3.3.1) in the CLP-4 text yields

$$\begin{aligned} \hat{\mathbf{n}}\,\mathrm{d}S &= \pm\Big(\frac{\partial x}{\partial\theta}, \frac{\partial y}{\partial\theta}, \frac{\partial z}{\partial\theta}\Big) \times \Big(\frac{\partial x}{\partial\varphi}, \frac{\partial y}{\partial\varphi}, \frac{\partial z}{\partial\varphi}\Big)\,\mathrm{d}\theta\mathrm{d}\varphi \\ &= \pm\big(-\sin\varphi\sin\theta\,,\ \sin\varphi\cos\theta\,,\ 0\big) \times (\cos\varphi\cos\theta\,,\ \cos\varphi\sin\theta\,,\ -\sin\varphi)\,\mathrm{d}\theta\mathrm{d}\varphi \\ &= \pm\big(-\sin^2\varphi\cos\theta\,,\ -\sin^2\varphi\sin\theta\,,\ -\sin\varphi\cos\varphi\big)\,\mathrm{d}\theta\mathrm{d}\varphi \\ &= \mp\sin\varphi\big(\sin\varphi\cos\theta\,,\ \sin\varphi\sin\theta\,,\ \cos\varphi\big)\,\mathrm{d}\theta\mathrm{d}\varphi \\ &= \mp\sin\varphi\big(x(\theta,\varphi)\,,\ y(\theta,\varphi)\,,\ z(\theta,\varphi)\big)\,\mathrm{d}\theta\mathrm{d}\varphi \end{aligned}$$

To get an outward pointing normal we need the $+$ sign, since then $\hat{\mathbf{n}}(\theta,\varphi)$ is a positive multiple, namely $\sin\varphi$, times $\mathbf{r}(\theta,\varphi)$. So, on S,

$$\begin{aligned} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \sin\varphi\,\overbrace{\big(\sin\varphi\cos\theta\,,\ \sin\varphi\sin\theta\,,\ \cos^2\varphi\big)}^{\mathbf{F}}\cdot\big(\sin\varphi\cos\theta\,,\ \sin\varphi\sin\theta\,,\ \cos\varphi\big)\,\mathrm{d}\theta\mathrm{d}\varphi \\ &= \sin\varphi\big(\sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta + \cos^3\varphi\big) \end{aligned}$$

and

$$\begin{aligned} \iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \int_0^\pi \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta\ \sin\varphi\big(\sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta + \cos^3\varphi\big) \\ &= \int_0^\pi \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta\ \big(\sin^3\varphi + \sin\varphi\cos^3\varphi\big) \\ &= 2\pi\left\{\int_0^\pi \mathrm{d}\varphi\ \sin^3\varphi\ +\ \left[-\frac{1}{4}\cos^4\varphi\right]_0^\pi\right\} \\ &= 2\pi\int_0^\pi \mathrm{d}\varphi\ \sin\varphi(1-\cos^2\varphi) = 2\pi\left[-\cos\varphi + \frac{1}{3}\cos^3\varphi\right]_0^\pi = 2\pi\left[\frac{4}{3}\right] \\ &= \frac{8\pi}{3} \end{aligned}$$

(b) Let V be the interior of S. Then, by the divergence theorem,

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V = \iiint_V (1+1+2z)\,\mathrm{d}V$$

By oddness under $z \to -z$, the z integral vanishes, so that

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = 2 \iiint_V \mathrm{d}V = 2 \, \text{Volume}(V) = 2\frac{4\pi}{3} = \frac{8\pi}{3}$$

S-4: (a) Let's use spherical coordinates. As S is the sphere of radius a centred on the origin, we can parametrize it by

$$\begin{aligned}
\mathbf{r}(\theta, \varphi) &= a \sin\varphi \cos\theta \, \hat{\boldsymbol{\imath}} + a \sin\varphi \sin\theta \, \hat{\boldsymbol{\jmath}} + a \cos\varphi \, \hat{\mathbf{k}} \\
\frac{\partial \mathbf{r}}{\partial \theta} &= -a \sin\varphi \sin\theta \, \hat{\boldsymbol{\imath}} + a \sin\varphi \cos\theta \, \hat{\boldsymbol{\jmath}} \\
\frac{\partial \mathbf{r}}{\partial \varphi} &= \quad a \cos\varphi \cos\theta \, \hat{\boldsymbol{\imath}} + a \cos\varphi \sin\theta \, \hat{\boldsymbol{\jmath}} - a \sin\varphi \, \hat{\mathbf{k}} \\
\hat{\mathbf{n}} \, \mathrm{d}S &= \pm \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi} \, \mathrm{d}\theta \, d\varphi \\
&= \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ -a \sin\varphi \sin\theta & a \sin\varphi \cos\theta & 0 \\ a \cos\varphi \cos\theta & a \cos\varphi \sin\theta & -a \sin\varphi \end{bmatrix} \mathrm{d}\theta \, d\varphi \\
&= \pm \Big(- a^2 \sin^2\varphi \cos\theta \, \hat{\boldsymbol{\imath}} - a^2 \sin^2\varphi \sin\theta \, \hat{\boldsymbol{\jmath}} - a^2 \sin\varphi \cos\varphi \, \hat{\mathbf{k}} \Big) \, \mathrm{d}\theta \, d\varphi \\
&= \mp a^2 \sin\varphi \big(\sin\varphi \cos\theta \, \hat{\boldsymbol{\imath}} + \sin\varphi \sin\theta \, \hat{\boldsymbol{\jmath}} + \cos\varphi \, \hat{\mathbf{k}} \Big) \, \mathrm{d}\theta \, d\varphi
\end{aligned}$$

For the outward normal, we want the $+$ sign, so

$$\begin{aligned}
\hat{\mathbf{n}} \, \mathrm{d}S &= a^2 \sin\varphi \big(\sin\varphi \cos\theta \, \hat{\boldsymbol{\imath}} + \sin\varphi \sin\theta \, \hat{\boldsymbol{\jmath}} + \cos\varphi \, \hat{\mathbf{k}} \Big) \, \mathrm{d}\theta \, d\varphi \\
\mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S &= z(\theta, \varphi) \, \hat{\mathbf{k}} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = a^3 \sin\varphi \cos^2\varphi \, \mathrm{d}\theta \, d\varphi
\end{aligned}$$

and

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S &= a^3 \int_0^{\pi} d\varphi \int_0^{2\pi} \mathrm{d}\theta \ \sin\varphi \ \cos^2\varphi \\
&= 2\pi a^3 \int_0^{\pi} d\varphi \ \sin\varphi \ \cos^2\varphi = 2\pi a^3 \left[-\frac{1}{3} \cos^3\varphi \right]_0^{\pi} \\
&= \frac{4}{3} \pi a^3
\end{aligned}$$

(b) Call the solid $x^2 + y^2 + z^2 \leqslant a^2$, V. As

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(z) = 1$$

the divergence theorem gives

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = \iiint_V \boldsymbol{\nabla} \cdot \mathbf{F} \, \mathrm{d}V = \iiint_V \mathrm{d}V = \text{Volume}(V) = \frac{4}{3} \pi a^3$$

S-5: (a) On D, $z=0$ and

$$\hat{\mathbf{n}}=-\hat{\mathbf{k}}\qquad \mathrm{d}S=\mathrm{d}x\,\mathrm{d}y\qquad \mathbf{F}\cdot\hat{\mathbf{n}}=-y^2$$

so that

$$\iint_D \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S=-\iint_D y^2\,\mathrm{d}x\,\mathrm{d}y$$

Switching to polar coordinates

$$\begin{aligned}\iint_D \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S&=-\int_0^3 dr\,r\int_0^{2\pi}\mathrm{d}\theta\,\big(r\sin\theta\big)^2=-\Big[\int_0^3 dr\,r^3\Big]\Big[\int_0^{2\pi}\mathrm{d}\theta\ \sin^2\theta\Big]\\&=-\frac{3^4}{4}\Big[\int_0^{2\pi}\mathrm{d}\theta\,\frac{1-\cos(2\theta)}{2}\Big]=-\frac{81}{4}\Big[\frac{\theta}{2}-\frac{\sin(2\theta)}{4}\Big]_0^{2\pi}=-\frac{81}{4}\pi\end{aligned}$$

For an efficient, sneaky, way to evaluate $\int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta$, see Example 2.4.4 in the CLP-4 text.

(b) Observe that $\boldsymbol{\nabla}\cdot\mathbf{F}=x+2$. Since x is odd and V is invariant under $x\to -x$, we have $\iiint_V x\,\mathrm{d}V=0$ (more details below) so that

$$\iiint_V \boldsymbol{\nabla}\cdot F\,\mathrm{d}V=\iiint_V (x+2)\,\mathrm{d}V=2\iiint_V \mathrm{d}V=2|V|$$

Here are two more detailed arguments showing that $\iiint_V x\,\mathrm{d}V=0$.

Argument 1: We may rewrite the equation $z=\frac{9-x^2-y^2}{9+x^2+y^2}$ of the curved boundary of V as

$$z(9+x^2+y^2)=9-x^2-y^2\iff x^2+y^2=\frac{9(1-z)}{1+z}$$

This is the equation of the circle of radius $r(z)=\sqrt{\frac{9(1-z)}{1+z}}$ centred on $x=y=0$. So z runs from 0 to 1, and for each fixed $0\leqslant z\leqslant 1$, y runs from $-r(z)$ to $r(z)$ and, for each fixed y and z, x runs from $-\sqrt{r(z)^2-y^2}$ to $\sqrt{r(z)^2-y^2}$. So

$$\iiint_V x\,\mathrm{d}V=\int_0^1\mathrm{d}z\int_{-r(z)}^{r(z)}\mathrm{d}y\int_{-\sqrt{r(z)^2-y^2}}^{\sqrt{r(z)^2-y^2}}\mathrm{d}x\,x=\int_0^1\mathrm{d}z\int_{-r(z)}^{r(z)}\mathrm{d}y\,0=0$$

since $\int_{-a}^{a}x\,\mathrm{d}x=0$ for any $a>0$.

Argument 2: As we have observed above, the curved boundary of V is $x^2+y^2=\frac{9(1-z)}{1+z}$ which is invariant under rotations about the z–axis. By that symmetry, the centroid of V lies on the z-axis. Recall that, for any solid V, the centroid of V is $(\bar{x},\bar{y},\bar{z})$ with

$$\bar{x}=\frac{\iiint_V x\mathrm{d}V}{\iiint_V \mathrm{d}V}\qquad \bar{y}=\frac{\iiint_V y\mathrm{d}V}{\iiint_V \mathrm{d}V}\qquad \bar{z}=\frac{\iiint_V z\mathrm{d}V}{\iiint_V \mathrm{d}V}$$

So

$$\iiint_V x\,\mathrm{d}V = \bar{x}\,\mathrm{Volume}(V) = 0 \qquad \text{and} \qquad \iiint_V y\,\mathrm{d}V = \bar{y}\,\mathrm{Volume}(V) = 0$$

(c) By the divergence theorem,

$$\iiint_V \boldsymbol{\nabla}\cdot F\,\mathrm{d}V = \iint_{\partial V} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \iint_D \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$$

so that

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint_V \boldsymbol{\nabla}\cdot F\,\mathrm{d}V - \iint_D \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 2|V| + \frac{81}{4}\pi$$

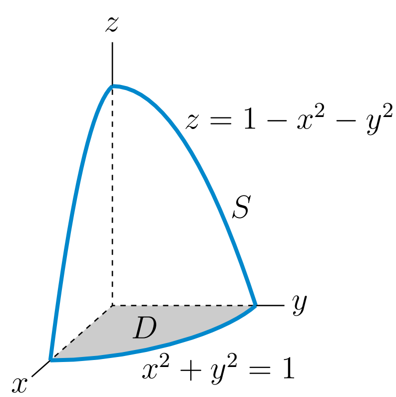
S-6: (a) Let $G(x,y,z) = x^2+y^2+z$. Then the surface is $G(x,y,z)=1$ and $\boldsymbol{\nabla}G(x,y,z) = 2x\,\hat{\imath} + 2y\,\hat{\jmath} + \hat{\mathbf{k}}$ so that, by (3.3.3) in the CLP-4 text,

$$\begin{aligned}\hat{\mathbf{n}}\,\mathrm{d}S &= \frac{\boldsymbol{\nabla}G}{\boldsymbol{\nabla}G\cdot\hat{\mathbf{k}}}\,\mathrm{d}x\,\mathrm{d}y = \frac{2x\,\hat{\imath}+2y\,\hat{\jmath}+\hat{\mathbf{k}}}{1}\,\mathrm{d}x\,\mathrm{d}y = (2x\,\hat{\imath}+2y\,\hat{\jmath}+\hat{\mathbf{k}})\,\mathrm{d}x\,\mathrm{d}y\\ \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \big[x\,\hat{\imath}+y\,\hat{\jmath}+\hat{\mathbf{k}}\big]\cdot\big[2x\,\hat{\imath}+2y\,\hat{\jmath}+\hat{\mathbf{k}}\big]\,\mathrm{d}x\,\mathrm{d}y = \big[2x^2+2y^2+1\big]\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

Switching to polar coordinates

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_0^1 dr\; r\int_0^{2\pi}\mathrm{d}\theta\;(2r^2+1) = 2\pi\left[\frac{2}{4}r^4+\frac{1}{2}r^2\right]_0^1 = 2\pi$$

(b) Call the solid $0 \leqslant z \leqslant 1-x^2-y^2$, V.



Let D denote the bottom surface of V. The disk D has radius 1, area π, $z=0$ and outward normal $-\hat{\mathbf{k}}$, so that

$$\iint_D \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = -\iint_D \mathbf{F}\cdot\hat{\mathbf{k}}\,\mathrm{d}x\,\mathrm{d}y = -\iint_D \mathrm{d}x\,\mathrm{d}y = -\pi$$

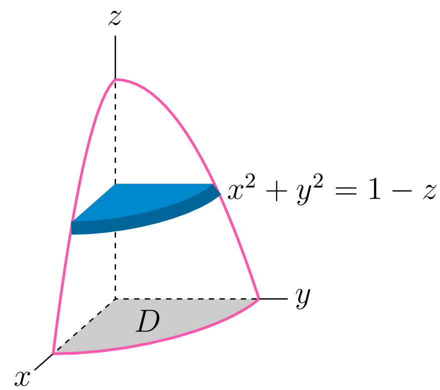
As

$$\boldsymbol{\nabla}\cdot\mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(1) = 2$$

the divergence theorem gives

$$\iint_{\mathcal{S}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V - \iint_D \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint_V 2\,\mathrm{d}V - (-\pi) = \pi + 2\iiint_V \mathrm{d}V$$

To evaluate the volume $\iiint_V \mathrm{d}V$, we slice the V into thin horizontal pancakes. Here is a sketch of the pancake at height z.



Its cross-section is a circular disk of radius $\sqrt{1-z}$, and hence of area $\pi(1-z)$. As the pancake has thickness $\mathrm{d}z$, it has volume $\pi(1-z)\,\mathrm{d}z$. So

$$\begin{aligned}\iint_{\mathcal{S}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \pi + 2\int_0^1 \mathrm{d}z \iint_{x^2+y^2\leqslant 1-z} \mathrm{d}x\,\mathrm{d}y = \pi + 2\int_0^1 \mathrm{d}z\ \pi(1-z)\\ &= \pi + 2\pi\left[z - \frac{1}{2}z^2\right]_0^1 = 2\pi\end{aligned}$$

S-7: (a) The divergence is

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{F} &= \frac{\partial}{\partial x}(z+\sin y) + \frac{\partial}{\partial y}(zy) + \frac{\partial}{\partial z}(\sin x\cos y)\\ &= z\end{aligned}$$

(b) Let

$$V = \{\ (x,y,z)\ |\ x^2+y^2+z^2 \leqslant 9\ \}$$

By the divergence theorem (assuming that we are to find the *outward* flux),

$$\iint_{\partial V} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V = \iiint_V z\,\mathrm{d}V = 0$$

since z is odd.

S-8: Call the silo V. Call the sides and top of the silo S. Call the base of the silo (namely, $x^2+y^2\leqslant 1, z=0$) B. By the divergence theorem,

$$\iint_S \mathbf{V}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \iint_B \mathbf{V}\cdot(-\hat{\mathbf{k}})\,\mathrm{d}S = \iiint_V \boldsymbol{\nabla}\cdot\mathbf{V}\,\mathrm{d}V$$

$$\iint_S \mathbf{V}\cdot\hat{\mathbf{n}}\,\mathrm{d}S - \iint_{x^2+y^2\leqslant 1} (x^2+y)\,\mathrm{d}x\,\mathrm{d}y = \iiint_V (2xyz+z)\,\mathrm{d}V$$

By oddness under $Y\rightarrow -y$, $\iint_{x^2+y^2\leqslant 1} y\,\mathrm{d}x\,\mathrm{d}y = \iiint_V xyz\,\mathrm{d}V = 0$, so

$$\begin{aligned}\iint_S \mathbf{V}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iint_{x^2+y^2\leqslant 1} x^2\,\mathrm{d}x\,\mathrm{d}y + \iiint_V z\,\mathrm{d}V \\ &= \int_0^1 \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\; r\,(\overbrace{r\cos\theta}^{x})^2 + \iiint_V z\,\mathrm{d}V\end{aligned}$$

We can evaluate the volume integral by decomposing V into thin horizontal pancakes. See Section 1.6 in the CLP-2 text. For $0\leqslant z\leqslant 1$, the horizontal cross-section of the silo at height z is a circle of radius 1 and hence of area π. For $z\geqslant 1$, the horizontal cross-section of the silo at height z is again a circle. Its radius is determined by the equation $x^2+y^2+z^2=2$ of the top of the silo. The radius is $\sqrt{2-z^2}$, so the cross-section has area $\pi(2-z^2)$. The biggest that z can get is $\sqrt{2}$. Thus

$$\begin{aligned}\iint_S \mathbf{V}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \int_0^1 \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\; r\,(r\cos\theta)^2 + \int_0^1 dz\;\pi z + \int_1^{\sqrt{2}} dz\;\pi(2-z^2)z \\ &= \left[\int_0^1 \mathrm{d}r\; r^3\right]\left[\int_0^{2\pi} \mathrm{d}\theta\;\frac{\cos(2\theta)+1}{2}\right] + \int_0^1 dz\;\pi z + \int_1^{\sqrt{2}} dz\;\pi(2-z^2)z \\ &= \left[\frac{r^4}{4}\right]_0^1 \pi + \left[\pi\frac{z^2}{2}\right]_0^1 + \pi\left[z^2-\frac{z^4}{4}\right]_1^{\sqrt{2}} \\ &= \frac{\pi}{4}+\frac{\pi}{2}+\pi\left[1-\frac{3}{4}\right] \\ &= \pi\end{aligned}$$

For an efficient, sneaky, way to evaluate $\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta$, see Example 2.4.4 in the CLP-4 text.

S-9: Apply the divergence theorem. The divergence of $\mathbf{F}$ is

$$\boldsymbol{\nabla}\cdot\mathbf{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(3z-yz) = 3+3x-y$$

So

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint_B \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V = \iiint_B (3+3x-y)\,\mathrm{d}V$$

To evaluate the integrals of x and y we use that, for any solid $\mathcal{V}$ in $\mathbb{R}^3$,

$$\iiint_{\mathcal{V}} \mathrm{d}V = \text{Volume}(\mathcal{V}) \qquad \bar{x} = \frac{\iiint_{\mathcal{V}} x\,\mathrm{d}V}{\text{Volume}(\mathcal{V})} \qquad \bar{y} = \frac{\iiint_{\mathcal{V}} y\,\mathrm{d}V}{\text{Volume}(\mathcal{V})} \qquad \bar{z} = \frac{\iiint_{\mathcal{V}} z\,\mathrm{d}V}{\text{Volume}(\mathcal{V})}$$

where $(\bar{x},\bar{y},\bar{z})$ is the centroid of $\mathcal{V}$. Our ball has volume V and centroid $(\bar{x},\bar{y},\bar{z}) = (x_0,y_0,z_0)$. So

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = V[3+3\bar{x}-\bar{y}] = [3+3x_0-y_0]\,V$$

S-10: Let

$$V = \left\{\ (x,y,z)\ \middle|\ x^2+y^2 \leqslant 1-z^4,\ 0\leqslant z\leqslant 1\ \right\}$$

Then the boundary, ∂V, of V, with the orientation that is used in the divergence theorem, consists of two parts

- the surface S, but with the upward pointing normal, and
- the disk $D = \left\{\ (x,y,z)\ \middle|\ x^2+y^2\leqslant 1,\ z=0\ \right\}$, with normal $-\hat{\mathbf{k}}$.

So the divergence theorem gives

$$\iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V = \iint_{\partial V} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = -\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \iint_D \mathbf{F}\cdot(-\hat{\mathbf{k}})\,\mathrm{d}S$$

As $\boldsymbol{\nabla}\cdot\mathbf{F} = 0$ and $\mathbf{F}(x,y,0) = (1\,,\,1\,,\,1)$

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_D \mathbf{F}\cdot(-\hat{\mathbf{k}})\,\mathrm{d}S = -\iint_D \mathrm{d}S = -\pi$$

S-11: Let V be the solid $x^2+y^2+2z^2\leqslant 2, z\geqslant 0$. The surface of V consists of the half-ellipsoid $S = \left\{\ (x,y,z)\ \middle|\ x^2+y^2+2z^2 = 2,\ z\geqslant 0\ \right\}$, on top with upward pointing normal, and the disk $D = \{(x,y,z)\}\,z=0,\ x^2+y^2\leqslant 2$, on the bottom with normal $-\hat{\mathbf{k}}$. Call the vector field $\mathbf{F}$. By the divergence theorem

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \iint_D \mathbf{F}\cdot(-\hat{\mathbf{k}})\,\mathrm{d}S = \iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V = \iiint_V 4\,\mathrm{d}V$$

The ellipsoid has $a = \sqrt{2}$, $b = \sqrt{2}$, $c = 1$ and volume $\frac{4}{3}\pi abc = \frac{8}{3}\pi$. So

$$\iiint_V 4\,\mathrm{d}V = 4 \times \tfrac{1}{2}(\text{Volume of the ellipsoid}) = \frac{16\pi}{3}$$

On D, $z = 0$ and $\iint_D x\,\mathrm{d}S = \iint_D y\,\mathrm{d}S = 0$ because x and y are odd. So

$$\iint_D \mathbf{F}\cdot(-\hat{\mathbf{k}})\,\mathrm{d}S = \iint_D (x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + 0\hat{\mathbf{k}})\cdot(-\hat{\mathbf{k}})\,\mathrm{d}S = 0$$

and the desired flux is

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint_V 4\,\mathrm{d}V = \frac{16}{3}\pi$$

S-12: (a) If $(x, y, z) \neq \mathbf{0}$,

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{F}(x,y,z) &= \frac{\partial}{\partial x}\frac{x}{\left[x^2+y^2+z^2\right]^{3/2}} + \frac{\partial}{\partial y}\frac{y}{\left[x^2+y^2+z^2\right]^{3/2}} + \frac{\partial}{\partial z}\frac{z}{\left[x^2+y^2+z^2\right]^{3/2}} \\
&= \frac{\left[x^2+y^2+z^2\right] - x\frac{3}{2}(2x)}{\left[x^2+y^2+z^2\right]^{5/2}} + \frac{\left[x^2+y^2+z^2\right] - y\frac{3}{2}(2y)}{\left[x^2+y^2+z^2\right]^{5/2}} + \frac{\left[x^2+y^2+z^2\right] - z\frac{3}{2}(2z)}{\left[x^2+y^2+z^2\right]^{5/2}} \\
&= \frac{3\left[x^2+y^2+z^2\right] - 3x^2 - 3y^2 - 3z^2}{\left[x^2+y^2+z^2\right]^{5/2}} = 0
\end{aligned}$$

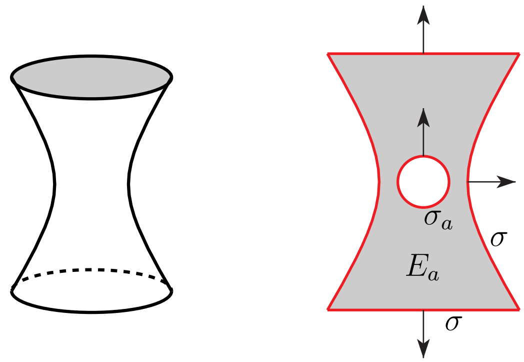
If $(x, y, z) = \mathbf{0}$, $\mathbf{F}(x, y, z)$ is not defined and hence $\boldsymbol{\nabla}\cdot\mathbf{F}(x, y, z)$ is also not defined.

(b) Let $a > 0$. Write $\sigma_a = \left\{ (x,y,z) \mid x^2+y^2+z^2 = a^2 \right\}$. The outward unit normal to σ_a is $\hat{\mathbf{n}} = \frac{\mathbf{r}}{|\mathbf{r}|}$ so that

$$\begin{aligned}
\int_{\sigma_a} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \int_{|\mathbf{r}|=a} \frac{\mathbf{r}}{|\mathbf{r}|^3}\cdot\frac{\mathbf{r}}{|\mathbf{r}|}\,\mathrm{d}S = \int_{|\mathbf{r}|=a} \frac{1}{|\mathbf{r}|^2}\,\mathrm{d}S = \frac{1}{a^2}\int_{|\mathbf{r}|=a} \mathrm{d}S = \frac{1}{a^2}\left(4\pi a^2\right) \\
&= 4\pi \neq 0
\end{aligned}$$

(c) No, the results of (a) and (b) do not contradict the divergence theorem. One hypothesis of the divergence theorem is that $\boldsymbol{\nabla}\cdot\mathbf{F}$ (in fact all first order derivatives of $\mathbf{F}$) be defined and continuous throughout the solid that $\boldsymbol{\nabla}\cdot\mathbf{F}$ is to be integrated over. That hypothesis is violated in this case.

(d) Let's first figure out what the surface $z^2 - x^2 - y^2 + 1 = 0$, i.e. the surface $x^2 + y^2 = 1 + z^2$, looks like. For each z_0, the $z = z_0$ cross-section of this surface is the circle $x^2 + y^2 = 1 + z_0^2$. The radius of this circle is 1 when $z_0 = 0$ and grows as $|z_0|$ increases. So the solid region E looks like an hourglass drum, as sketched in the figure on the left below.



We are going to use the divergence theorem to compute the flux of $\mathbf{F}$ out through the surface σ of E. However we cannot apply the divergence theorem using E as the solid, because $\mathbf{F}$ is not defined at the origin, $(0,0,0)$, which is a point in E. So we pick any $0 < a < 1$, and define the auxiliary solid

$$E_a = \left\{ (x,y,z) \,\middle|\, x^2+y^2+z^2 \geqslant a^2,\ x^2+y^2 \leqslant 1+z^2,\ -1 \leqslant z \leqslant 1 \right\}$$

The solid E_a is constructed from the solid E by removing the ball $x^2+y^2+z^2 \leqslant a^2$ from it. A side view of E_a is sketched in the figure on the right above. As in part (b), denote by σ_a the surface $x^2+y^2+z^2 = a^2$ with *outward* pointing normal. Then the boundary of E_a is $\partial E_a = \sigma - \sigma_a$, meaning that it consists of two parts. One part is the boundary, σ, of E, with outward pointing normal. The other part is the surface $x^2+y^2+z^2 = a^2$, but with normal pointing into the sphere, opposite to the normals for σ_a. Consequently the divergence theorem gives

$$0 = \iiint_{E_a} \boldsymbol{\nabla} \cdot \mathbf{F}\, \mathrm{d}V = \iint_{\partial E_a} \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iint_{\sigma} \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S - \iint_{\sigma_a} \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$$

so that, by part (b)

$$\iint_{\sigma} \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iint_{\sigma_a} \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = 4\pi$$

(e) The equation $z^2 - x^2 - y^2 + 4y - 3 = 0$ can be rewritten as

$$x^2 + (y-2)^2 = 1 + z^2$$

As is part (d), for each z_0, the $z = z_0$ cross-section of this surface is a circle $x^2 + (y-2)^2 = 1 + z_0^2$ of radius $\sqrt{1+z_0^2}$. But this circle is centred at $(0,2,z_0)$, whereas the corresponding circle in part (d) was centred at $(0,0,z_0)$. The solid R again has the shape of an hourglass drum. But while the origin $(0,0,0)$ was in E, it is *not* in

$$R = \left\{ (x,y,z) \,\middle|\, x^2 + (y-2)^2 \leqslant 1 + z^2,\ -1 \leqslant z \leqslant 1 \right\}$$

So $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$ *throughout all* of R and the divergence theorem gives

$$\iint_{\Sigma} \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iint_{\partial R} \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iiint_{R} \boldsymbol{\nabla} \cdot \mathbf{F}\, \mathrm{d}V = 0$$

S-13: (a) If the surface were the sphere $x^2 + y^2 + z^2 = 1$, we could parametrize it using the spherical coordinates θ and φ (with the radial spherical coordinate $\rho = 1$).

$$\begin{aligned} x &= \sin\varphi\cos\theta \\ y &= \sin\varphi\sin\theta \\ z &= \cos\varphi \end{aligned}$$

with $0 \leqslant \theta < 2\pi$, $0 \leqslant \varphi \leqslant \pi$. Our surface is not a sphere, but the equation looks like the equation of the sphere with the units of the y- and z-coordinates changed. In particular, if we define $\tilde{y} = y/2$ and $\tilde{z} = z/2$, so that $y = 2\tilde{y}$ and $z = 2\tilde{z}$, then on our surface

$$1 = x^2 + \frac{y^2}{4} + \frac{z^2}{4} = x^2 + \frac{(2\tilde{y})^2}{4} + \frac{(2\tilde{z})^2}{4} = x^2 + \tilde{y}^2 + \tilde{z}^2$$

and we can parametrize

$$\begin{aligned} x &= \sin\varphi\cos\theta \\ \tilde{y} &= \sin\varphi\sin\theta \\ \tilde{z} &= \cos\varphi \end{aligned}$$

and then

$$\begin{aligned} x &= \sin\varphi\cos\theta \\ y &= 2\tilde{y} = 2\sin\varphi\sin\theta \\ z &= 2\tilde{z} = 2\cos\varphi \end{aligned}$$

or

$$\mathbf{r}(\theta, \varphi) = \sin\varphi\cos\theta\,\hat{\boldsymbol{\imath}} + 2\sin\varphi\sin\theta\,\hat{\boldsymbol{\jmath}} + 2\cos\varphi\,\hat{\mathbf{k}} \qquad 0 \leqslant \theta < 2\pi,\ 0 \leqslant \varphi \leqslant \pi$$

(b) Considering part (c) in this question, we are presumably to evaluate the flux integral directly. Since

$$\begin{aligned} \left(\frac{\partial x}{\partial\theta}, \frac{\partial y}{\partial\theta}, \frac{\partial z}{\partial\theta}\right) &= \big(-\sin\varphi\sin\theta\,,\ 2\sin\varphi\cos\theta\,,\ 0\big) \\ \left(\frac{\partial x}{\partial\varphi}, \frac{\partial y}{\partial\varphi}, \frac{\partial z}{\partial\varphi}\right) &= (\cos\varphi\cos\theta\,,\ 2\cos\varphi\sin\theta\,,\ -2\sin\varphi) \end{aligned}$$

(3.3.1) in the CLP-4 text yields

$$\begin{aligned} \hat{\mathbf{n}}\,\mathrm{d}S &= \pm\left(\frac{\partial x}{\partial\theta}, \frac{\partial y}{\partial\theta}, \frac{\partial z}{\partial\theta}\right) \times \left(\frac{\partial x}{\partial\varphi}, \frac{\partial y}{\partial\varphi}, \frac{\partial z}{\partial\varphi}\right) \mathrm{d}\theta\mathrm{d}\varphi \\ &= \pm\big(-\sin\varphi\sin\theta\,,\ 2\sin\varphi\cos\theta\,,\ 0\big) \times (\cos\varphi\cos\theta\,,\ 2\cos\varphi\sin\theta\,,\ -2\sin\varphi)\,\mathrm{d}\theta\mathrm{d}\varphi \\ &= \pm\Big(-4\sin^2\varphi\cos\theta\,,\ -2\sin^2\varphi\sin\theta\,,\ -2\sin\varphi\cos\varphi\Big)\,\mathrm{d}\theta\mathrm{d}\varphi \\ &= \mp 2\sin\varphi\Big(2\sin\varphi\cos\theta\,,\ \sin\varphi\sin\theta\,,\ \cos\varphi\Big)\,\mathrm{d}\theta\mathrm{d}\varphi \end{aligned}$$

To get an outward pointing normal we need the $+$ sign. For example, with the $+$ sign, the z-component is $2\sin\varphi\cos\varphi = \sin(2\varphi)$ so that the normal is pointing upward when

$0 < \varphi < \frac{\pi}{2}$, i.e. in the northern hemisphere, and is pointing downward when $\frac{\pi}{2} < \varphi < \pi$, i.e. in the southern hemisphere. So

$$\begin{aligned}
\mathbf{F}\cdot\hat{\mathbf{n}}\mathrm{d}S &= \big\{(\sin\varphi\cos\theta)(4\sin^2\varphi\cos\theta) + (2\sin\varphi\sin\theta)(2\sin^2\varphi\sin\theta) \\
&\qquad\qquad\qquad\qquad + (2\cos\varphi)(2\sin\varphi\cos\varphi)\big\}\mathrm{d}\theta\mathrm{d}\varphi \\
&= \big\{4\sin^3\varphi\cos^2\theta + 4\sin^3\varphi\sin^2\theta + 4\sin\varphi\cos^2\varphi\big\}\mathrm{d}\theta\mathrm{d}\varphi \\
&= 4\sin\varphi\big(\sin^2\varphi + \cos^2\varphi\big)\mathrm{d}\theta\mathrm{d}\varphi \\
&= 4\sin\varphi\,\mathrm{d}\theta\mathrm{d}\varphi
\end{aligned}$$

and the flux is

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_0^\pi \mathrm{d}\varphi \int_0^{2\pi}\mathrm{d}\theta\, 4\sin\varphi = 8\pi\int_0^\pi \mathrm{d}\varphi\ \sin\varphi = 16\pi$$

(c) Set

$$V = \big\{\, (x,y,z) \bigm| x^2 + \tfrac{y^2}{4} + \tfrac{z^2}{4} \leqslant 1 \,\big\}$$

Since $\boldsymbol{\nabla}\cdot\mathbf{F} = 3$, the divergence theorem gives

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V = 3\mathrm{Volume}(V)$$

The volume contained in the ellipsoid, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, of semiaxes a, b and c is $\frac{4}{3}\pi abc$. In our case $a = 1$, $b = c = 2$, so

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 3\mathrm{Volume}(V) = 3\times\frac{4}{3}\pi(1)(2)(2) = 16\pi$$

which is exactly what we found in part (b).

The volume of the ellipsoid V can also be found by observing that, in V,

- x runs from -1 to 1 and
- for each fixed $-1 \leqslant x \leqslant 1$, (y,z) runs over the disk $y^2 + z^2 \leqslant 4(1-x^2)$, which has area $4\pi(1-x^2)$.

That is

$$V = \big\{\, (x,y,z) \bigm| -1 \leqslant x \leqslant 1,\ y^2 + z^2 \leqslant 4(1-x^2) \,\big\}$$

so that

$$\begin{aligned}
\mathrm{Volume}(V) &= \int_{-1}^1 \mathrm{d}x \iint_{y^2+z^2\leqslant 4(1-x^2)} \mathrm{d}y\,\mathrm{d}z \\
&= \int_{-1}^1 \mathrm{d}x\, 4\pi(1-x^2) = 2\times 4\pi\int_0^1 \mathrm{d}x\,(1-x^2) = 8\pi\left[1 - \frac{1}{3}\right] \\
&= \frac{16\pi}{3}
\end{aligned}$$

S-14: Set

$$V = \{ (x,y,z) \mid x^2+y^2 \leqslant 2,\ 0 \leqslant z \leqslant 2x+3 \}$$

Let's try the divergence theorem. Since

$$\begin{aligned}
\nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}\big(x^3+\cos(y^2)\big) + \frac{\partial}{\partial y}\big(y^3+ze^x\big) + \frac{\partial}{\partial z}\big(z^2+\arctan(xy)\big) \\
&= 3x^2+3y^2+2z
\end{aligned}$$

the divergence theorem (Theorem 4.2.2 of the CLP-4 text) gives

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iiint_V \nabla\cdot\mathbf{F}\,\mathrm{d}V \\
&= \int_{x^2+y^2\leqslant 2} \mathrm{d}x\mathrm{d}y \int_0^{2x+3} \mathrm{d}z\ \big(3x^2+3y^2+2z\big) \\
&= \int_{x^2+y^2\leqslant 2} \mathrm{d}x\mathrm{d}y\ \big\{3(x^2+y^2)(2x+3)+(2x+3)^2\big\} \\
&= \int_{x^2+y^2\leqslant 2} \mathrm{d}x\mathrm{d}y\ \big\{9+12x+13x^2+9y^2+6x^3+6xy^2\big\} \\
&= 9(2\pi) + \int_{x^2+y^2\leqslant 2} \mathrm{d}x\mathrm{d}y\ \big\{13x^2+9y^2\big\}
\end{aligned}$$

because $12x$, $6x^3$ and $6xy^2$ are all odd under $x \to -x$. To evaluate the final remaining integral, let's switch to polar coordinates.

$$\begin{aligned}
\iint_{x^2+y^2\leqslant 2} \big\{13x^2+9y^2\big\}\,\mathrm{d}x\mathrm{d}y &= \int_0^{\sqrt{2}} \mathrm{d}r\ r \int_0^{2\pi} \mathrm{d}\theta\ \big\{13\big(r\cos\theta\big)^2+9\big(r\sin\theta\big)^2\big\} \\
&= \int_0^{\sqrt{2}} \mathrm{d}r\ r^3 \int_0^{2\pi} \mathrm{d}\theta\ \big\{13\cos^2\theta+9\sin^2\theta\big\}
\end{aligned}$$

Since

$$\begin{aligned}
\int_0^{2\pi} \cos^2\theta\ \mathrm{d}\theta &= \int_0^{2\pi} \frac{\cos(2\theta)+1}{2}\ \mathrm{d}\theta = \left[\frac{\sin(2\theta)}{4}+\frac{\theta}{2}\right]_0^{2\pi} = \pi \\
\int_0^{2\pi} \sin^2\theta\ \mathrm{d}\theta &= \int_0^{2\pi} \frac{1-\cos(2\theta)}{2}\ \mathrm{d}\theta = \left[\frac{\theta}{2}-\frac{\sin(2\theta)}{4}\right]_0^{2\pi} = \pi
\end{aligned}$$

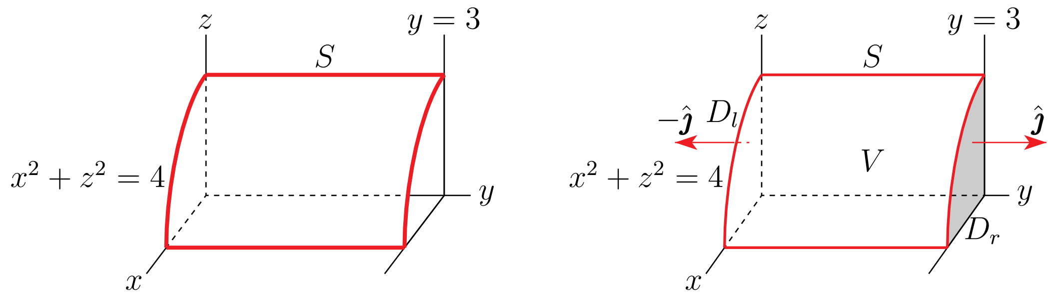
we finally have

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 18\pi + \frac{(\sqrt{2})^4}{4}\pi\big\{13+9\big\} = (18+22)\pi = 40\pi$$

For an efficient, sneaky, way to evaluate $\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta$ and $\int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta$, see Example 2.4.4 in the CLP-4 text.

S-15: *Solution 1 (divergence theorem)*: Set $\mathbf{F} = (x+y, x+z, y+z)$. Then $\boldsymbol{\nabla}\cdot\mathbf{F} = 2$. That's really simple. So let's try using the divergence theorem.

- Set $S = \{\,(x,y,z)\mid x^2+z^2=4,\ 0\leqslant y\leqslant 3\,\}$. We are to compute $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$, with $\hat{\mathbf{n}}$ denoting the outward normal to S. S is *not* the boundary of a solid, so we cannot compute $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ by applying the divergence theorem directly. The figure on the left below shows the part of S that is in the first octant.



- On the other hand S, is "almost" the boundary of

$$V = \{\,(x,y,z)\mid x^2+z^2\leqslant 4,\ 0\leqslant y\leqslant 3\,\}$$

The boundary, ∂V of V consists of three pieces — S and the two disks

$$D_l = \{\,(x,y,z)\mid x^2+z^2\leqslant 4,\ y=0\,\} \qquad D_r = \{\,(x,y,z)\mid x^2+z^2\leqslant 4,\ y=3\,\}$$

The figure on the right above shows the parts of S, V, D_l and D_r that are in the first octant.

The outward normal to D_r is $\hat{\boldsymbol{\jmath}}$ and the outward normal to D_l is $-\hat{\boldsymbol{\jmath}}$, to the divergence theorem gives

$$\begin{aligned}\iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V &= \iint_{\partial V}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S\\ &= \iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \iint_{D_r}\mathbf{F}\cdot\hat{\boldsymbol{\jmath}}\,\mathrm{d}S + \iint_{D_l}\mathbf{F}\cdot(-\hat{\boldsymbol{\jmath}})\,\mathrm{d}S\end{aligned}$$

Since $\boldsymbol{\nabla}\cdot\mathbf{F} = 2$ and $\mathbf{F}\cdot\hat{\boldsymbol{\jmath}} = x+z$,

$$\begin{aligned}\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iiint_V 2\,\mathrm{d}V - \iint_{x^2+z^2\leqslant 4}(x+z)\,\mathrm{d}x\mathrm{d}z - \iint_{x^2+z^2\leqslant 4}(-x-z)\,\mathrm{d}x\mathrm{d}z\\ &= \iiint_V 2\,\mathrm{d}V\\ &= 2\,\mathrm{volume}(V) = 2(\pi 2^2)3 = 24\pi\end{aligned}$$

Solution 2 (direct evaluation): Let's parametrize the surface by

$$\mathbf{r}(\theta,y) = 2\cos\theta\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + 2\sin\theta\,\hat{\mathbf{k}} \qquad 0\leqslant\theta<2\pi,\ 0\leqslant y\leqslant 3$$

Then

$$\frac{\partial \mathbf{r}}{\partial \theta} = \big(-2\sin\theta\,,\,0\,,\,2\cos\theta\big)$$
$$\frac{\partial \mathbf{r}}{\partial y} = \big(0\,,\,1\,,\,0\big)$$
$$\hat{\mathbf{n}}\mathrm{d}S = \pm\frac{\partial \mathbf{r}}{\partial \theta}\times\frac{\partial \mathbf{r}}{\partial y}\mathrm{d}\theta\mathrm{d}y = \pm\big(-2\cos\theta\,,\,0\,,\,-2\sin\theta\big)\mathrm{d}\theta\mathrm{d}y$$

To get the outward normal, we want the minus sign. So

$$\hat{\mathbf{n}}\mathrm{d}S = \big(2\cos\theta\,,\,0\,,\,2\sin\theta\big)\mathrm{d}\theta\mathrm{d}y$$

and, since

$$\mathbf{F}\big(\mathbf{r}(\theta,y)\big) = \big(2\cos\theta + y\,,\,2\cos\theta+2\sin\theta\,,\,y+2\sin\theta\big)$$

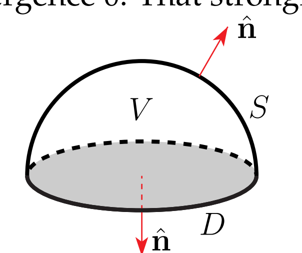
the specified flux is

$$\begin{aligned}\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \int_0^{2\pi}\mathrm{d}\theta\int_0^3\mathrm{d}y\ \big(2\cos\theta + y\,,\,2\cos\theta+2\sin\theta\,,\,y+2\sin\theta\big)\cdot\big(2\cos\theta\,,\,0\,,\,2\sin\theta\big)\\ &= \int_0^{2\pi}\mathrm{d}\theta\int_0^3\mathrm{d}y\ \big(4\cos^2\theta+2y\cos\theta+2y\sin\theta+4\sin^2\theta\big)\\ &= \int_0^{2\pi}\mathrm{d}\theta\int_0^3\mathrm{d}y\ \big(4+2y\cos\theta+2y\sin\theta\big)\end{aligned}$$

Since $\int_0^{2\pi}\mathrm{d}\theta\ \cos\theta = \int_0^{2\pi}\mathrm{d}\theta\ \sin\theta = 0$,

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 4\int_0^{2\pi}\mathrm{d}\theta\int_0^3\mathrm{d}y = 4(2\pi)3 = 24\pi$$

S-16: The question highlights that the vector field has divergence 0. That strongly suggests that we use the divergence theorem. Set



$$V = \big\{\ (x,y,z)\ \big|\ 0\leqslant z\leqslant 1-\big(x^2+y^2\big)^2\ \big\}$$

Then the boundary, ∂V, of V consists of two parts, namely S (with normal pointing upwards) and the disk

$$D = \big\{\ (x,y,0)\ \big|\ x^2+y^2\leqslant 1\ \big\}$$

(with normal pointing downwards). The divergence theorem (Theorem 4.2.2 of the CLP-4 text) gives

$$\begin{aligned}\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V - \iint_D \mathbf{F}\cdot(-\hat{\mathbf{k}})\,\mathrm{d}S\\ &= \iint_D (x^2+y^2)\,\mathrm{d}x\mathrm{d}y\end{aligned}$$

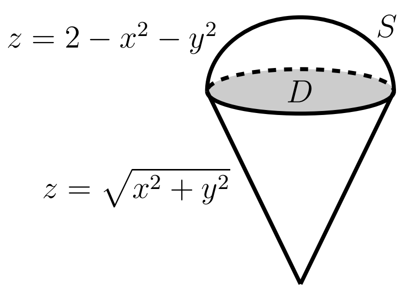
Switching to polar coordinates, the flux is

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_0^1 \mathrm{d}r\, r\int_0^{2\pi}\mathrm{d}\theta\, r^2 = 2\pi\int_0^1 \mathrm{d}r\, r^3 = 2\pi\tfrac{1}{4} = \tfrac{\pi}{2}$$

S-17: As $\mathbf{F}$ looks complicated, we will probably want to avoid evaluating the flux integral directly. Let's first compute the divergence of $\mathbf{F}$, to see if it looks wise to use the divergence theorem instead.

$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}\big(\tan\sqrt{z}+\sin(y^3)\big) + \frac{\partial}{\partial y}\big(e^{-x^2}\big) + \frac{\partial}{\partial z}(z) = 1$$

Looks good! We cannot yet apply the divergence theorem, since S is not the boundary of a solid region V. To help us choose a solid V whose boundary at least includes S, here is a sketch. S is the top of the "ice cream cone"



Note that the the paraboloid $z = 2 - x^2 - y^2$ and the cone $z = \sqrt{x^2+y^2}$ intersect along the circle $x^2 + y^2 = 1$, $z = 1$. Probably the simplest solid whose boundary includes S is

$$V = \big\{\ (x,y,z)\ \big|\ 1 \leqslant z \leqslant 2 - x^2 - y^2,\ x^2+y^2 \leqslant 1\ \big\}$$

The boundary ∂V of V consists of S (with upward pointing normal) and the disk

$$D = \big\{\ (x,y,z)\ \big|\ x^2+y^2 \leqslant 1,\ z = 1\ \big\}$$

with normal $-\hat{\mathbf{k}}$. So the divergence theorem gives

$$\begin{aligned}\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iiint_V \nabla\cdot\mathbf{F}\,\mathrm{d}V - \iint_D \mathbf{F}\cdot(-\hat{\mathbf{k}})\,\mathrm{d}S \\ &= \iiint_V \overbrace{1}^{\nabla\cdot\mathbf{F}}\,\mathrm{d}V + \iint_D \overbrace{1}^{\mathbf{F}\cdot\hat{\mathbf{k}}=z}\,\mathrm{d}S\end{aligned}$$

As D is a disk of radius 1, $\iint_D \mathrm{d}S = \pi$. To compute the volume of V, we'll slice it into a stack of horizontal "pancakes". Since $z = 2 - x^2 - y^2$ is equivalent to $\sqrt{x^2+y^2} = \sqrt{2-z}$, the pancake at height z is a circular disk of radius $\sqrt{2-z}$ and hence of cross-sectional area $\pi(2-z)$. So the volume of V is

$$\iiint_V \mathrm{d}V = \int_1^2 \pi(2-z)\,\mathrm{d}z = -\frac{\pi}{2}(2-z)^2\Big|_1^2 = \frac{\pi}{2}$$

and the flux

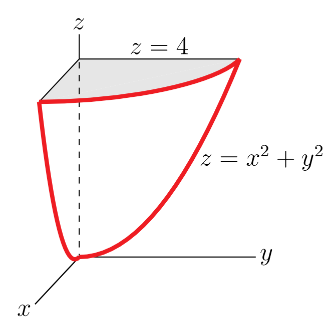
$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \frac{\pi}{2} + \pi = \frac{3}{2}\pi$$

S-18: As $\mathbf{F}$ looks complicated, we will probably want to avoid evaluating the flux integral directly. Let's first compute the divergence of $\mathbf{F}$, to see if it looks wise to use the divergence theorem instead.

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}\big(\cos z + xy^2\big) + \frac{\partial}{\partial y}\big(xe^{-z}\big) + \frac{\partial}{\partial z}\big(\sin y + x^2 z\big) = y^2 + x^2$$

Looks promising. Furthermore S is the boundary of the solid region

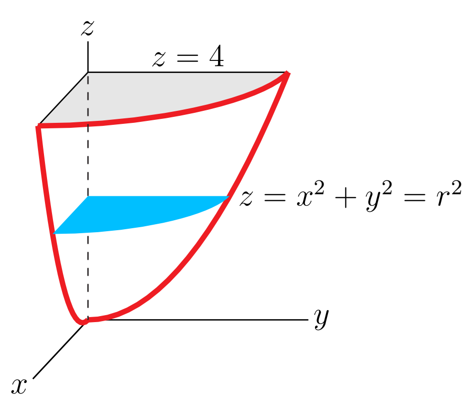
$$V = \{\ (x, y, z)\ \big|\ x^2 + y^2 \leqslant z \leqslant 4\ \}$$



So the divergence theorem gives

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iiint_V \nabla \cdot \mathbf{F}\, dV = \iiint_V (x^2 + y^2)\, dV$$

To compute the triple integral, we'll use the cylindrical coordinates (r, θ, z). The z-coordinate runs from 0 to 4. For each fixed $0 \leqslant z \leqslant 4$ (see the blue disk in the figure below — which shows the part of V in the first octant), (x, y) runs over $0 \leqslant x^2 + y^2 \leqslant z$,



which in cylindrical coordinates is $0 \leqslant r^2 \leqslant z$ or $0 \leqslant r \leqslant \sqrt{z}$. So the flux and the triple integral are

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iiint_V (x^2+y^2)\,\mathrm{d}V \\
&= \int_0^4 \mathrm{d}z \int_0^{\sqrt{z}} \mathrm{d}r\; r \int_0^{2\pi} \mathrm{d}\theta\; r^2 \\
&= 2\pi \int_0^4 \mathrm{d}z \int_0^{\sqrt{z}} \mathrm{d}r\; r^3 \\
&= 2\pi \int_0^4 \mathrm{d}z\; \frac{z^2}{4} = 2\pi \frac{4^3}{3\times 4} \\
&= \frac{32}{3}\pi
\end{aligned}$$

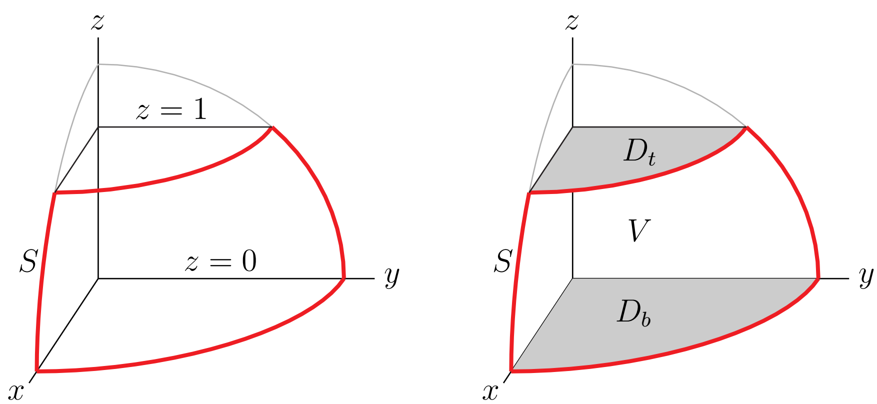
S-19: If we were to evaluate this integral directly using, for example, spherical coordinates, our integrand would contain

$$\tan(x) = \tan\big(2\sin\varphi\cos\theta\big)$$

That's not very friendly looking. So let's consider using the divergence theorem instead. To start,

$$\boldsymbol{\nabla}\cdot\mathbf{F} = \frac{\partial}{\partial x}\big(e^y + xz\big) + \frac{\partial}{\partial y}\big(zy + \tan(x)\big) + \frac{\partial}{\partial z}\big(z^2-1\big) = 4z$$

That's nice and simple. So let's move on to consideration of S. The part of S in the first octant is outlined in red in the figure on the left below.



The surface S is not closed, and so is not the boundary of a solid, so we cannot apply the divergence theorem directly. But we can easily come up with a solid whose boundary contains S. Let

$$V = \big\{\,(x,y,z)\;\big|\;x^2+y^2+z^2 \leqslant 4,\ 0\leqslant z\leqslant 1\,\big\}$$

The boundary ∂V of V consists of three parts — S, the bottom disk

$$D_b = \big\{\,(x,y,z)\;\big|\;x^2+y^2 \leqslant 4,\ z=0\,\big\}$$

and the top disk

$$D_t = \{ (x,y,z) \mid x^2 + y^2 \leqslant 3,\ z = 1 \}$$

The outward normal to D_t is $\hat{\mathbf{k}}$ and the outward normal to D_b is $-\hat{\mathbf{k}}$. So the divergence theorem gives

$$\iiint_V \nabla \cdot \mathbf{F}\, \mathrm{d}V = \iint_{\partial V} \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S + \iint_{D_t} \mathbf{F} \cdot \hat{\mathbf{k}}\, \mathrm{d}S + \iint_{D_b} \mathbf{F} \cdot (-\hat{\mathbf{k}})\, \mathrm{d}S$$

On D_b, $z = 0$ so that $\mathbf{F} \cdot (-\hat{\mathbf{k}}) = -(0^2 - 1) = 1$ and on D_t, $z = 1$ so that $\mathbf{F} \cdot \hat{\mathbf{k}} = 1^2 - 1 = 0$. So

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iiint_V \overbrace{4z}^{\nabla\cdot\mathbf{F}}\, \mathrm{d}V - \iint_{D_b} \mathrm{d}S$$

The constant z cross-section of V is a disk of radius $\sqrt{4 - z^2}$ and hence of area $\pi(4 - z^2)$ and D_b is a disk of radius 2 and hence of area 4π. So

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \int_0^1 (4z)\, \pi(4 - z^2)\, \mathrm{d}z - 4\pi = 4\pi\Big[2z^2 - \frac{z^4}{4}\Big]_0^1 - 4\pi = 3\pi$$

S-20: The divergence of $\mathbf{F}$, namely,

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}(x^2 z + \cos \pi y) + \frac{\partial}{\partial y}(yz + \sin \pi z) + \frac{\partial}{\partial z}(x - y^2) \\ &= 2xz + z \end{aligned}$$

is a lot simpler than $\mathbf{F}$ itself. So let's use the divergence theorem (Theorem 4.2.2 of the CLP-4 text).

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iiint_B \nabla \cdot \mathbf{F}\, \mathrm{d}V = \iiint_B (2xz + z)\, \mathrm{d}V$$

As B is invariant under $x \to -x$ while $2xz$ is odd under $x \to -x$, the integral $\iiint_B 2xz\, \mathrm{d}V$ is zero. To help set up the limits of integration for $\iiint_B z\, \mathrm{d}V$, note that, in B,

- (x, y) runs over the rectangle $-1 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 2$ and
- for each fixed (x, y), z runs over $0 \leqslant z \leqslant 3 - y$.

So

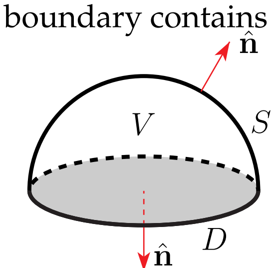
$$\begin{aligned}
\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \int_{-1}^{1}\mathrm{d}x\int_0^2\mathrm{d}y\int_0^{3-y}\mathrm{d}z\;z\\
&= \frac{1}{2}\int_{-1}^{1}\mathrm{d}x\int_0^2\mathrm{d}y\;(3-y)^2\\
&= -\frac{1}{2}\int_{-1}^{1}\mathrm{d}x\int_3^1\mathrm{d}u\;u^2 \qquad \text{with } u=3-y,\ \mathrm{d}u=-\mathrm{d}y\\
&= -\frac{1}{2}\int_{-1}^{1}\mathrm{d}x\;\Big[\frac{1^3}{3}-\frac{3^3}{3}\Big]_{-1}^{1}\\
&= \frac{26}{3}
\end{aligned}$$

S-21: The vector field $\mathbf{F}$ looks very complicated. That strongly suggests that we not evaluate the integral directly. So let's start by computing

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{F} &= \frac{\partial}{\partial x}\big(x+\cos(z^2)\big)+\frac{\partial}{\partial y}\Big(y+\ln(x^2+z^5)\Big)+\frac{\partial}{\partial z}\big(\sqrt{x^2+y^2}\big)\\
&= 2
\end{aligned}$$

That's really simple, which suggest that we use the divergence theorem. But the surface S is not closed, and so is not the boundary of a solid. So we cannot apply the divergence theorem directly. But we can easily come up with a solid whose boundary contains S. Let

$$V=\big\{\,(x,y,z)\ \big|\ 0\leqslant z\leqslant\sqrt{1-x^2-y^2},\ x^2+y^2\leqslant 1\,\big\}$$



Then the boundary, ∂V, of V consists of two parts, namely S (with normal pointing upwards) and the disk

$$D=\big\{\,(x,y,0)\ \big|\ x^2+y^2\leqslant 1\,\big\}$$

(with normal $-\hat{\mathbf{k}}$). The divergence theorem (Theorem 4.2.2 of the CLP-4 text) gives

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V-\iint_D \mathbf{F}\cdot(-\hat{\mathbf{k}})\,\mathrm{d}S\\
&= \iiint_V 2\,\mathrm{d}V+\iint_D\sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y \qquad = 2\,\frac{1}{2}\,\frac{4}{3}\pi 1^3+\iint_D\sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y
\end{aligned}$$

Switching to polar coordinates, the flux is

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S=\frac{4}{3}\pi+\int_0^1\mathrm{d}r\,r\int_0^{2\pi}\mathrm{d}\theta\;r=\frac{4}{3}\pi+2\pi\int_0^1\mathrm{d}r\,r^2=\frac{4}{3}\pi+2\pi\frac{1}{3}=2\pi$$

S-22: (a) By the divergence theorem (Theorem 4.2.2 of the CLP-4 text), the outward flux of $\mathbf{F}$ through the boundary of E is

$$\begin{aligned}\iint_{\partial E} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iiint_E \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V\\ &= \iiint_E \big(-x^2-y^2+4\big)\,\mathrm{d}V\end{aligned}$$

To evaluate this integral we switch to cylindrical coordinates. In cylindrical coordinates

$$E = \big\{\ (r\cos\theta\,,\ r\sin\theta\,,\ z)\ \big|\ 0\leqslant z\leqslant 4,\ r^2\leqslant z\ \big\}$$

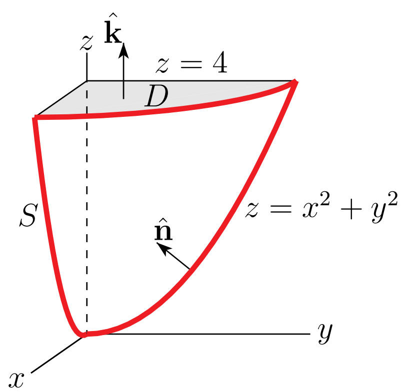
So

$$\begin{aligned}\iint_{\partial E} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \int_0^4 \mathrm{d}z\int_0^{\sqrt{z}}\mathrm{d}r\,r\int_0^{2\pi}\mathrm{d}\theta\,\big(-r^2+4\big)\\ &= 2\pi\int_0^4\mathrm{d}z\int_0^{\sqrt{z}}\mathrm{d}r\,\big(4r-r^3\big)\\ &= 2\pi\int_0^4\mathrm{d}z\,\Big(2z-\frac{z^2}{4}\Big)\\ &= 2\pi\Big[z^2-\frac{z^3}{12}\Big]_0^4 = 2\pi\Big[16-\frac{16}{3}\Big] = \frac{64}{3}\pi\end{aligned}$$

(b) The boundary of S consists of two parts — S, but with downward pointing normal, on the bottom and the disk

$$D = \big\{\ (x,y,z)\ \big|\ z=4,\ x^2+y^2\leqslant 4\ \big\}$$

with normal $\hat{\mathbf{k}}$, on top.



So, by part (a),

$$\frac{64}{3}\pi = \iint_{\partial E}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = -\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \iint_D \mathbf{F}\cdot\hat{\mathbf{k}}\,\mathrm{d}S = -\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \iint_D \overbrace{4z}^{\mathbf{F}\cdot\hat{\mathbf{k}}}\,\mathrm{d}S$$

Since $z = 4$ on D, and D is a disk of radius 2,

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = -\frac{64}{3}\pi + 16 \iint_D \mathrm{d}S = -\frac{64}{3}\pi + 16(4\pi) = \frac{128}{3}\pi$$

S-23: (a) Since

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{x}{[x^2+y^2+z^2]^{3/2}} &= \frac{1}{[x^2+y^2+z^2]^{3/2}} - \frac{3}{2}\frac{x(2x)}{[x^2+y^2+z^2]^{5/2}} = \frac{-2x^2+y^2+z^2}{[x^2+y^2+z^2]^{5/2}} \\
\frac{\partial}{\partial y}\frac{y}{[x^2+y^2+z^2]^{3/2}} &= \frac{1}{[x^2+y^2+z^2]^{3/2}} - \frac{3}{2}\frac{y(2y)}{[x^2+y^2+z^2]^{5/2}} = \frac{x^2-2y^2+z^2}{[x^2+y^2+z^2]^{5/2}} \\
\frac{\partial}{\partial z}\frac{z}{[x^2+y^2+z^2]^{3/2}} &= \frac{1}{[x^2+y^2+z^2]^{3/2}} - \frac{3}{2}\frac{z(2z)}{[x^2+y^2+z^2]^{5/2}} = \frac{x^2+y^2-2z^2}{[x^2+y^2+z^2]^{5/2}}
\end{aligned}$$

the specified divergence is

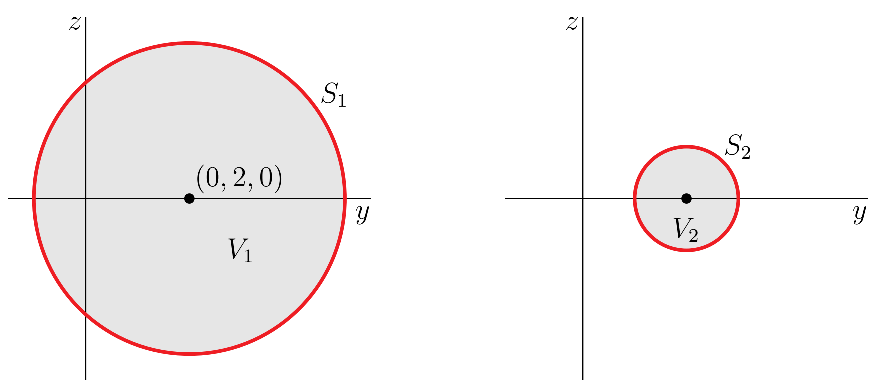
$$\nabla \cdot \mathbf{F} = \frac{(-2x^2+y^2+z^2)+(x^2-2y^2+z^2)+(x^2+y^2-2z^2)}{[x^2+y^2+z^2]^{5/2}} = 0$$

if $(x,y,z) \neq \mathbf{0}$ and is not defined if $(x,y,z) = \mathbf{0}$.

(b), (c) Set

$$\begin{aligned}
V_1 &= \left\{ (x,y,z) \,\middle|\, x^2+(y-2)^2+z^2 \leqslant 9 \right\} \\
V_2 &= \left\{ (x,y,z) \,\middle|\, x^2+(y-2)^2+z^2 \leqslant 1 \right\}
\end{aligned}$$

Here are side views of both V_1 and V_2. Both V_1 and V_2 are spherical balls centred on



$(0,2,0)$. The difference between them is that V_1 has radius 3 while V_2 has radius 1. In particular $(0,0,0)$ is not in V_2. So $\nabla \cdot \mathbf{F}$ is well-defined and zero throughout V_2 and, by the divergence theorem (Theorem 4.2.2 of the CLP-4 text),

$$\iint_{S_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iiint_{V_2} \nabla \cdot \mathbf{F}\, \mathrm{d}V = 0$$

On the other hand, $(0,0,0)$ is in V_1. We cannot blindly apply the divergence theorem to V_1 — $\nabla \cdot \mathbf{F}(x,y,z)$ is not defined at the point $(x,y,z) = (0,0,0)$ in V_1. We can work around this obstruction by

- choosing a number $\rho > 0$ that is small enough that the sphere

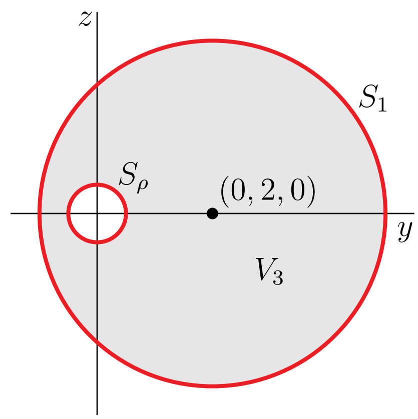
$$S_\rho = \{\ (x,y,z)\ |\ x^2+y^2+z^2=\rho^2\ \}$$

is completely contained inside V_1 (for example, $\rho = \frac{1}{2}$ is fine)
- and then removing the interior of S_ρ from V_1.

This produces

$$V_3 = \{\ (x,y)\ |\ x^2+(y-2)^2+z^2 \leqslant 9,\ x^2+y^2+z^2 \geqslant \rho^2\ \}$$

whose side view is sketched below.



The boundary of V_3 consists of two parts

- the sphere S_1, with outward normal and
- the sphere S_ρ with *inward* normal $\hat{\mathbf{n}} = -\frac{\mathbf{r}}{|\mathbf{r}|}$

The divergence $\boldsymbol{\nabla}\cdot\mathbf{F}$ is well-defined and zero throughout V_3 so that, by the divergence theorem,

$$0 = \iiint_{V_3} \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V = \iint_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \iint_{S_\rho} \mathbf{F}\cdot\left(-\frac{\mathbf{r}}{|\mathbf{r}|}\right)\mathrm{d}S$$

So

$$\iint_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_{S_\rho} \mathbf{F}\cdot\left(\frac{\mathbf{r}}{|\mathbf{r}|}\right)\mathrm{d}S = \iint_{S_\rho} \left(\frac{\mathbf{r}}{|\mathbf{r}|^3}\right)\cdot\left(\frac{\mathbf{r}}{|\mathbf{r}|}\right)\mathrm{d}S = \iint_{S_\rho} \frac{1}{|\mathbf{r}|^2}\,\mathrm{d}S = \iint_{S_\rho} \frac{1}{\rho^2}\,\mathrm{d}S$$
$$= \frac{1}{\rho^2}\,4\pi\rho^2 = 4\pi$$

since S_ρ is a sphere of radius ρ and hence of surface area $4\pi\rho^2$.

(d) The flux integrals $\iint_{S_1}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ and $\iint_{S_2}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ are different, because the one point, $(0,0,0)$, where $\boldsymbol{\nabla}\cdot\mathbf{F}$ fails to be well-defined and zero, is contained inside S_1 but is not contained inside S_2.

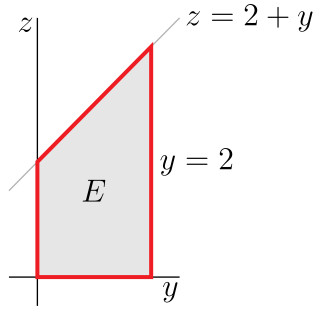
S-24: The vector field $\mathbf{F}$ looks pretty complicated. But its divergence

$$\boldsymbol{\nabla} \cdot \mathbf{F} = 2 + 3 + 1 = 6$$

is very simple. So let's use the divergence theorem (Theorem 4.2.9 of the CLP-4 text). It says

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iiint_E \boldsymbol{\nabla} \cdot \mathbf{F}\, \mathrm{d}V = \iiint_E 6\, \mathrm{d}V = 6\, \text{Volume}(E)$$

For any fixed $0 \leqslant X \leqslant 2$, the cross-section of E with $x = X$ has side view



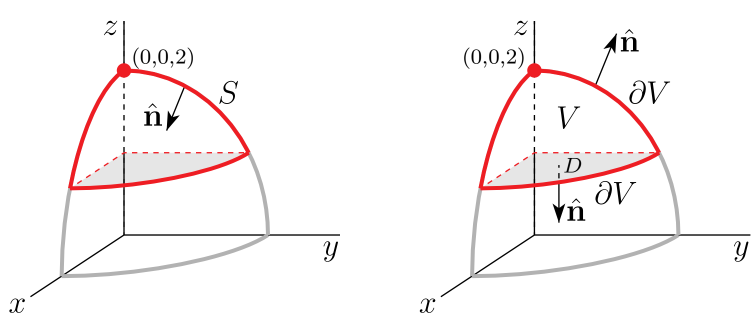
That cross-section has area $2 \times \frac{2+4}{2} = 6$. Consequently the volume of E is $2 \times 6 = 12$ and

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = 6 \times 12 = 72$$

S-25: (a) The divergence is

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{\partial}{\partial x}\big(z \arctan(y^2)\big) + \frac{\partial}{\partial y}\big(z^3 \ln(x^2+1)\big) + \frac{\partial}{\partial z}(3z) = 3$$

(b) The complexity of $\mathbf{F}$ and the simplicity of $\boldsymbol{\nabla} \cdot \mathbf{F}$ strongly suggest that we use the divergence theorem to evaluate $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$. However, S is *not* a closed surface and is *not* the boundary of a solid. The figure on the left below is a sketch of the part of S in the first octant.



On the other hand S is part of the surface of the solid

$$V = \big\{\ (x, y, z)\ \big|\ x^2 + y^2 + z^2 \leqslant 4,\ z \geqslant 1\ \big\}$$

which is sketched on the right above. The boundary of V consists of two parts:

- the original surface S, but with upward, rather than downward, normal and
- the disk $D = \{ (x,y,z) \mid x^2+y^2 \leqslant 3,\ z=1 \}$ with normal $-\hat{\mathbf{k}}$.

So the divergence theorem (Theorem 4.2.9 in the CLP-4 text) gives

$$\begin{aligned} \iint_{\partial V} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V = 3\iiint_V \mathrm{d}V \\ \implies -\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \iint_D \mathbf{F}\cdot(-\hat{\mathbf{k}})\,\mathrm{d}S &= 3\,\text{Volume}(V) \end{aligned}$$

Thus

$$\begin{aligned} \iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= -3\,\text{Volume}(V) + \iint_D \overbrace{-3}^{-\mathbf{F}\cdot\hat{\mathbf{k}}}\,\mathrm{d}S \\ &= -3\,\text{Volume}(V) - 3\,\text{Area}(D) \\ &= -3\,\text{Volume}(V) - 9\pi \end{aligned}$$

since D is a circular disk of radius $\sqrt{3}$. To compute the volume of V, we slice V into thin horizontal pancakes each of thinkness $\mathrm{d}z$. The pancake at height z has cross-section the circular disk $x^2+y^2 \leqslant 4-z^2$. As this disk has area $\pi(4-z^2)$, the pancake has volume $\pi(4-z^2)\,\mathrm{d}z$. All together

$$\text{Volume}(V) = \int_1^2 \mathrm{d}z\ \pi(4-z^2) = \pi\left[4z - \frac{z^3}{3}\right]_1^2 = \pi\left[4-\frac{7}{3}\right] = \frac{5\pi}{3}$$

and

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = -3\frac{5\pi}{3} - 9\pi = -14\pi$$

S-26: Let's try the divergence theorem. Set

$$V = \{ (x,y,z) \mid x^2+y^2+z^2 \leqslant 3 \}$$

Then the boundary of V is S, but with outward pointing normal. Since

$$\boldsymbol{\nabla}\cdot\mathbf{F} = \frac{\partial}{\partial x}(xy^2+y^4z^6) + \frac{\partial}{\partial y}(yz^2+x^4z) + \frac{\partial}{\partial z}(zx^2+xy^4) = y^2+z^2+x^2$$

and because S is oriented *inward*, the divergence theorem (Theorem 4.2.2 of the CLP-4 text) gives

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = -\iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V = -\iiint_V (x^2+y^2+z^2)\,\mathrm{d}V$$

Switching to spherical coordinates (see Appendix F.3 in the CLP-4 text)

$$\begin{aligned}
\iint\limits_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= -\int_0^{\sqrt{3}}\mathrm{d}\rho\int_0^{\pi}\mathrm{d}\varphi\int_0^{2\pi}\mathrm{d}\theta\ \rho^4\sin\varphi\\
&= -2\pi\left[\int_0^{\sqrt{3}}\mathrm{d}\rho\ \rho^4\right]\left[\int_0^{\pi}\mathrm{d}\varphi\ \sin\varphi\right]\\
&= -2\pi\left[\frac{\rho^5}{5}\right]_0^{\sqrt{3}}\left[-\cos\varphi\right]_0^{\pi}\\
&= -\frac{36\sqrt{3}}{5}\pi
\end{aligned}$$

S-27: (a) The divergence of $\mathbf{F}$ is

$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}\big(-2xy\big) + \frac{\partial}{\partial y}\big(y^2+\sin(xz)\big) + \frac{\partial}{\partial z}\big(x^2+y^2\big) = -2y+2y+0 = 0$$

(b) Call the specified surface S and set

$$V = \left\{\ (x,y,z)\ \middle|\ x^2+y^2+(z-12)^2\leqslant 13^2,\ z\geqslant 0\ \right\}$$

The boundary, ∂V, of V consists of two parts — S, with outward normal, and the disk

$$D = \left\{\ (x,y,z)\ \middle|\ x^2+y^2\leqslant 13^2-12^2=5^2,\ z=0\ \right\}$$

with normal $-\hat{\mathbf{k}}$. By the divergence theorem, the desired flux is

$$\begin{aligned}
\iint\limits_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}s &= \iiint\limits_V \nabla\cdot\mathbf{F}\,\mathrm{d}V - \iint\limits_D \mathbf{F}\cdot(-\hat{\mathbf{k}})\,\mathrm{d}S\\
&= \iiint\limits_V 0\,\mathrm{d}V + \iint\limits_D (x^2+y^2)\mathrm{d}x\mathrm{d}y\\
&= 0 + \int_0^5 \mathrm{d}r\ r\int_0^{2\pi}\mathrm{d}\theta\ r^2\\
&= 2\pi\frac{5^4}{4} = \frac{625}{2}\pi
\end{aligned}$$

S-28: The boundary of the solid V enclosed by S and $z=\pm 1$ consists of three pieces: S, the top disk

$$S_1 = \left\{\ (x,y,z)\ \middle|\ x^2+y^2\leqslant 2,\ z=1\ \right\}$$

and the bottom disk

$$S_2 = \left\{\ (x,y,z)\ \middle|\ x^2+y^2\leqslant 2,\ z=-1\ \right\}$$

On S_1, $\hat{\mathbf{n}} = \hat{\mathbf{k}}$ and

$$\mathbf{F}\cdot\hat{\mathbf{n}} = \mathbf{F}\cdot\hat{\mathbf{k}} = xy - z - z^2\Big|_{z=1} = xy - 2$$

so that, denoting $D = \{\ (x,y)\ \big|\ x^2+y^2 \leqslant 2\ \}$,

$$\iint_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_D (xy-2)\,\mathrm{d}x\mathrm{d}y = -2\,\mathrm{Area}(D) = -4\pi$$

Here we have used that the integral $\iint_D xy\,\mathrm{d}x\mathrm{d}y = 0$ because xy is odd under $x \to -x$. On S_2, $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$ and

$$\mathbf{F}\cdot\hat{\mathbf{n}} = -\mathbf{F}\cdot\hat{\mathbf{k}} = -(xy - z - z^2)\Big|_{z=-1} = -xy$$

so that

$$\iint_{S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_D (-xy)\,\mathrm{d}x\mathrm{d}y = 0$$

By the divergence theorem (Theorem 4.2.2 in the CLP-4 text),

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\mathrm{d}S = \iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V - \iint_{S_1}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S - \iint_{S_2}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 0 - (-4\pi) - 0 = 4\pi$$

since

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{F} &= \frac{\partial}{\partial x}(x + e^{yz}) + \frac{\partial}{\partial y}\big(2yz + \sin(xz)\big) + \frac{\partial}{\partial z}(xy - z - z^2)\\ &= 1 + 2z - 1 - 2z\\ &= 0\end{aligned}$$

S-29: *Direct Solution.* The surface is given by the implicit equation $f(x,y,z) = 0$ with $f(x,y,z) = x^2 + y^2 + 2z^2 - 1$. Hence, by (3.3.3) in the CLP-4 text,

$$\hat{\mathbf{n}}\,\mathrm{d}S = \frac{\boldsymbol{\nabla} f}{\boldsymbol{\nabla} f\cdot\hat{\mathbf{k}}}\mathrm{d}x\mathrm{d}y = \frac{2x\hat{\boldsymbol{\imath}} + 2y\hat{\boldsymbol{\jmath}} + 4z\hat{\mathbf{k}}}{4z}\mathrm{d}x\mathrm{d}y$$

This $\hat{\mathbf{n}}$ has positive $\hat{\mathbf{k}}$ component. Assume that it is the desired $\hat{\mathbf{n}}$, though this was not specified in the question. Since

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{F} &= \det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2 - y - 1 & e^{\cos y} + z^3 & 2xz + z^5\end{bmatrix}\\ &= -3z^2\,\hat{\boldsymbol{\imath}} - 2z\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\end{aligned}$$

we have

$$\iint\limits_S \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint\limits_{x^2+y^2\leqslant 1} \big(-3z(x,y)^2\,\hat{\boldsymbol{\imath}} - 2z(x,y)\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\big)\cdot\frac{2x\,\hat{\boldsymbol{\imath}} + 2y\,\hat{\boldsymbol{\jmath}} + 4z(x,y)\,\hat{\mathbf{k}}}{4z(x,y)}\,\mathrm{d}x\mathrm{d}y$$

$$= \iint\limits_{x^2+y^2\leqslant 1} \Big(-\frac{3}{2}x\,z(x,y) - y + 1\Big)\,\mathrm{d}x\mathrm{d}y$$

Since y is an odd function of y and $x\,z(x,y) = x\sqrt{\frac{1}{2}(1-x^2-y^2)}$ is an odd function of x, they both integrate to zero. Hence

$$\iint\limits_S \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint\limits_{x^2+y^2\leqslant 1} 1\,\mathrm{d}x\mathrm{d}y = \pi$$

Tricky Solution. Let V be the solid $x^2+y^2+2z^2\leqslant 1$, $z\geqslant 0$. The surface of V consists of S with upward pointing normal and the disk $D = \big\{\,(x,y,z)\,\big|\,z=0,\ x^2+y^2\leqslant 1\,\big\}$ with normal $-\hat{\mathbf{k}}$. By the divergence theorem, Theorem 4.2.2 in the CLP-4 text,

$$\iint\limits_S \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \iint\limits_D \nabla\times\mathbf{F}\cdot(-\hat{\mathbf{k}})\,\mathrm{d}S = \iiint\limits_V \nabla\cdot\nabla\times\mathbf{F}\,\mathrm{d}V = \iiint\limits_V 0\,\mathrm{d}V = 0$$

Hence

$$\iint\limits_S \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint\limits_D \nabla\times\mathbf{F}\cdot\hat{\mathbf{k}}\,\mathrm{d}S = \iint\limits_D \mathrm{d}S = \pi$$

S-30: Let S' be the disk $x^2+y^2\leqslant 3$, $z=0$ (with $\hat{\mathbf{n}}$ the downward pointing normal) and let V be the portion of the ball $x^2+y^2+(z-1)^2\leqslant 4$ with $z\geqslant 0$. Then, by the divergence theorem,

$$\iint\limits_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint\limits_V \nabla\cdot\mathbf{F}\,\mathrm{d}V - \iint\limits_{S'} \mathbf{F}\cdot(-\hat{\mathbf{k}})\,\mathrm{d}S$$

$$= \iiint\limits_V (2x+2y)\,\mathrm{d}V + \iint\limits_{S'} (4+5x)\,\mathrm{d}x\mathrm{d}y$$

Because x is odd under $x\to -x$ and y is odd under $y\to -y$,

$$\iiint\limits_V x\,\mathrm{d}V = \iiint\limits_V y\,\mathrm{d}V = \iint\limits_{S'} x\,\mathrm{d}x\mathrm{d}y = 0$$

so that

$$\iint\limits_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 4\iint\limits_{S'} \mathrm{d}x\mathrm{d}y = 4\,\mathrm{Area}(S') = 4\times\pi\big(\sqrt{3}\big)^2 = 12\pi$$

S-31: Call the hemisphere $0 \leqslant z \leqslant \sqrt{4-x^2-y^2}$, H. Call the bottom surface of the hemisphere D and the top surface S. The disk D has radius 2, area 4π, $z=0$ and the outward normal $-\hat{\mathbf{k}}$, so that

$$\iint\limits_D \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = -\iint\limits_D \mathbf{F}\cdot\hat{\mathbf{k}}\,\mathrm{d}x\mathrm{d}y = -\iint\limits_D \mathrm{d}x\mathrm{d}y = -4\pi$$

As

$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(x^2y) + \frac{\partial}{\partial z}(1) = x^2+y^2$$

the divergence theorem (Theorem 4.2.2 of the CLP-4 text) gives

$$\iint\limits_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint\limits_H \nabla\cdot\mathbf{F}\,\mathrm{d}V - \iint\limits_D \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint\limits_R (x^2+y^2)\,\mathrm{d}V - (-4\pi)$$

To evaluate the remaining integral, let's switch to the cylindrical coordinates (r,θ,z). In cylindrical coordinates, the equation $x^2+y^2+z^2=4$ becomes $r^2+z^2=4$. So

$$\begin{aligned}\iint\limits_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= 4\pi + \int_0^2 \mathrm{d}z \int_0^{\sqrt{4-z^2}} dr\,r\int_0^{2\pi}\mathrm{d}\theta\, r^2 = 4\pi + 2\pi\int_0^2 \mathrm{d}z\,\frac{1}{4}\big(\sqrt{4-z^2}\big)^4 \\ &= 4\pi + \frac{\pi}{2}\int_0^2 \mathrm{d}z\,\big(16-8z^2+z^4\big) = 4\pi + \frac{\pi}{2}\Big[16z - \frac{8}{3}z^3 + \frac{1}{5}z^5\Big]_0^2 \\ &= \frac{188}{15}\pi \approx 39.37\end{aligned}$$

S-32: Let S_t, S_b and S_c denote the top, bottom and curved surfaces of D respectively. On the top surface, $z=5$ and the outward normal to D is $\hat{\mathbf{k}}$, so that

$$\iint\limits_{S_t} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint\limits_{x^2+y^2\leqslant 1} (15-5ye^5)\,\mathrm{d}x\mathrm{d}y = 15\iint\limits_{x^2+y^2\leqslant 1}\mathrm{d}x\mathrm{d}y = 15\pi$$

The integral over y was zero because y is odd under $y\to -y$. On the bottom surface, $z=0$ and the outward normal to D is $-\hat{\mathbf{k}}$, so that

$$\iint\limits_{S_b} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = -\iint\limits_{x^2+y^2\leqslant 1} (3\times 0 - 0\times ye^0)\,\mathrm{d}x\mathrm{d}y = 0$$

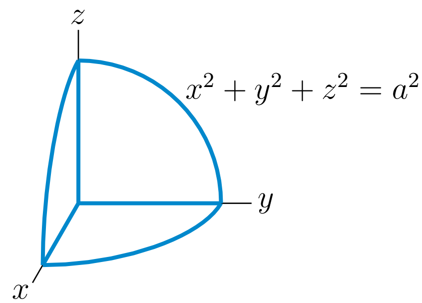
Again, the integral over y was zero because y is odd under $y\to -y$. As

$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(x+xye^z) + \tfrac{1}{2}\frac{\partial}{\partial y}(y^2ze^z) + \frac{\partial}{\partial z}(3z-yze^z) = (1+ye^z)+yze^z+(3-yze^z-ye^z) = 4$$

the divergence theorem gives

$$\begin{aligned}\iint\limits_{S_c} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iiint\limits_D \nabla\cdot\mathbf{F}\,\mathrm{d}V - \iint\limits_{S_t}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S - \iint\limits_{S_b}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S \\ &= \iiint\limits_D 4\,\mathrm{d}V - 15\pi - 0 = 4\times\pi 1^2\times 5 - 15\pi = 5\pi\end{aligned}$$

S-33: Let $V = \left\{\, (x,y,z) \;\middle|\; x^2+y^2+z^2 \leqslant a^2, x \geqslant 0, y \geqslant 0,\ z \geqslant 0 \,\right\}$.



Then ∂V consists of

- the $x = 0$ face $\left\{\, (x,y,z) \;\middle|\; y^2+z^2 \leqslant a^2, x=0, y \geqslant 0,\ z \geqslant 0 \,\right\}$ with normal $\hat{\mathbf{n}} = -\hat{\imath}$,
- the $y = 0$ face $\left\{\, (x,y,z) \;\middle|\; x^2+z^2 \leqslant a^2, x \geqslant 0, y = 0,\ z \geqslant 0 \,\right\}$ with normal $\hat{\mathbf{n}} = -\hat{\jmath}$,
- the $z = 0$ face $\left\{\, (x,y,z) \;\middle|\; x^2+y^2 \leqslant a^2, x \geqslant 0, y \geqslant 0,\ z = 0 \,\right\}$ with normal $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$,
- and the first octant part of the sphere. Call it S.

Then

$$\iiint_V \boldsymbol{\nabla} \cdot \mathbf{F}\, \mathrm{d}V = \iiint_V \big[z+1+z-2z\big]\, \mathrm{d}V = \iiint_V \mathrm{d}V = \frac{1}{8}\,\frac{4}{3}\pi a^3 = \frac{1}{6}\pi a^3$$

$$\iint_{\substack{z=0\\ \text{face}}} \mathbf{F}\cdot(-\hat{\mathbf{k}})\, \mathrm{d}x\, \mathrm{d}y = \iint_{\substack{z=0\\ \text{face}}} (2x+0^2)\, \mathrm{d}x\, \mathrm{d}y = 2\int_0^a dr\, r \int_0^{\pi/2} \mathrm{d}\theta\, r\cos\theta = 2\int_0^a r^2\, dr = \frac{2a^3}{3}$$

$$\iint_{\substack{y=0\\ \text{face}}} \mathbf{F}\cdot(-\hat{\jmath})\, \mathrm{d}x\, \mathrm{d}z = -\iint_{\substack{y=0\\ \text{face}}} (0+0z)\, \mathrm{d}x\, \mathrm{d}z = 0$$

$$\iint_{\substack{x=0\\ \text{face}}} \mathbf{F}\cdot(-\hat{\imath})\, \mathrm{d}y\, \mathrm{d}z = \iint_{\substack{x=0\\ \text{face}}} -(y+0z)\, \mathrm{d}y\, \mathrm{d}z = -\int_0^a dr\, r \int_0^{\pi/2} \mathrm{d}\theta\, r\sin\theta = -\int_0^a r^2\, dr = -\frac{a^3}{3}$$

By the divergence theorem

$$\begin{aligned}\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\, \mathrm{d}x\, \mathrm{d}y &= \iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}\, \mathrm{d}V - \iint_{\substack{x=0\\ \text{face}}} \mathbf{F}\cdot(-\hat{\imath})\, \mathrm{d}y\, \mathrm{d}z - \iint_{\substack{y=0\\ \text{face}}} \mathbf{F}\cdot(-\hat{\jmath})\, \mathrm{d}x\, \mathrm{d}z - \iint_{\substack{z=0\\ \text{face}}} \mathbf{F}\cdot(-\hat{\mathbf{k}})\, \mathrm{d}x\, \mathrm{d}y \\ &= \left[\frac{\pi}{6} - \frac{1}{3}\right] a^3\end{aligned}$$

S-34: (a) On the cylindrical surface S_1, use (surprise!) cylindrical coordinates. Since the cylinder has radius $\sqrt{2}$, we may parametrize it by

$$\begin{aligned}
\mathbf{r}(\theta,z) &= \sqrt{2}\cos\theta\,\hat{\boldsymbol{\imath}} + \sqrt{2}\sin\theta\,\hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}} \\
\frac{\partial\mathbf{r}}{\partial\theta}(\theta,z) &= -\sqrt{2}\sin\theta\,\hat{\boldsymbol{\imath}} + \sqrt{2}\cos\theta\,\hat{\boldsymbol{\jmath}} \\
\frac{\partial\mathbf{r}}{\partial z}(\theta,z) &= \hat{\mathbf{k}} \\
\hat{\mathbf{n}}\,\mathrm{d}S &= \pm\frac{\partial\mathbf{r}}{\partial\theta}(\theta,z)\times\frac{\partial\mathbf{r}}{\partial z}(\theta,z)\,\mathrm{d}\theta\,\mathrm{d}z \\
&= \pm\det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ -\sqrt{2}\sin\theta\,\hat{\boldsymbol{\imath}} & \sqrt{2}\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\,\mathrm{d}\theta\,\mathrm{d}z \\
&= \pm\big(\sqrt{2}\cos\theta\,\hat{\boldsymbol{\imath}} + \sqrt{2}\sin\theta\,\hat{\boldsymbol{\jmath}}\big)\,\mathrm{d}\theta\,\mathrm{d}z
\end{aligned}$$

To get the inward pointing normal, choose the minus sign. So

$$\begin{aligned}
\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \big[\sqrt{2}\big(\cos\theta - z\sin\theta\big)\hat{\boldsymbol{\imath}} + \sqrt{2}\big(\sin\theta + z\cos\theta\big)\hat{\boldsymbol{\jmath}} + (\cdots)\hat{\mathbf{k}}\big]\cdot\big[-\sqrt{2}\cos\theta\,\hat{\boldsymbol{\imath}} - \sqrt{2}\sin\theta\,\hat{\boldsymbol{\jmath}}\big]\,\mathrm{d}\theta\,\mathrm{d}z \\
&= -2\big[\big(\cos\theta - z\sin\theta\big)\cos\theta + \big(\sin\theta + z\cos\theta\big)\sin\theta\big]\,\mathrm{d}\theta\,\mathrm{d}z \\
&= -2\,\mathrm{d}\theta\,\mathrm{d}z
\end{aligned}$$

On the intersection of the sphere and cylinder

$$z^2 = 4 - x^2 - y^2 = 4 - 2 = 2$$

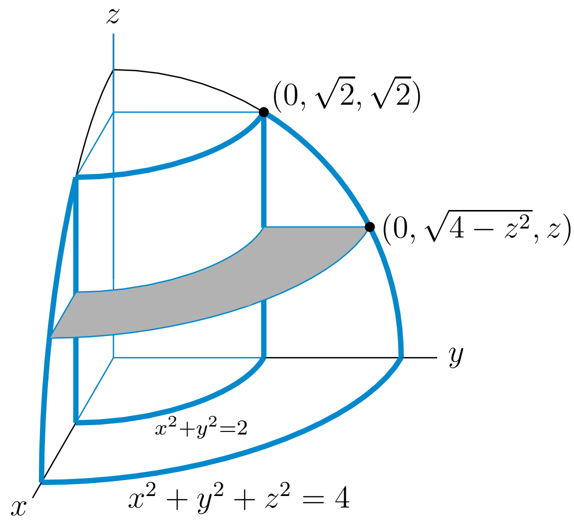
so z runs from $-\sqrt{2}$ to $\sqrt{2}$ (see the figure below) and

$$\iint_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = -2\int_{-\sqrt{2}}^{\sqrt{2}}\mathrm{d}z\int_0^{2\pi}\mathrm{d}\theta = -8\sqrt{2}\pi$$

(b) Observe that $\boldsymbol{\nabla}\cdot\mathbf{F} = 3$. So

$$\iiint_V \boldsymbol{\nabla}\cdot F\,\mathrm{d}V = \iiint_V 3\,\mathrm{d}V$$

The horizontal cross-section of V at height z is a washer with outer radius $\sqrt{4-z^2}$ (determined by the equation of the sphere) and inner radius $\sqrt{2}$ (determined by the equation of the cylinder).



So the cross-section has area $\pi\big(\sqrt{4-z^2}\big)^2 - \pi\big(\sqrt{2}\big)^2 = \pi\big(2-z^2\big)$ and

$$\iiint_V \nabla\cdot F\,\mathrm{d}V = 3\iiint_V \mathrm{d}V = 3\int_{-\sqrt{2}}^{\sqrt{2}} \pi\big(2-z^2\big)\,\mathrm{d}z = 6\pi\int_0^{\sqrt{2}}\big(2-z^2\big)\,\mathrm{d}z = 6\pi\big(2\sqrt{2}-\frac{2^{3/2}}{3}\big)$$
$$= 8\sqrt{2}\pi$$

(c) By the divergence theorem

$$\iint_{S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint_V \nabla\cdot F\,\mathrm{d}V - \iint_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 16\sqrt{2}\pi$$

S-35: By the divergence theorem

$$\iint_{\partial V} \mathbf{E}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint_V \nabla\cdot\mathbf{E}\,\mathrm{d}V$$

So by Gauss' law

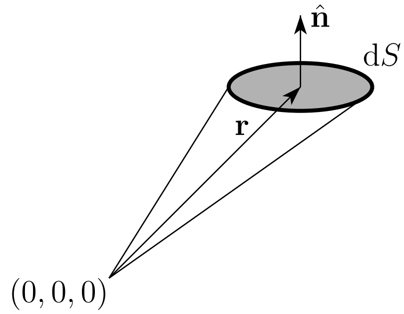
$$\iiint_V \nabla\cdot\mathbf{E}\,\mathrm{d}V = 4\pi\iiint_V \rho\,\mathrm{d}V \qquad\Rightarrow\qquad \iiint_V \big[\nabla\cdot\mathbf{E}-4\pi\rho\big]\,\mathrm{d}V = 0$$

This is true for all solids V for which the divergence theorem applies. If there were some point in $\mathbb{R}^3$ for which $\nabla\cdot\mathbf{E}-4\pi\rho$ were, say, strictly bigger than zero, then, by continuity, we could find a ball B_ϵ centered on that point with $\nabla\cdot\mathbf{E}-4\pi\rho > 0$ everywhere on B_ϵ. This would force $\iiint_{B_\epsilon}\big[\nabla\cdot\mathbf{E}-4\pi\rho\big]\,\mathrm{d}V > 0$, which violates $\iiint_V\big[\nabla\cdot\mathbf{E}-4\pi\rho\big]\,\mathrm{d}V = 0$ with V set equal to B_ϵ. Hence $\nabla\cdot\mathbf{E}-4\pi\rho$ must be zero everywhere.

S-36: By the divergence theorem

$$\iint_{\partial V} \mathbf{r}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint_V \nabla\cdot\mathbf{r}\,\mathrm{d}V = \iiint_V \nabla\cdot(x\,\hat{\imath}+y\,\hat{\jmath}+z\,\hat{\mathbf{k}})\,\mathrm{d}V = \iiint_V 3\,\mathrm{d}V = 3\,\text{Volume}(V)$$

Our gemetric explanation starts with the observation that the volume of the cone with vertex $(0, 0, 0)$ and base a tiny piece of surface $\mathrm{d}S$ is $\frac{1}{3}$ times the area of the base times the height of the cone. The height of the cone is $|\hat{\mathbf{n}} \cdot \mathbf{r}|$, where $\mathbf{r}$ is a point in $\mathrm{d}S$. So the volume of the cone is $\frac{1}{3}|\hat{\mathbf{n}} \cdot \mathbf{r}|\,\mathrm{d}S$.



First assume that $(0, 0, 0)$ is in V and V is convex. Then

- $\hat{\mathbf{n}} \cdot \mathbf{r} > 0$, and the volume is $\frac{1}{3}\hat{\mathbf{n}} \cdot \mathbf{r}\,\mathrm{d}S$.
- the cone is contained in V and
- V is the union of all the tiny conical pieces with $\mathrm{d}S$ running over ∂V.

So

$$\text{Volume}(V) = \frac{1}{3}\iint_{\partial V} \mathbf{r} \cdot \hat{\mathbf{n}}\,\mathrm{d}S$$

To generalise to the case that V is not convex or $(0, 0, 0)$ is not in V, write V as the difference between a large convex solid and one or more smaller convex solids.

S-37: (a) We'll parametrize the sphere using the spherical coordinates θ and φ.

$$\begin{aligned} x &= 3\sin\varphi\cos\theta \\ y &= 3\sin\varphi\sin\theta \\ z &= 3\cos\varphi \end{aligned}$$

with $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant \varphi \leqslant \pi$. Since

$$\begin{aligned} \Big(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta}\Big) &= \big(-3\sin\varphi\sin\theta\,,\,3\sin\varphi\cos\theta\,,\,0\big) \\ \Big(\frac{\partial x}{\partial \varphi}, \frac{\partial y}{\partial \varphi}, \frac{\partial z}{\partial \varphi}\Big) &= (3\cos\varphi\cos\theta\,,\,3\cos\varphi\sin\theta\,,\,-3\sin\varphi) \end{aligned}$$

(3.3.1) in the CLP-4 text yields

$$\begin{aligned} \hat{\mathbf{n}}\,\mathrm{d}S &= \pm\Big(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta}\Big) \times \Big(\frac{\partial x}{\partial \varphi}, \frac{\partial y}{\partial \varphi}, \frac{\partial z}{\partial \varphi}\Big)\,\mathrm{d}\theta\mathrm{d}\varphi \\ &= \pm\big(-3\sin\varphi\sin\theta\,,\,3\sin\varphi\cos\theta\,,\,0\big) \times (3\cos\varphi\cos\theta\,,\,3\cos\varphi\sin\theta\,,\,-3\sin\varphi)\,\mathrm{d}\theta\mathrm{d}\varphi \\ &= \pm\Big(-9\sin^2\varphi\cos\theta\,,\,-9\sin^2\varphi\sin\theta\,,\,-9\sin\varphi\cos\varphi\Big)\,\mathrm{d}\theta\mathrm{d}\varphi \\ &= \mp 9\sin\varphi\Big(\sin\varphi\cos\theta\,,\,\sin\varphi\sin\theta\,,\,\cos\varphi\Big)\,\mathrm{d}\theta\mathrm{d}\varphi \end{aligned}$$

To get an outward pointing normal we need the + sign. For example, with the + sign, the z-component is $9\sin\varphi\cos\varphi = \frac{9}{2}\sin(2\varphi)$ so that the normal is pointing upward when $0 < \varphi < \frac{\pi}{2}$, i.e. in the northern hemisphere, and is pointing downward when $\frac{\pi}{2} < \varphi < \pi$, i.e. in the southern hemisphere. (As a further consistency check, note that $\hat{\mathbf{n}}(\theta,\varphi)$ is parallel to $\mathbf{r}(\theta,\varphi)$.) So

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= 9\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\mathrm{d}\varphi\ \sin\varphi\,(0,0,3\sin\varphi\cos\theta+3\cos\varphi)\cdot\big(\sin\varphi\cos\theta\,,\ \sin\varphi\sin\theta\,,\ \cos\varphi\big)\\
&= 27\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\mathrm{d}\varphi\ \big(\sin^2\varphi\cos\varphi\cos\theta+\sin\varphi\cos^2\varphi\big)\\
&= 54\pi\int_0^{\pi}\mathrm{d}\varphi\ \sin\varphi\cos^2\varphi \qquad \text{since } \int_0^{2\pi}\cos\theta\,\mathrm{d}\theta = 0\\
&= -18\pi\big[\cos^3\varphi\big]_0^{\pi}\\
&= 36\pi
\end{aligned}$$

(b) Set

$$V = \{\,(x,y,z)\in\mathbb{R}^3\ \big|\ x^2+y^2+z^2\leqslant 9\,\}$$

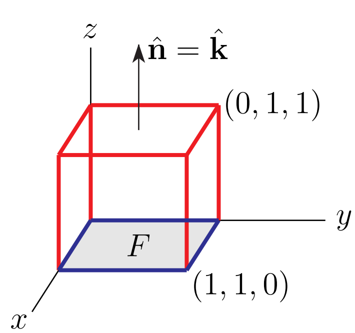
Since

$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial z}(x+z) = 1$$

the divergence theorem (Theorem 4.2.2 of the CLP-4 text) gives

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint_V \nabla\cdot\mathbf{F}\,\mathrm{d}V = \iiint_V \mathrm{d}V = \frac{4}{3}\pi 3^3 = 36\pi$$

S-38: Denote by V the cube specified in the problem. Then ∂V consists of S together with the face F in the plane $z=0$, oriented with the normal being $-\hat{\mathbf{k}}$.



As

$$\begin{aligned}
\nabla\cdot\mathbf{F} &= \frac{\partial}{\partial x}\big(y\cos(y^2)+z-1\big)+\frac{\partial}{\partial y}\left(\frac{z}{x+1}+1\right)+\frac{\partial}{\partial z}\big(xye^{z^2}\big)\\
&= \frac{\partial}{\partial z}\big(xye^{z^2}\big)
\end{aligned}$$

the divergence theorem (Theorem 4.2.2 of the CLP-4 text) gives

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V - \iint_F \mathbf{F}\cdot(-\hat{\mathbf{k}})\,\mathrm{d}S \\
&= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\, \frac{\partial}{\partial z}\big(xye^{z^2}\big) + \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\; xye^{z^2}\Big|_{z=0} \\
&= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\; xye^{z^2}\Big|_{z=0}^{z=1} + \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\; xye^{z^2}\Big|_{z=0} \\
&= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\; xye^{z^2}\Big|_{z=1} \\
&= e\left[\frac{x^2}{2}\right]_0^1 \left[\frac{y^2}{2}\right]_0^1 \\
&= \frac{e}{4}
\end{aligned}$$

S-39: (a) The equation of the surface is $G(x, y, z) = z - xy = 0$. So one normal to the surface at $(1, 1, 1)$ is $(\boldsymbol{\nabla}G)(1, 1, 1) = (-y, -x, 1)\big|_{(x,y,z)=(1,1,1)} = (-1, -1, 1)$ and a unit upward pointing normal at $(1, 1, 1)$ is $\frac{(-1,-1,1)}{|(-1,-1,1)|} = \frac{1}{\sqrt{3}}(-1, -1, 1)$.

(b) For the surface $G(x, y, z) = z - xy$, so that, by (3.3.3) in the CLP-4 text,

$$\hat{\mathbf{n}}\,\mathrm{d}S = \pm\frac{\boldsymbol{\nabla}G(x,y,z)}{\boldsymbol{\nabla}G(x,y,z)\cdot\hat{\mathbf{k}}}\mathrm{d}x\mathrm{d}y = \pm(-y, -x, 1)\mathrm{d}x\mathrm{d}y$$

The "+" sign gives the upward normal, so the specified upward flux is

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_{x^2+y^2\leqslant 9} (y, x, 3)\cdot(-y - x, 1)\,\mathrm{d}x\mathrm{d}y = \iint_{x^2+y^2\leqslant 9} (3 - x^2 - y^2)\,\mathrm{d}x\mathrm{d}y$$

Switching to polar coordinates, the flux is

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_0^3 \mathrm{d}r\, r\int_0^{2\pi} \mathrm{d}\theta\,(3 - r^2) = 2\pi\int_0^3 \mathrm{d}r\,(3r - r^3) = 2\pi\big(\tfrac{3}{2}3^2 - \tfrac{1}{4}3^4\big) = -\tfrac{27\pi}{2}$$

(c) *by direct evaluation:* Parametrize the specified surface using the cylindrical coordinates θ and z.

$$\begin{aligned}
x &= 3\cos\theta \\
y &= 3\sin\theta \\
z &= z
\end{aligned}$$

with $0 \leqslant \theta \leqslant 2\pi$ and $9\sin\theta\cos\theta \leqslant z \leqslant 10$. Then, using (3.3.1) in the CLP-4 text,

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial \theta} &= \big(-3\sin\theta\,,\, 3\cos\theta\,,\, 0\big) \\
\frac{\partial \mathbf{r}}{\partial z} &= \big(0\,,\, 0\,,\, 1\big) \\
\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial z} &= 3\big(\cos\theta\,,\, \sin\theta\,,\, 0\big) \\
\hat{\mathbf{n}}\, \mathrm{d}S &= \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial z} \mathrm{d}\theta\, \mathrm{d}z = 3\big(\cos\theta\,,\, \sin\theta\,,\, 0\big)\, \mathrm{d}\theta\, \mathrm{d}z
\end{aligned}$$

(We have taken the $+$ sign in $\hat{\mathbf{n}}\, \mathrm{d}S = \pm \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial z} \mathrm{d}\theta\, \mathrm{d}z$ to give the outward pointing normal.) So the specified flux is

$$\begin{aligned}
\iint \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S &= 3\int_0^{2\pi} \mathrm{d}\theta \int_{9\cos\theta\sin\theta}^{10} \mathrm{d}z\ \overbrace{\big(3\sin\theta\,,\, 3\cos\theta\,,\, 3\big)}^{\mathbf{F}=(y,x,3)} \cdot \big(\cos\theta\,,\, \sin\theta\,,\, 0\big) \\
&= 18\int_0^{2\pi} \mathrm{d}\theta \int_{9\cos\theta\sin\theta}^{10} \mathrm{d}z\ \sin\theta\ \cos\theta \\
&= 18\int_0^{2\pi} \mathrm{d}\theta\ \big[10 - 9\cos\theta\sin\theta\big] \sin\theta\ \cos\theta \\
&= -9 \times 18 \int_0^{2\pi} \mathrm{d}\theta\ \sin^2\theta\ \cos^2\theta \\
&\qquad \text{since} \int_0^{2\pi} \sin\theta\cos\theta\ \mathrm{d}\theta = \frac{1}{2}\int_0^{2\pi} \sin(2\theta)\ \mathrm{d}\theta = 0 \\
&= -9 \times 18 \times \frac{1}{4}\int_0^{2\pi} \mathrm{d}\theta\ \sin^2(2\theta) \\
&= -\frac{81}{2}\int_0^{2\pi} \mathrm{d}\theta\ \frac{1-\cos(4\theta)}{2} \\
&= -\frac{81}{2}\left[\frac{\theta}{2} - \frac{\sin(4\theta)}{8}\right]_0^{2\pi} \\
&= -\frac{81}{2}\pi
\end{aligned}$$

For an efficient, sneaky, way to evaluate $\int_0^{2\pi} \mathrm{d}\theta\ \sin^2(2\theta)$ see Example 2.4.4 in the CLP-4 text.

(c) *using the divergence theorem:* Note that if $x^2 + y^2 \leqslant 9$, then $|x| \leqslant 3$ and $y \leqslant 3$ so that $|xy| \leqslant 9 < 10$. Set

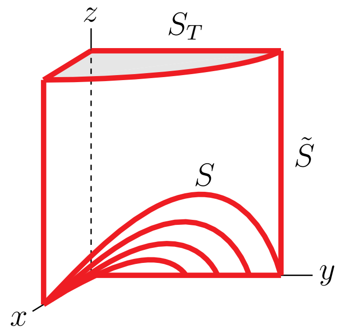
$$\begin{aligned}
\tilde{S} &= \big\{\ (x,y,z)\ \big|\ x^2 + y^2 = 9,\ xy \leqslant z \leqslant 10\ \big\} \\
V &= \big\{\ (x,y,z)\ \big|\ x^2 + y^2 \leqslant 9,\ xy \leqslant z \leqslant 10\ \big\}
\end{aligned}$$

Note that the boundary, ∂V, of V consists of three parts:

- the side $\tilde{S}$, with outward pointing normal (which is the surface and the normal specified in part (c) of the question)

- the bottom, which is the surface S of part (b), with downward pointing normal (which is opposite the normal specified in part (b)) and
- the top, which is the surface $S_T = \{\, (x,y,z) \mid x^2+y^2 \leqslant 9,\ z=10 \,\}$, with normal $\hat{\mathbf{n}} = \hat{\mathbf{k}}$.

Here is a sketch of the part of ∂V that is in the first octant.



Note that $\nabla \cdot \mathbf{F} = 0$. So the divergence theorem yields

$$\begin{aligned} 0 &= \iiint_V \nabla \cdot \mathbf{F}\, \mathrm{d}V \\ &= \iint_{\tilde{\partial} V} \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S \\ &= \iint_{\tilde{S}} \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S - \iint_{S} \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S + \iint_{S_T} \mathbf{F} \cdot \hat{\mathbf{k}}\, \mathrm{d}S \end{aligned}$$

This implies

$$\begin{aligned} \iint_{\tilde{S}} \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S &= \iint_{S} \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S - \iint_{S_T} \mathbf{F} \cdot \hat{\mathbf{k}}\, \mathrm{d}S \\ &= -\tfrac{27\pi}{2} - \iint_{x^2+y^2 \leqslant 9} 3\, \mathrm{d}S \\ &= -\tfrac{27\pi}{2} - 3\pi 3^2 = -\tfrac{81\pi}{2} \end{aligned}$$

S-40: (a) The divergence of $\mathbf{F}$ is

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \tfrac{\partial}{\partial x}(x + \sin y) + \tfrac{\partial}{\partial y}(z+y) + \tfrac{\partial}{\partial z}(z^2) \\ &= 2 + 2z \end{aligned}$$

(b) Set

$$\begin{aligned}
V &= \{\, (x,y,z) \mid x^2+y^2 \leqslant 25,\ 0 \leqslant z \leqslant \sqrt{25-x^2-y^2} \,\} \\
S_T &= \{\, (x,y,z) \mid x^2+y^2+z^2 = 25,\ z \geqslant 0 \,\} \\
S_B &= \{\, (x,y,z) \mid x^2+y^2 \leqslant 25,\ z = 0 \,\}
\end{aligned}$$

Note that the boundary, ∂V, of V consists ot two parts — S_T with upward normal, and S_B with normal $-\hat{\mathbf{k}}$. We are to find the flux through S_T with upward normal. By the divergence theorem, it is

$$\begin{aligned}
\iint_{S_T} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iiint_V \nabla\cdot\mathbf{F}\,\mathrm{d}V - \iint_{S_B} \mathbf{F}\cdot(-\hat{\mathbf{k}})\,\mathrm{d}S \\
&= \iiint_V (2+2z)\,\mathrm{d}V
\end{aligned}$$

since $\mathbf{F}\cdot\hat{\mathbf{k}} = z^2 = 0$ on S_B. We'll compute the volume integral by expressing it as an iterated integral, with the z integration on the outside. In V, z ranges for 0 to 5. The set of points at exactly height z in V is $\{\, (x,y,z) \mid x^2+y^2 \leqslant 25-z^2 \,\}$. So

$$\begin{aligned}
\iint_{S_T} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \int_0^5 \mathrm{d}z \iint_{x^2+y^2\leqslant 25-z^2} \mathrm{d}x\,\mathrm{d}y\ (2+2z) = \int_0^5 \mathrm{d}z\ (2+2z) \iint_{x^2+y^2\leqslant 25-z^2} \mathrm{d}x\,\mathrm{d}y \\
&= \int_0^5 \mathrm{d}z\ \pi(25-z^2)(2+2z)
\end{aligned}$$

since $\iint_{x^2+y^2\leqslant 25-z^2} \mathrm{d}x\,\mathrm{d}y$ is the area of a disk of radius $\sqrt{25-z^2}$

$$\begin{aligned}
&= \pi\int_0^5 \mathrm{d}z\ (50 - 2z^2 + 50z - 2z^3) \\
&= \pi\Big(50\times 5 - 2\frac{5^3}{3} + 50\frac{5^2}{2} - 2\frac{5^4}{4}\Big) = \pi 5^3\Big(2 - \frac{2}{3} + 5 - \frac{5}{2}\Big) = \pi\, 5^3\Big(\frac{4}{3} + \frac{5}{2}\Big) \\
&= \pi\frac{23}{6}5^3 = 479\frac{1}{6}\pi
\end{aligned}$$

(c) To start, consider any closed surface S that is the boundary of a solid V. Use

- the outward pointing normal for S,
- $|V|$ to denote the volume of V, and
- $\bar{z} = \frac{1}{|V|}\iiint_V z\,\mathrm{d}V$ to denote the z-component of the centroid (i.e. centre of mass with constant density) of V.

Then, by the divergence theorem

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iiint_V \nabla\cdot\mathbf{F}\,\mathrm{d}V = \iiint_V (2+2z)\,\mathrm{d}V = 2\iiint_V \mathrm{d}V + 2\iiint_V z\,\mathrm{d}V \\
&= 2|V| + 2|V|\bar{z}
\end{aligned}$$

This takes the value -9 if and only if

$$2|V|\bar{z} = -9 - 2|V| \iff \bar{z} = -\frac{9}{2|V|} - 1$$

One surface which obeys this condition is the unit cube (with outward normal) centred on $\big(0, 0, -\frac{11}{2}\big)$.

S-41: (a) The constant z cross-section of the cone at height $0 \leqslant z \leqslant 1$ is a circle of radius $2z$. So we may parametrize the cone by

$$\mathbf{r}(\theta, z) = 2z\cos\theta\,\hat{\boldsymbol{\imath}} + 2z\sin\theta\,\hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}} \qquad 0 \leqslant \theta < 2\pi,\ 0 \leqslant z \leqslant 1$$

Since

$$\begin{aligned} \tfrac{\partial \mathbf{r}}{\partial \theta} &= \big(-2z\sin\theta\,,\,2z\cos\theta\,,\,0\big) \\ \tfrac{\partial \mathbf{r}}{\partial z} &= \big(2\cos\theta\,,\,2\sin\theta\,,\,1\big) \\ \tfrac{\partial \mathbf{r}}{\partial \theta} \times \tfrac{\partial \mathbf{r}}{\partial z} &= \big(2z\cos\theta\,,\,2z\sin\theta\,,\,-4z\big) \end{aligned}$$

(3.3.1) in the CLP-4 text yields that the element of surface area for this parametrization is

$$\begin{aligned} \mathrm{d}S &= \big|\tfrac{\partial \mathbf{r}}{\partial \theta} \times \tfrac{\partial \mathbf{r}}{\partial z}\big|\mathrm{d}\theta\mathrm{d}z \ = \ 2z\big|(\cos\theta\,,\,\sin\theta\,,\,-2)\big|\mathrm{d}\theta\mathrm{d}z \\ &= 2\sqrt{5}z\,\mathrm{d}\theta\mathrm{d}z \end{aligned}$$

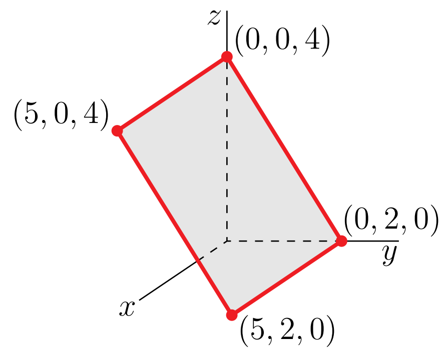
In our parametrization the condition $x \leqslant y$ becomes $2z\cos\theta \leqslant 2z\sin\theta$, which, for $z > 0$, is equivalent to $\tan\theta \geqslant 1$. So the specified integral is

$$\iint_S z^2\,\mathrm{d}S = 2\sqrt{5}\int_0^1 \mathrm{d}z \int_{\pi/4}^{\pi/2} \mathrm{d}\theta\, z^3 = \frac{\sqrt{5}\pi}{2}\int_0^1 \mathrm{d}z\, z^3 = \frac{\sqrt{5}\pi}{8}$$

(b) Let's first do some strategizing. We have to compute a flux integral over a surface that is not closed. There are two potential sneaky attacks that come to mind.

- The first uses Stokes' theorem. But the flux integral in Stokes' theorem is of the form $\iint_S \boldsymbol{\nabla} \times \mathbf{A} \cdot \hat{\mathbf{n}}\,\mathrm{d}S$. So to be able to apply Stokes' theorem in the current problem, $\mathbf{F}$ has to be of the form $\boldsymbol{\nabla} \times \mathbf{A}$. That is, $\mathbf{F}$ has to have a vector potential. We know that in order for $\mathbf{F}$ to have a vector potential, it must pass the screening test $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$. Our $\mathbf{F} = z\,\hat{\mathbf{k}}$ fails this screening test. So we can't use Stokes' theorem.
- The second uses the divergence theorem. But the flux integral in the divergence theorem is over the boundary of a solid. That is not the case for our S. So in order to apply the divergence theorem in the current problem, we have to enlarge S to the boundary of a solid. There are many ways to do this. But they all appear fairly complicated. So it does not seem wise to use the divergence theorem.

So it looks like we have to evaluate the flux integral directly. To do so, we have to determine $\hat{\mathbf{n}}\,\mathrm{d}S$ for the specified rectangle. Look at the sketch of S below. It is part of a plane, and that plane is invariant under translations parallel to the x axis. As the plane



does not pass through the origin, the equation of the plane has to be of the form $by + cz = 1$. For $(0,0,4)$ to be on the plane, we need $c = \frac{1}{4}$. For $(0,2,0)$ to be on the plane, we need $b = \frac{1}{2}$. So S is contained in the plane $G(x,y,z) = \frac{y}{2} + \frac{z}{4} = 1$ and equation (3.3.3) in the CLP-4 text gives that

$$\hat{\mathbf{n}}\,\mathrm{d}S = \pm\frac{\boldsymbol{\nabla} G(x,y,z)}{\boldsymbol{\nabla} G(x,y,z)\cdot\hat{\mathbf{k}}}\mathrm{d}x\mathrm{d}y = \pm\frac{(0,1/2,1/4)}{1/4}\mathrm{d}x\mathrm{d}y = \pm(0,2,1)\,\mathrm{d}x\mathrm{d}y$$

The problem specifies that the normal is to be upward, i.e. is to have a positive z-component. So

$$\hat{\mathbf{n}}\,\mathrm{d}S = (0,2,1)\,\mathrm{d}x\mathrm{d}y$$

Again looking at the sketch of S above we see, as (x,y,z) runs over S, (x,y) runs over

$$R = \left\{\ (x,y)\ \middle|\ 0 \leqslant x \leqslant 5,\ 0 \leqslant y \leqslant 2\ \right\}$$

Thus our flux integral is

$$\begin{aligned}\iint\limits_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iint\limits_R \overbrace{(4-2y)}^{z}\,\hat{\mathbf{k}}\cdot\overbrace{(0,2,1)\,\mathrm{d}x\mathrm{d}y}^{\hat{\mathbf{n}}\,\mathrm{d}S} = \int_0^2 \mathrm{d}y\int_0^5 \mathrm{d}x\ (4-2y)\\ &= \int_0^2 \mathrm{d}y\ 5(4-2y) = 5\Big[4y-y^2\Big]_0^2 = 20\end{aligned}$$

(c) The divergence of the given vector field is $\boldsymbol{\nabla}\cdot\mathbf{F} = 2z$, which is pretty simple. So let's use the divergence theorem. If $V = \left\{\ (x,y,z)\ \middle|\ 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 2, 0 \leqslant z \leqslant 3\ \right\}$, the divergence theorem says that

$$\iint\limits_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint\limits_V \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V = 2\iiint\limits_V z\,\mathrm{d}V$$

This integral would be easy enough to evaluate directly, but we don't need to. The average value of z (i.e. the z-coordinate of the centre of mass with constant density) is $\frac{3}{2}$, by symmetry. Since V has volume 6, that average value of z is also

$$\bar{z} = \frac{1}{6}\iiint\limits_V z\,\mathrm{d}V = \frac{3}{2}$$

So $\iiint_V z\,\mathrm{d}V = 9$

$$\iint\limits_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 2\iiint\limits_V z\,\mathrm{d}V = 18$$

S-42: (a) For the surface $z = f(x,y) = 1 - x^2 - y^2$, with an upwards pointing normal,

$$\hat{\mathbf{n}}\,\mathrm{d}S = \big[-f_x(x,y)\,\mathrm{d}x - f_y(x,y) + \hat{\mathbf{k}}\big]\mathrm{d}x\mathrm{d}y = \big[2x\,\hat{\imath} + 2y\,\hat{\jmath} + \hat{\mathbf{k}}\big]\,\mathrm{d}x\mathrm{d}y$$

by (3.3.2) in the CLP-4 text. So the specified upward flux is

$$\begin{aligned}
&\iint\limits_{\sigma_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S \\
&\qquad = \iint\limits_{x^2+y^2\leqslant 1} \big\{[a(y^2+z^2)+bxz]\,\hat{\imath} + [c(x^2+z^2)+dyz]\,\hat{\jmath} + x^2\,\hat{\mathbf{k}}\big\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot \big\{2x\,\hat{\imath} + 2y\,\hat{\jmath} + \hat{\mathbf{k}}\big\}_{z=1-x^2-y^2}\,\mathrm{d}x\mathrm{d}y \\
&\qquad = \iint\limits_{x^2+y^2\leqslant 1} \big\{[2ax(y^2+z^2) + 2bx^2z] + [2cy(x^2+z^2) + 2dy^2z] + x^2\big\}_{z=1-x^2-y^2}\,\mathrm{d}x\mathrm{d}y
\end{aligned}$$

Now

$$\iint\limits_{x^2+y^2\leqslant 1} \big\{2ax(y^2+z^2)\big\}_{z=1-x^2-y^2}\,\mathrm{d}x\mathrm{d}y = 0$$

because the integrand is odd under $x\to -x$ and

$$\iint\limits_{x^2+y^2\leqslant 1} \big\{2cy(x^2+z^2)\big\}_{z=1-x^2-y^2}\,\mathrm{d}x\mathrm{d}y = 0$$

because the integrand is odd under $y \to -y$. So that leaves

$$\iint\limits_{\sigma_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint\limits_{x^2+y^2\leqslant 1} \big\{2bx^2z + 2dy^2z + x^2\big\}_{z=1-x^2-y^2}\,\mathrm{d}x\mathrm{d}y$$

We'll switch to polar coordinates to evaluate the remaining integral.

$$\iint\limits_{\sigma_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_0^1 \mathrm{d}r\,r\int_0^{2\pi}\mathrm{d}\theta\,\big\{2br^2z\cos^2\theta + 2dr^2z\sin^2\theta + r^2\cos^2\theta\big\}_{z=1-r^2}$$

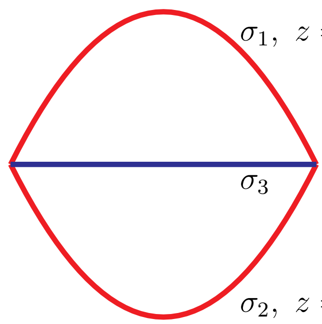
Now

$$\begin{aligned}
\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta &= \int_0^{2\pi}\frac{\cos(2\theta)+1}{2}\,\mathrm{d}\theta = \left[\frac{\sin(2\theta)}{4} + \frac{\theta}{2}\right]_0^{2\pi} = \pi \\
\int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta &= \int_0^{2\pi}\frac{1-\cos(2\theta)}{2}\,\mathrm{d}\theta = \left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right]_0^{2\pi} = \pi
\end{aligned}$$

For an efficient, sneaky, way to evaluate $\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta$ and $\int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta$, see Example 2.4.4 in the CLP-4 text. So, we finally have

$$\begin{aligned}\iint_{\sigma_1}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \int_0^1 \mathrm{d}r\,\left\{2\pi br^3(1-r^2)+2\pi dr^3(1-r^2)+\pi r^3\right\}\\ &= 2\pi b\Big[\frac{1}{4}-\frac{1}{6}\Big]+2\pi d\Big[\frac{1}{4}-\frac{1}{6}\Big]+\pi\frac{1}{4}=\frac{\pi}{4}+\frac{\pi(b+d)}{6}\end{aligned}$$

(b), (c) Here is a side view of σ_1, σ_2 and σ_3.



Set

$$\begin{aligned}V_b &= \left\{\,(x,y,z)\;\middle|\;0\leqslant z\leqslant 1-x^2-y^2,\ x^2+y^2\leqslant 1\,\right\}\\ V_c &= \left\{\,(x,y,z)\;\middle|\;x^2+y^2-1\leqslant z\leqslant 1-x^2-y^2,\ x^2+y^2\leqslant 1\,\right\}\end{aligned}$$

Then $\partial V_b=\sigma_1\cup\sigma_3$ and $\partial V_c=\sigma_1\cup\sigma_2$, all with outward pointing normals. Since the divergence of $\mathbf{F}$ is

$$\boldsymbol{\nabla}\cdot\mathbf{F}=\frac{\partial}{\partial x}[a(y^2+z^2)+bxz]+\frac{\partial}{\partial y}[c(x^2+z^2)+dyz]+\frac{\partial}{\partial z}[x^2]=(b+d)z$$

the divergence theorem gives

$$\begin{aligned}\iint_{\sigma_1\cup\sigma_3}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iiint_{V_b}\boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V=(b+d)\iiint_{V_b}z\,\mathrm{d}V\\ \iint_{\sigma_1\cup\sigma_2}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iiint_{V_c}\boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V=(b+d)\iiint_{V_c}z\,\mathrm{d}V\end{aligned}$$

Now on V_b, $z\geqslant 0$ and $z>0$ except on σ_3. So $\iiint_{V_b}z\,\mathrm{d}V>0$ and $\iint_{\sigma_1\cup\sigma_3}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ is zero if and only if $d=-b$. That's the answer to part (b).

On the other hand, V_c is even under $z\to -z$ so that $\iiint_{V_c}z\,\mathrm{d}V=0$. Consequently $\iint_{\sigma_1\cup\sigma_3}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ is zero for all a, b, c, d. That's the answer to part (c).

S-43: We will be using the divergence theorem in both parts (a) and b. So as a prelimary calculation, let's find the divergence of $\mathbf{H}(x,y,z) = \frac{(x,y,z)-(a,b,c)}{[(x-a)^2+(y-b)^2+(z-b)^2]^{3/2}}$ for any (a,b,c). If $(x,y,z) \neq (a,b,c)$,

$$\begin{aligned}
&\boldsymbol{\nabla}\cdot\mathbf{H}(x,y,z)\\
&\quad= \frac{\partial}{\partial x}\frac{x-a}{[(x-a)^2+(y-b)^2+(z-c)^2]^{3/2}} + \frac{\partial}{\partial y}\frac{y-b}{[(x-a)^2+(y-b)^2+(z-c)^2]^{3/2}}\\
&\qquad\qquad + \frac{\partial}{\partial z}\frac{z-c}{[(x-a)^2+(y-b)^2+(z-c)^2]^{3/2}}\\
&\quad= \frac{\big[(x-a)^2+(y-b)^2+(z-c)^2\big] \,-\, (x-a)\,\frac{3}{2}(2(x-a))}{\big[(x-a)^2+(y-b)^2+(z-c)^2\big]^{5/2}}\\
&\qquad\qquad + \frac{\big[(x-a)^2+(y-b)^2+(z-c)^2\big] \,-\, (y-b)\,\frac{3}{2}(2y)}{\big[(x-a)^2+(y-b)^2+(z-c)^2\big]^{5/2}}\\
&\qquad\qquad + \frac{\big[(x-a)^2+(y-b)^2+z^2\big] \,-\, (z-c)\,\frac{3}{2}(2(z-c))}{\big[(x-a)^2+(y-b)^2+(z-c)^2\big]^{5/2}}\\
&\quad= \frac{3\big[(x-a)^2+(y-b)^2+(z-c)^2\big] \,-\, 3(x-a)^2-3(y-b)^2-3(z-c)^2}{\big[(x-a)^2+(y-b)^2+(z-c)^2\big]^{5/2}}\\
&\quad= 0
\end{aligned}$$

If $(x,y,z) = (a,b,c)$, $\mathbf{H}(x,y,z)$ is not defined and hence $\boldsymbol{\nabla}\cdot\mathbf{H}(x,y,z)$ is also not defined.

(b) By the above preliminary computation with $(a,b,c) = (3,2,2)$, $\boldsymbol{\nabla}\cdot\mathbf{G}$ is defined and zero for all $(x,y,z) \neq (3,2,2)$, and, in particular for all (x,y,z) in

$$V = \big\{\ (x,y,z)\ \big|\ x^2+2y^2+3z^2 \leqslant 16\ \big\}$$

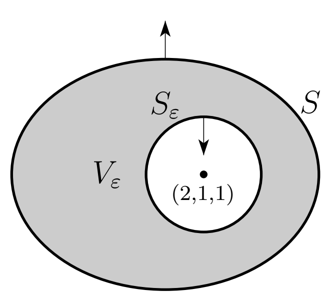
So, by the divergence theorem,

$$\iint_S \mathbf{G}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint_V \boldsymbol{\nabla}\cdot\mathbf{G}\,\mathrm{d}V = 0$$

(a) Because $(1,1,2)$ is inside V, we cannot use the argument of part (b), to conclude that the integral is zero. Let $\varepsilon > 0$ be small enough that

$$S_\varepsilon = \big\{\ (x,y,z)\ \big|\ (x-2)^2+(y-1)^2+(z-1)^2 = \varepsilon^2\ \big\}$$

is completely contained inside V, as in the sketch below.



Set
$$V_\varepsilon = \left\{ \, (x, y, z) \mid x^2 + 2y^2 + 3z^2 \leqslant 16, \ (x-2)^2 + (y-1)^2 + (z-1)^2 \geqslant \varepsilon^2 \, \right\}$$
The boundary, ∂V_ε, of V consists of two parts — S and S_ε, with the normals as in the figure above. The divergence $\boldsymbol{\nabla} \cdot \mathbf{F}$ of $\mathbf{F}$ is well-defined and zero throughout V_ε. Consequently, the divergence theorem gives

$$0 = \iiint_V \boldsymbol{\nabla} \cdot \mathbf{F} \, \mathrm{d}V = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S + \iint_{S_\varepsilon} \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$$

So
$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = -\iint_{S_\varepsilon} \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$$
The unit normal to S_ε at the point (x, y, z) on S_ε is

$$\hat{\mathbf{n}} = -\frac{1}{\varepsilon}\big[(x-2)\,\hat{\boldsymbol{\imath}} + (y-1)\,\hat{\boldsymbol{\jmath}} + (z-1)\,\hat{\mathbf{k}}\big]$$

(Recall that $|(2-1)\,\hat{\boldsymbol{\imath}} + (y-1)\,\hat{\boldsymbol{\jmath}} + (z-1)\,\hat{\mathbf{k}}| = \varepsilon$ on S_ε. So, on S_ε,

$$\begin{aligned}
\mathbf{F} \cdot \hat{\mathbf{n}} &= -\frac{1}{\varepsilon}\left(\frac{(x,y,z)-(2,1,1)}{\big[(x-2)^2+(y-1)^2+(z-1)^2\big]^{3/2}}\right) \cdot \big[(x-2)\,\hat{\boldsymbol{\imath}} + (y-1)\,\hat{\boldsymbol{\jmath}} + (z-1)\,\hat{\mathbf{k}}\big] \\
&= -\frac{1}{\varepsilon}\left(\frac{(x-2)^2+(y-1)^2+(z-1)^2}{\big[(x-2)^2+(y-1)^2+(z-1)^2\big]^{3/2}}\right) \\
&= -\frac{1}{\varepsilon^2}
\end{aligned}$$

Hence
$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = -\iint_{S_\varepsilon} \left(-\frac{1}{\varepsilon^2}\right) \mathrm{d}S = \frac{1}{\varepsilon^2}(4\pi\varepsilon^2) = 4\pi$$

S-44: This was part of Theorem 4.2.9 in the CLP-4 text. To prove it apply the divergence theorem, but with $\mathbf{F}$ replaced by $\mathbf{a} \times \mathbf{F}$, where $\mathbf{a}$ is any constant vector.

$$\begin{aligned}
\iint_{\partial\Omega} (\mathbf{a} \times \mathbf{F}) \cdot \hat{\mathbf{n}} \, \mathrm{d}S &= \iiint_V \boldsymbol{\nabla} \cdot (\mathbf{a} \times \mathbf{F}) \, \mathrm{d}V \\
&= \iiint_\Omega \big[\mathbf{F} \cdot \underbrace{(\boldsymbol{\nabla} \times \mathbf{a})}_{=\mathbf{0}} - \mathbf{a} \cdot (\boldsymbol{\nabla} \times \mathbf{F})\big] \, \mathrm{d}V \\
&= -\iiint_\Omega \mathbf{a} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) \, \mathrm{d}V = -\mathbf{a} \cdot \iiint_\Omega \boldsymbol{\nabla} \times \mathbf{F} \, \mathrm{d}V
\end{aligned}$$

To get the second line we used the vector identity Theorem 4.1.4.d in the CLP-4 text. To get the third line, we used that $\mathbf{a}$ is a constant, so that all of its derivatives are zero. For all vectors $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ (in case you don't remember this, it was Lemma 4.1.8.a in the CLP-4 text) so that

$$(\mathbf{a} \times \mathbf{F}) \cdot \hat{\mathbf{n}} = \mathbf{a} \cdot (\mathbf{F} \times \hat{\mathbf{n}})$$

and

$$\begin{aligned}
\mathbf{a} \cdot \iint_{\partial\Omega} \mathbf{F} \times \mathbf{n}\, \mathrm{d}S &= -\mathbf{a} \cdot \iiint_{\Omega} \boldsymbol{\nabla} \times \mathbf{F}\, \mathrm{d}V \\
\implies \mathbf{a} \cdot \left\{ \iint_{\partial\Omega} \mathbf{F} \times \mathbf{n}\, \mathrm{d}S + \iiint_{\Omega} \boldsymbol{\nabla} \times \mathbf{F}\, \mathrm{d}V \right\} &= 0
\end{aligned}$$

In particular, choosing $\mathbf{a} = \hat{\imath}, \hat{\jmath}$ and $\hat{\mathbf{k}}$, we see that all three components of the vector $\iint_{\partial\Omega} \mathbf{F} \times \mathbf{n}\, \mathrm{d}S + \iiint_{\Omega} \boldsymbol{\nabla} \times \mathbf{F}\, \mathrm{d}V$ are zero. So

$$\iiint_{\Omega} \boldsymbol{\nabla} \times \mathbf{F}\, \mathrm{d}V = -\iint_{\partial\Omega} \mathbf{F} \times \mathbf{n}\, \mathrm{d}S = \iint_{\partial\Omega} \hat{\mathbf{n}} \times \mathbf{F}\, \mathrm{d}S$$

which is what we wanted show.

S-45: Pressure is force per unit surface area acting normally into a surface. So the force per unit surface area is $-p\hat{\mathbf{n}}$. The total force acting on S is

$$-\iint_{S} p\hat{\mathbf{n}}\, \mathrm{d}S = -\iiint_{E} \nabla p\, \mathrm{d}V$$

We are assuming that p is a constant, so that $\nabla p = 0$ and the total force is zero.

S-46: Let S_a denote the sphere $x^2 + y^2 + z^2 = a^2$ and V_a denote the solid inside it, which is the ball $x^2 + y^2 + z^2 \leqslant a^2$. Then, by the divergence theorem, Theorem 4.2.2 in the CLP-4 text,

$$\pi(a^3 + 2a^4) = \iint_{S_a} \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iiint_{V_a} \boldsymbol{\nabla} \cdot \mathbf{F}\, \mathrm{d}V$$

Now, for very small a, $\boldsymbol{\nabla} \cdot \mathbf{F}$ is almost equal to $\boldsymbol{\nabla} \cdot \mathbf{F}(0,0,0)$ on all of V_a, and the integral $\iiint_{V_a} \boldsymbol{\nabla} \cdot \mathbf{F}\, \mathrm{d}V$ will be

$$\boldsymbol{\nabla} \cdot \mathbf{F}(0,0,0)\mathrm{Volume}(V_a) + O(a^4) = \frac{4}{3}\pi a^3 \boldsymbol{\nabla} \cdot \mathbf{F}(0,0,0) + O(a^4)$$

Here $O(a^4)$ is an error term that is bounded by a constant times a^4. This is consistent with the above equation if and only if $\boldsymbol{\nabla} \cdot \mathbf{F}(0,0,0) = \frac{3}{4}$.

S-47: Note that, since $z^2 - 2az = (z-a)^2 - a^2$,

$$S = \{\ (x, y, z \mid x^2 + y^2 + (z-a)^2 = 4a^2,\ z \geqslant 0\ \}$$

Let V be the solid

$$V = \{ \, (x,y,z) \mid x^2 + y^2 + (z-a)^2 \leqslant 4a^2, \; z \geqslant 0 \, \}$$

It is the interior of the sphere of radius $2a$ centred on $(0,0,a)$. The surface of V (with outward normal) is the union of S (with normal pointing away from the origin) and the disk

$$B = \{ \, (x,y,0) \mid x^2 + y^2 \leqslant 3a^2 \, \}$$

with normal $-\hat{\mathbf{k}}$. Hence, by the Divergence Theorem

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S &= \iiint_V \mathbf{n} \cdot \mathbf{F} \, \mathrm{d}V - \iint_B \mathbf{F} \cdot (-\hat{\mathbf{k}}) \, \mathrm{d}S \\
&= \iiint_V (2x + 2y + 1) \, \mathrm{d}V - \iint_B (-3 - x) \, \mathrm{d}S
\end{aligned}$$

Both V and B are invariant under $x \to -x$ and under $y \to -y$, so $\iiint_V x \, \mathrm{d}V = \iiint_V y \, \mathrm{d}V = \iint_B x \, \mathrm{d}S = 0$ and

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = \iiint_V \mathrm{d}V + 3 \iint_B \mathrm{d}S$$

To evaluate the integral over V, we note that z runs from 0 to $3a$ and that the cross section of

$$V = \{ \, (x,y,z) \mid 0 \leqslant z \leqslant 3a, \; x^2 + y^2 \leqslant 4a^2 - (z-a)^2, \; z \geqslant 0 \, \}$$

with fixed z is the circular disk $x^2 + y^2 \leqslant 4a^2 - (z-a)^2 = 3a^2 + 2az - z^2$, which has area $\pi \big(\sqrt{3a^2 + 2az - z^2} \, \big)^2$. So

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S &= \int_0^{3a} \pi \big(\sqrt{3a^2 + 2az - z^2} \, \big)^2 \, \mathrm{d}z + 3 \, \mathrm{Area}(B) \\
&= \pi \int_0^{3a} (3a^2 + 2az - z^2) \, \mathrm{d}z + 3\pi(3a^2) \\
&= \pi \left(3a^2 \times 3a + 2a \times \frac{9a^2}{2} - \frac{27a^3}{3} \right) + 9\pi a^2 \\
&= 9\pi a^3 + 9\pi a^2
\end{aligned}$$

S-48: (a) Let $\mathcal{S}$ denote the boundary of $\mathcal{R}$. Then "the total flux of $\mathbf{F} = \nabla u$ out through the boundary of $\mathcal{R}$" is given by the integral

$$I = \iint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$$

Thanks to the divergence theorem,

$$I = \iiint_{\mathcal{R}} \nabla \cdot \mathbf{F} \, \mathrm{d}V = \iiint_{\mathcal{R}} \nabla \cdot \nabla u \, \mathrm{d}V = \iiint_{\mathcal{R}} \Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \Big) \, \mathrm{d}V = 0$$

(b) Similarly, "the total flux of $\mathbf{G} = u\nabla u$ out through the boundary of $\mathcal{R}$" equals

$$J = \iint_{\mathcal{S}} \mathbf{G}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint_{\mathcal{R}} \nabla\cdot\mathbf{G}\,\mathrm{d}V$$

Here $\mathbf{G} = u\mathbf{F}$ (using the notation from part (a)), so by the vector identity of Theorem 4.1.4.c in the CLP-4 text,

$$\nabla\cdot\mathbf{G} = \nabla\cdot(u\mathbf{F}) = (\nabla u)\cdot\mathbf{F} + u(\nabla\cdot\mathbf{F})$$

But $\mathbf{F} = \nabla u$, so $\nabla\cdot\mathbf{F} = \Delta u = 0$ as in part (a), giving

$$\nabla\cdot\mathbf{G} = |\nabla u|^2 + 0$$

In conclusion,

$$J = \iiint_{\mathcal{R}} \nabla\cdot\mathbf{G}\,\mathrm{d}V = \iiint_{\mathcal{R}} \Big[\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial u}{\partial y}\Big)^2 + \Big(\frac{\partial u}{\partial z}\Big)^2\Big]\,\mathrm{d}V$$

S-49: (a) This is a classic case for the divergence theorem. The flux we want equals

$$I = \iint_{\mathcal{S}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint_{\mathcal{R}} \nabla\cdot\mathbf{F}\,\mathrm{d}V = \iiint_{\mathcal{R}} (2x+2-2)\,\mathrm{d}V = 2\iiint_{\mathcal{R}} x\,\mathrm{d}V$$

The solid $\mathcal{R}$ clearly has reflection symmetry across the plane $x = 0$. So the x-coordinate of the centre of mass of $\mathcal{R}$, i.e. the average value of x over $\mathcal{R}$, i.e.

$$\bar{x} = \frac{\iiint_{\mathcal{R}} x\,\mathrm{d}V}{\iiint_{\mathcal{R}} \mathrm{d}V} = \frac{\iiint_{\mathcal{R}} x\,\mathrm{d}V}{\mathrm{Vol}(\mathcal{R})}$$

is zero. Hence

$$I = 2\bar{x}\mathrm{Vol}(\mathcal{R}) = 0$$

Alternatively, here is a direct evaluation of $2\iiint_{\mathcal{R}} x\,\mathrm{d}V$. The base region $x^2 + (y-1)^2 \leqslant 1$ is the circular disk of radius 1 centred on $(0,1)$. In polar coordinates it is

$$r^2\cos^2\theta + (r\sin\theta - 1)^2 \leqslant 1 \qquad \text{or} \qquad r^2 - 2r\sin\theta + 1 \leqslant 1 \qquad \text{or} \qquad r \leqslant 2\sin\theta$$

Because the disk is contained in the upper half plane, the polar angle θ is restricted to $0 \leqslant \theta \leqslant \pi$. So, in cylindrical coordinates, the solid $\mathcal{R}$ is described by

$$0 \leqslant \theta \leqslant \pi, \qquad 0 \leqslant r \leqslant 2\sin\theta, \qquad 0 \leqslant z \leqslant r^2\sin^2\theta$$

Hence

$$\begin{aligned}
I &= 2\int_{\theta=0}^{\pi}\int_{r=0}^{2\sin\theta}\int_{z=0}^{r^2\sin^2\theta} (r\cos\theta)\,\mathrm{d}z\,r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= 2\int_{\theta=0}^{\pi}\int_{r=0}^{2\sin\theta} r^4\sin^2\theta\cos\theta\,\mathrm{d}r\,\mathrm{d}\theta \\
&= 2\int_{\theta=0}^{\pi} \sin^2\theta\cos\theta\Big[\frac{2^5\sin^5\theta}{5}\Big]\,\mathrm{d}\theta = \frac{64}{5}\int_{\theta=0}^{\pi}\sin^7\theta\cos\theta\,\mathrm{d}\theta = \frac{64}{5}\Big[\frac{\sin^8\theta}{8}\Big]_{\theta=0}^{\pi} \\
&= 0
\end{aligned}$$

(b) *using part (a):* We have

$$\iint_{\mathcal{S}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_{\mathcal{S}_{\text{bottom}}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \iint_{\mathcal{S}_{\text{top}}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \iint_{\mathcal{S}_{\text{side}}} \mathbf{F}\cdot\hat{\mathbf{n}}\mathrm{d}S$$

On $\mathcal{S}_{\text{bottom}}$, $z = 0$ and the outward unit normal is $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$, so $\mathbf{F}\cdot\hat{\mathbf{n}} = 0$. Hence

$$\iint_{\mathcal{S}_{\text{bottom}}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_{\mathcal{S}_{\text{bot}}} 0\,\mathrm{d}S = 0$$

On $\mathcal{S}_{\text{top}}$, $z = y^2$, so $\mathbf{F} = (2x, 2y, -2y^2)$ and, by (3.3.2) of the CLP-4 text,

$$\hat{\mathbf{n}}\mathrm{d}S = (0, -2y, 1)\,\mathrm{d}x\mathrm{d}y$$

Hence (by the Hint)

$$\begin{aligned}
\iint_{\mathcal{S}_{\text{bot}}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iint_{\mathcal{D}} [-4y^2 - 2y^2]\,\mathrm{d}x\mathrm{d}y \\
&= -6\int_{\theta=0}^{\pi}\int_{r=0}^{2\sin\theta} \big(r^2\sin^2\theta\big)\,r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= -6\frac{2^4}{4}\int_{\theta=0}^{\pi}\sin^6\theta\,\mathrm{d}\theta = -24\frac{5}{6}\int_{\theta=0}^{\pi}\sin^4\theta\,\mathrm{d}\theta = -24\frac{5}{6}\frac{3}{4}\int_{\theta=0}^{\pi}\sin^2\theta\,\mathrm{d}\theta \\
&= -24\frac{5}{6}\frac{3}{4}\frac{1}{2}\int_{\theta=0}^{\pi}\mathrm{d}\theta = -24\Big[\frac{5}{6}\frac{3}{4}\frac{1}{2}\pi\Big] = -\frac{15}{2}\pi
\end{aligned}$$

The conclusion is

$$\iint_{\mathcal{S}_{\text{side}}} \mathbf{F}\cdot\hat{\mathbf{n}}\mathrm{d}S = \iint_{\mathcal{S}} \mathbf{F}\cdot\hat{\mathbf{n}}\mathrm{d}S - \iint_{\mathcal{S}_{\text{top}}} \mathbf{F}\cdot\hat{\mathbf{n}}\mathrm{d}S - \iint_{\mathcal{S}_{\text{bot}}} \mathbf{F}\cdot\hat{\mathbf{n}}\mathrm{d}S = \frac{15}{2}\pi$$

(b) *by direct evaluation:* Use the polar equation $r = 2\sin\theta$ to parametrize $\mathcal{S}_{\text{side}}$:

$$\mathbf{r}(\theta,t) = (r\cos\theta, r\sin\theta, t) = (2\sin\theta\cos\theta, 2\sin^2\theta, t), \qquad 0 \leqslant \theta \leqslant \pi,\ 0 \leqslant t \leqslant y^2 = 4\sin^4\theta$$

Then using (3.3.1) in the CLP-4 text,

$$\begin{aligned}
\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \mathbf{F}\cdot\Big(\frac{\partial\mathbf{r}}{\partial\theta}\times\frac{\partial\mathbf{r}}{\partial t}\Big)\,\mathrm{d}\theta\,\mathrm{d}t &= \det\begin{bmatrix} 4\sin^2\theta\cos^2\theta & 4\sin^2\theta & -2t \\ 2(\cos^2\theta - \sin^2\theta) & 4\sin\theta\cos\theta & 0 \\ 0 & 0 & 1\end{bmatrix}\,\mathrm{d}\theta\,\mathrm{d}t \\
&= \det\begin{bmatrix} 4\sin^2\theta\cos^2\theta & 4\sin^2\theta \\ 2(\cos^2\theta - \sin^2\theta) & 4\sin\theta\cos\theta\end{bmatrix}\,\mathrm{d}\theta\,\mathrm{d}t \\
&= \big[16\sin^3\theta\cos^3\theta - 8\sin^2\theta\big(\cos^2\theta - \sin^2\theta\big)\big]\,\mathrm{d}\theta\,\mathrm{d}t \\
&= \big[16\sin^3\theta\big(1 - \sin^2\theta\big)\cos\theta - 8\sin^2\theta\big(1 - 2\sin^2\theta\big)\big]\,\mathrm{d}\theta\,\mathrm{d}t \\
&= 8\big[2\sin^3\theta\cos\theta - 2\sin^5\theta\cos\theta - \sin^2\theta + 2\sin^4\theta\big]\,\mathrm{d}\theta\,\mathrm{d}t
\end{aligned}$$

so

$$\iint\limits_{\mathcal{S}_{\rm side}} \mathbf{F}\cdot\hat{\mathbf{n}}\mathrm{d}S = 8\int_{\theta=0}^{\pi}\int_{t=0}^{4\sin^4\theta}\big[2\sin^3\theta\cos\theta - 2\sin^5\theta\cos\theta - \sin^2\theta + 2\sin^4\theta\big]\,\mathrm{d}t\,\mathrm{d}\theta$$

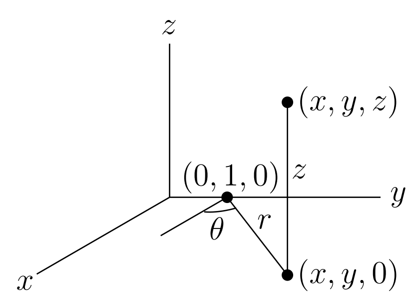
$$= 32\int_{\theta=0}^{\pi}\big[2\sin^7\theta\cos\theta - 2\sin^9\theta\cos\theta - \sin^6\theta + 2\sin^8\theta\big]\,\mathrm{d}\theta$$

$$= 32\Big[2\frac{\sin^8\theta}{8} - 2\frac{\sin^{10}\theta}{10}\Big]_0^{\pi} - 32\int_{\theta=0}^{\pi}\sin^6\theta\,\mathrm{d}\theta + 64\int_{\theta=0}^{\pi}\sin^8\theta\,\mathrm{d}\theta$$

$$= -32\frac{5}{6}\frac{3}{4}\frac{1}{2}\pi + 64\frac{7}{8}\frac{5}{6}\frac{3}{4}\frac{1}{2}\pi \quad \text{(by the Hint as above)}$$

$$= \frac{15}{2}\pi.$$

(b) *Offset polar alternative:* We can also parametrize $\mathcal{S}$ using cylindrical coordinates translated so that the centre of the base of the cylinder, namely $(0,1,0)$, plays the role of the origin. Then, looking at the figure



we see that

$$x = r\cos\theta \qquad y = 1 + r\sin\theta \qquad z = z$$

In these coordinates, the base region, $x^2 + (y-1)^2 \leqslant 1$, $z = 0$, of the cylinder is $0 \leqslant r \leqslant 1$, $z = 0$. So we can parametrize $\mathcal{S}$ by

$$x = \cos\theta,\ y = 1 + \sin\theta,\ z = t, \qquad 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant t \leqslant (1+\sin\theta)^2$$

By (3.3.1) in the CLP-4 text,

$$\frac{\partial\mathbf{r}}{\partial t}\times\frac{\partial\mathbf{r}}{\partial\theta} = \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ 0 & 0 & 1\\ -\sin\theta & \cos\theta & 0\end{bmatrix} = \big(-\cos\theta, -\sin\theta, 0\big),$$

$$\hat{\mathbf{n}}\,\mathrm{d}S = -\frac{\partial\mathbf{r}}{\partial t}\times\frac{\partial\mathbf{r}}{\partial\theta}\,\mathrm{d}t\,\mathrm{d}\theta = (\cos\theta, \sin\theta, 0)\mathrm{d}t\,\mathrm{d}\theta$$

where we have chosen the sign to give the outward pointing normal. So

$$\begin{aligned}
\iint_{\mathcal{S}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \int_{\theta=0}^{2\pi}\int_{t=0}^{(1+\sin\theta)^2} \left[\cos^3\theta + 2(1+\sin\theta)\sin\theta\right]\mathrm{d}t\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \left[(1+\sin\theta)^2\cos^3\theta + 2(1+\sin\theta)^3\sin\theta\right]\mathrm{d}\theta \\
&= \int_0^{2\pi} \left[2\sin\theta\cos^3\theta + 6\sin^2\theta + 2\sin^4\theta\right]\mathrm{d}\theta \\
&= -\frac{1}{2}\cos^4\theta\Big|_0^{2\pi} + 12\int_0^{\pi}\sin^2\theta\,\mathrm{d}\theta + 4\int_0^{\pi}\sin^4\theta\,\mathrm{d}\theta \\
&= 0 + 12\frac{\pi}{2} + 4\frac{3}{4}\frac{\pi}{2} = \frac{15}{2}\pi
\end{aligned}$$

To get the third line, we used that the integral over $0 \leqslant \theta \leqslant 2\pi$ of any odd power of $\sin\theta$ or $\cos\theta$ is zero.

S-50: The circle $x^2 + y^2 = 4y$, or equivalently, $x^2 + (y-2)^2 = 4$, has radius 2 and centre $(0, 2)$. On the bottom surface, $z = 0$ and the outward normal is $-\hat{\mathbf{k}}$, so that

$$\iint_D \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = -\iint_D \mathbf{F}\cdot\hat{\mathbf{k}}\,\mathrm{d}x\mathrm{d}y = -\iint_D (2x+3y)\,\mathrm{d}x\mathrm{d}y$$

By symmetry, the centre of mass, $(\bar{x}, \bar{y})$, of the circle is $(0, 2)$. Here $\bar{x}$ and $\bar{y}$ are the average values

$$\bar{x} = \frac{\iint_D x\,\mathrm{d}x\mathrm{d}y}{\iint_D \mathrm{d}x\mathrm{d}y} \qquad \bar{y} = \frac{\iint_D y\,\mathrm{d}x\mathrm{d}y}{\iint_D \mathrm{d}x\mathrm{d}y}$$

of x and y over D. As the disk D has area 4π,

$$\iint_D x\,\mathrm{d}x\mathrm{d}y = 4\pi\bar{x} = 0 \qquad \iint_D y\,\mathrm{d}x\mathrm{d}y = 4\pi\bar{y} = 8\pi$$

and

$$\iint_D \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = -4\pi(2\bar{x} + 3\bar{y}) = -4\pi(2\times 0 + 3\times 2) = -24\pi$$

As

$$\begin{aligned}
\nabla\cdot\mathbf{F} &= \frac{\partial}{\partial x}(x + x^2y) + \frac{\partial}{\partial y}(y - xy^2) + \frac{\partial}{\partial z}(z + 2x + 3y) = (1 + 2xy) + (1 - 2xy) + (1) \\
&= 3
\end{aligned}$$

the divergence theorem gives

$$\begin{aligned}
\iint_{\mathcal{S}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iiint_R \nabla\cdot\mathbf{F}\,\mathrm{d}V - \iint_D \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S \\
&= \iiint_R 3\,\mathrm{d}V - (-24\pi) = 3\mathrm{Vol}(R) + 24\pi = 3\times 10 + 24\pi \\
&= 30 + 24\pi
\end{aligned}$$

Solutions to Exercises 4.3 — Jump to TABLE OF CONTENTS

S-1: (a) Expressing the left hand side as an iterated integral, with y as the inner integration variable, we have

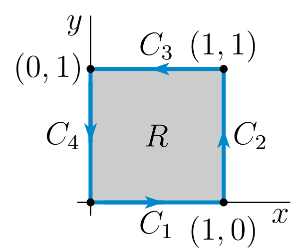
$$\begin{aligned}\iint_R \frac{\partial f}{\partial y}(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^1 \mathrm{d}x\left[\int_0^1 \mathrm{d}y\,\frac{\partial f}{\partial y}(x,y)\right]\\ &= \int_0^1 \mathrm{d}x\,\big[f(x,1)-f(x,0)\big]\\ &\qquad\text{by the fundamental theorem of calculus}\\ &= \int_0^1 f(x,1)\,\mathrm{d}x - \int_0^1 f(x,0)\,\mathrm{d}x\end{aligned}$$

(b) Define $F_1(x,y) = f(x,y)$ and $F_2(x,y) = 0$. Then. by Green's theorem

$$\begin{aligned}\iint_R \frac{\partial f}{\partial y}(x,y)\,\mathrm{d}x\,\mathrm{d}y &= -\iint_R \left[\frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y)\right]\mathrm{d}x\,\mathrm{d}y\\ &= -\int_{\partial R}\big[F_1(x,y)\,\mathrm{d}x + F_2(x,y)\,\mathrm{d}y\big]\\ &= -\int_{\partial R} f(x,y)\,\mathrm{d}x\end{aligned}$$

The boundary of R, oriented counterclockwise, is the union of four line segments.

C_1 from $(0,0)$ to $(1,0)$
C_2 from $(1,0)$ to $(1,1)$
C_3 from $(1,1)$ to $(0,1)$
C_4 from $(0,1)$ to $(0,0)$



Now x is constant on C_2 and C_4 so that

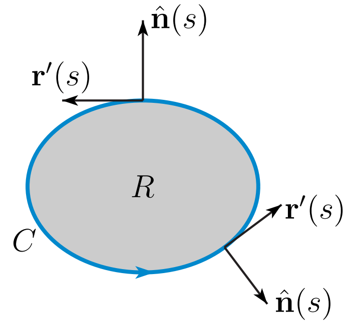
$$\int_{C_2} f(x,y)\,\mathrm{d}x = \int_{C_4} f(x,y)\,\mathrm{d}x = 0$$

So, using $-C_3$ to denote the line segment from $(0,1)$ to $(1,1)$

$$\begin{aligned}\iint_R \frac{\partial f}{\partial y}(x,y)\,\mathrm{d}x\,\mathrm{d}y &= -\left[\int_{C_1} f(x,y)\,\mathrm{d}x + \int_{C_3} f(x,y)\,\mathrm{d}x\right]\\ &= \int_{-C_3} f(x,y)\,\mathrm{d}x - \int_{C_1} f(x,y)\,\mathrm{d}x\\ &= \int_0^1 f(x,1)\,\mathrm{d}x - \int_0^1 f(x,0)\,\mathrm{d}x\end{aligned}$$

S-2: Let $\mathbf{r}(s) = x(s)\,\hat{\imath} + y(s)\,\hat{\jmath}$ be a counterclockwise parametrization of C by arc length. Then $\hat{\mathbf{T}}(s) = \mathbf{r}'(s) = x'(s)\,\hat{\imath} + y'(s)\,\hat{\jmath}$ is the forward pointing unit tangent vector to C at $\mathbf{r}(s)$ and $\hat{\mathbf{n}}(s) = \mathbf{r}'(s) \times \hat{\mathbf{k}} = y'(s)\,\hat{\imath} - x'(s)\,\hat{\jmath}$. To see that $\mathbf{r}'(s) \times \hat{\mathbf{k}}$ really is $\hat{\mathbf{n}}(s)$, note that $y'(s)\,\hat{\imath} - x'(s)\,\hat{\jmath}$

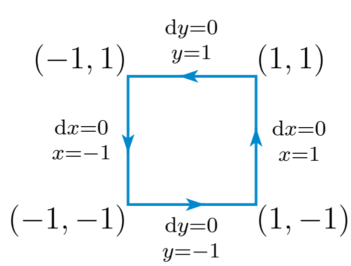
- has the same length, namely 1, as $\mathbf{r}'(s)$ (recall that $\mathbf{r}(s)$ is a parametrization by arc length),
- lies in the xy-plane and
- is perpendicular to $\mathbf{r}'(s)$. (Check that $\mathbf{r}'(s) \cdot \big[y'(s)\,\hat{\imath} - x'(s)\,\hat{\jmath}\big] = 0$.)
- Use the right hand rule to check that $\mathbf{r}'(s) \times \hat{\mathbf{k}}$ is $\hat{\mathbf{n}}$ rather than $-\hat{\mathbf{n}}$.



S-3: (a) Parametrize the circle by $x = a\cos\theta,\ y = a\sin\theta$, $0 \leqslant \theta \leqslant 2\pi$. Then $\mathrm{d}x = -a\sin\theta\,\mathrm{d}\theta$ and $\mathrm{d}y = a\cos\theta\,\mathrm{d}\theta$ so that

$$\frac{1}{2\pi}\oint_C \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2} = \frac{1}{2\pi}\int_0^{2\pi} \frac{a^2\cos^2\theta\,\mathrm{d}\theta + a^2\sin^2\theta\,\mathrm{d}\theta}{a^2\cos^2\theta + a^2\sin^2\theta} = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta = 1$$

(b) The boundary of the square has four sides — one with $y = -1$, one with $x = 1$, one with $y = 1$ and one with $x = -1$.



To evaluate the integrals over the four sides

- parametrize the $y = -1$ part by x so that $\mathbf{r}(x) = x\,\hat{\imath} - \hat{\jmath}$, $\mathbf{r}'(x) = \hat{\imath}$, with x running from -1 to 1,
- parametrize the $x = +1$ part by y so that $\mathbf{r}(y) = \hat{\imath} + y\,\hat{\jmath}$, $\mathbf{r}'(y) = \hat{\jmath}$, with y running from -1 to 1,
- parametrize the $y = +1$ part by x so that $\mathbf{r}(x) = x\,\hat{\imath} + \hat{\jmath}$, $\mathbf{r}'(x) = \hat{\imath}$, with x running from 1 to -1, and
- parametrize the $x = -1$ part by y so that $\mathbf{r}(y) = -\hat{\imath} + y\,\hat{\jmath}$, $\mathbf{r}'(y) = \hat{\jmath}$, with y running from 1 to -1,

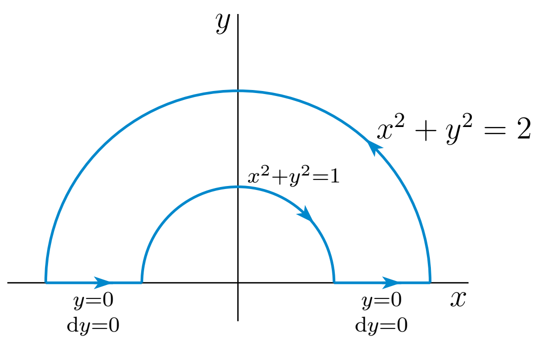
so that the integral

$$\frac{1}{2\pi}\oint_C \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2} = \frac{1}{2\pi}\overbrace{\int_{-1}^{1}\frac{-(-1)\mathrm{d}x}{x^2+1}}^{y=-1\text{ part}} + \frac{1}{2\pi}\overbrace{\int_{-1}^{1}\frac{(1)\mathrm{d}y}{1+y^2}}^{x=+1\text{ part}} + \frac{1}{2\pi}\overbrace{\int_{1}^{-1}\frac{-(1)\mathrm{d}x}{x^2+1}}^{y=+1\text{ part}} + \frac{1}{2\pi}\overbrace{\int_{1}^{-1}\frac{(-1)\mathrm{d}y}{1+y^2}}^{x=-1\text{ part}}$$

$$= 4\frac{1}{2\pi}\arctan x\Big|_{-1}^{1} = \frac{2}{\pi}\left[\frac{\pi}{4}+\frac{\pi}{4}\right] = 1$$

(c) As in part (a) with $a=\sqrt{2}$, but with θ running from 0 to π, the outer semicircle gives

$$\frac{1}{2\pi}\int_0^{\pi}\frac{a^2\cos^2\theta\,\mathrm{d}\theta + a^2\sin^2\theta\,\mathrm{d}\theta}{a^2\cos^2\theta + a^2\sin^2\theta} = \frac{1}{2\pi}\int_0^{\pi}\mathrm{d}\theta = \frac{1}{2}$$



As in part (a) with $a=1$, but with θ running from π to 0, the inner semicircle gives

$$\frac{1}{2\pi}\int_{\pi}^{0}\frac{a^2\cos^2\theta\,\mathrm{d}\theta + a^2\sin^2\theta\,\mathrm{d}\theta}{a^2\cos^2\theta + a^2\sin^2\theta} = \frac{1}{2\pi}\int_{\pi}^{0}\mathrm{d}\theta = -\frac{1}{2}$$

The two flat pieces each give zero, since on them $y=0$ and $\mathrm{d}y=0$. So

$$\frac{1}{2\pi}\oint_C \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2} = \frac{1}{2}+0-\frac{1}{2}+0 = 0$$

S-4: The two partial derivatives

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$
$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{-(x^2+y^2)-(-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

are well-defined and equal everywhere except at the origin $(0,0)$.

Short discussion: Were it not for the singularity at $(0,0)$, the vector field of the last problem would be conservative and the integral $\int \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ around any closed curve would be zero. But as we saw in parts (a) and (b) of Q[3], this is not the case. On the other hand, by Green's theorem (Theorem 4.3.2 in the CLP-4 text), the integral around the boundary of any region that does not contain $(0,0)$ is zero, as happened in part (c) of Q[3].

Long discussion: First consider part (c) of Q[3]. The curve C is the boundary of the region

$$R = \left\{ (x,y) \mid 1 \leqslant x^2 + y^2 \leqslant 2,\ y \geqslant 0 \right\}$$

The partial derivatives $\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$ and $\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)$ are well-defined and equal everywhere in R. So by Green's theorem

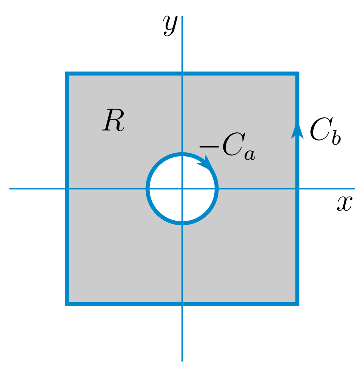
$$\frac{1}{2\pi}\oint_C \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2} = \frac{1}{2\pi}\iint_R \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right]\mathrm{d}x\,\mathrm{d}y = 0$$

which is the answer we got before.

We cannot apply Green's theorem in this way for parts (a) and (b) of Q[3] because the singularity at $(0,0)$ is inside the curve C for both parts (a) and (b). On the other hand suppose, for simplicity, that $0 < a < 1$. Denote by C_a, C_b the curves of parts (a) and (b), respectively. Define R to be the set of points that are inside C_b and outside C_a. That is,

$$R = \left\{ (x,y) \mid -1 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 1,\ x^2+y^2 \geqslant a^2 \right\}$$

Then the boundary, ∂R, of R consists of two parts. One part is C_b. The other part is C_a, but oriented clockwise rather than counterclockwise. We'll call it $-C_a$.



Again the partial derivatives $\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$ and $\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)$ are well-defined and equal everywhere in R. So by Green's theorem

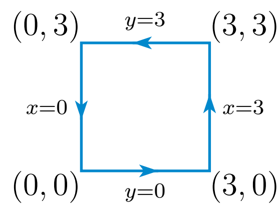
$$\frac{1}{2\pi}\oint_{\partial R} \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2} = \frac{1}{2\pi}\iint_R \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right]\mathrm{d}x\,\mathrm{d}y = 0$$

Consequently

$$\begin{aligned} 0 &= \frac{1}{2\pi}\oint_{\partial R} \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2} = \frac{1}{2\pi}\oint_{C_b} \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2} + \frac{1}{2\pi}\oint_{-C_a} \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2} \\ &= \frac{1}{2\pi}\oint_{C_b} \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2} - \frac{1}{2\pi}\oint_{C_a} \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2} \end{aligned}$$

and we conclude that the answers to parts (a) and (b) should be the same.We did indeed see that in Q[3].

S-5: *Solution 1 (direct evaluation)*: Here is a sketch of C.



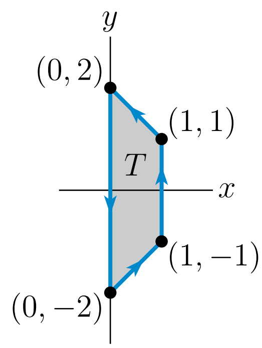
The square consists of four line segments.

- The bottom line segment may be parametrized $\mathbf{r}(x) = (x, 0)$, $0 \leqslant x \leqslant 3$. So the line integral along this segment is
$$\int_0^3 \mathbf{F}(\mathbf{r}(x)) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}x}\, \mathrm{d}x = \int_0^3 (0, 0) \cdot (1, 0)\, \mathrm{d}x = 0$$
- The second line segment may be parametrized $\mathbf{r}(y) = (3, y)$, $0 \leqslant y \leqslant 3$. So the line integral along this segment is
$$\int_0^3 \mathbf{F}(\mathbf{r}(y)) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}y}\, \mathrm{d}y = \int_0^3 (9y^2, 6y) \cdot (0, 1)\, \mathrm{d}y = \int_0^3 6y\, \mathrm{d}y = 27$$
- The third line segment may be parametrized $\mathbf{r}(t) = (3 - t, 3)$, $0 \leqslant t \leqslant 3$. So the line integral along this segment is
$$\int_0^3 \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\, \mathrm{d}t = \int_0^3 \big(9(3-t)^2\,,\, 6(3-t)\big) \cdot (-1\,,\, 0)\, \mathrm{d}t = -\int_0^3 9(3-t)^2\, \mathrm{d}t = -81$$
- The final line segment may be parametrized $\mathbf{r}(t) = (0, 3 - t)$, $0 \leqslant t \leqslant 3$. So the line integral along this segment is
$$\int_0^3 \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\, \mathrm{d}t = \int_0^3 (0, 0) \cdot (0, -1)\, \mathrm{d}t = 0$$

The full line integral is
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0 + 27 - 81 + 0 = -54$$

Solution 2 (Green's theorem): We apply Green's Theorem.
$$\begin{aligned}
\oint_C x^2y^2\, \mathrm{d}x + 2xy\, \mathrm{d}y &= \int_0^3 \mathrm{d}x \int_0^3 \mathrm{d}y\, \left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2y^2)\right] \\
&= \int_0^3 \mathrm{d}x \int_0^3 \mathrm{d}y\, \left[2y - 2x^2y\right] \\
&= \int_0^3 \mathrm{d}x\, \left[9 - 9x^2\right] \\
&= 27 - 9\frac{3^3}{3} = -54
\end{aligned}$$

S-6: Call the trapezoid T.



By Green's theorem,

$$\begin{aligned}\oint_C (x\sin y^2 - y^2)\,\mathrm{d}x + (x^2y\cos y^2 + 3x)\,\mathrm{d}y \\ &= \iint_T \left\{\frac{\partial}{\partial x}(x^2y\cos y^2 + 3x) - \frac{\partial}{\partial y}(x\sin y^2 - y^2)\right\}\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_T (2xy\cos y^2 + 3 - 2xy\cos y^2 + 2y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_T (3+2y)\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

The integral $\iint_T (2y)\,\mathrm{d}x\,\mathrm{d}y$ vanishes because $2y$ changes sign under $y \to -y$ while the domain of integration is invariant under $y \to -y$. The integral $\iint_T 3\,\mathrm{d}x\,\mathrm{d}y$ is 3 times the area of the trapezoid, which is its width (1) times the average of its heights $(\frac{1}{2}[2+4]) = 3$. So

$$\oint_C (x\sin y^2 - y^2)\,\mathrm{d}x + (x^2y\cos y^2 + 3x)\,\mathrm{d}y = 3\times 1\times 3 = 9$$

S-7: *(Using Green's theorem:)* By Green's theorem (Theorem 4.3.2 in the CLP-4 text), using D to denote the half-disk $0 \leqslant y \leqslant \sqrt{4-x^2}$,

$$\begin{aligned}\oint_C \left(\frac{1}{3}x^2y^3 - x^4y\right)\mathrm{d}x + (xy^4 + x^3y^2)\,\mathrm{d}y &= \iint_D \left[\frac{\partial}{\partial x}(xy^4 + x^3y^2) - \frac{\partial}{\partial y}\left(\frac{1}{3}x^2y^3 - x^4y\right)\right]\mathrm{d}x\mathrm{d}y \\ &= \iint_D (x^4 + 2x^2y^2 + y^4)\,\mathrm{d}x\mathrm{d}y = \iint_D (x^2+y^2)^2\,\mathrm{d}x\mathrm{d}y\end{aligned}$$

Switching to polar coordinates

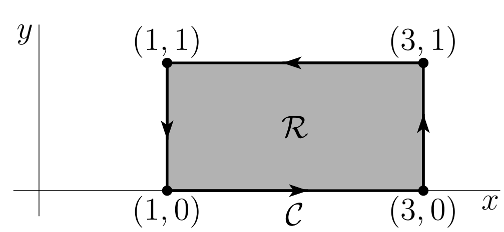
$$\oint_{\mathcal{C}} \Big(\frac{1}{3}x^2y^3 - x^4y\Big)\,\mathrm{d}x + \big(xy^4 + x^3y^2\big)\,\mathrm{d}y = \int_0^2 dr\, r \int_0^\pi \mathrm{d}\theta\, r^4 = \pi\frac{r^6}{6}\bigg|_0^2 = \frac{32}{3}\pi$$

(Using direct evaluation:) Write $\mathcal{C}$ as the union of $\mathcal{C}_1$, the straight line from $(-2,0)$ to $(2,0)$, and $\mathcal{C}_2$, the half-circle $\mathbf{r}(\theta) = x(\theta)\,\hat{\boldsymbol{\imath}} + y(\theta)\,\hat{\boldsymbol{\jmath}} = 2\cos\theta\,\hat{\boldsymbol{\imath}} + 2\sin\theta\,\hat{\boldsymbol{\jmath}}$, $0 \leqslant \theta \leqslant \pi$. As $y = 0$ at every point of $\mathcal{C}_1$, $\int_{\mathcal{C}_1} \big(\frac{1}{3}x^2y^3 - x^4y\big)\,\mathrm{d}x + \big(xy^4 + x^3y^2\big)\,\mathrm{d}y = 0$ and

$$\begin{aligned}
I &= \int_{\mathcal{C}_2} \Big(\frac{1}{3}x^2y^3 - x^4y\Big)\,\mathrm{d}x + \big(xy^4 + x^3y^2\big)\,\mathrm{d}y \\
&= \int_0^\pi \Big[\Big(\frac{1}{3}x(\theta)^2y(\theta)^3 - x(\theta)^4y(\theta)\Big)\,x'(\theta) + \big(x(\theta)y(\theta)^4 + x(\theta)^3y(\theta)^2\big)y'(\theta)\Big]\mathrm{d}\theta \\
&= \int_0^\pi \Big[\Big(\frac{1}{3}2^5\cos^2\theta\sin^3\theta - 2^5\cos^4\theta\sin\theta\Big)\,(-2\sin\theta) \\
&\qquad\qquad + \big(2^5\cos\theta\sin^4\theta + 2^5\cos^3\theta\sin^2\theta\big)\,(2\cos\theta)\Big]\mathrm{d}\theta \\
&= 2^5 \int_0^\pi \Big(\frac{4}{3}\cos^2\theta\sin^4\theta + 4\cos^4\theta\sin^2\theta\Big)\mathrm{d}\theta \\
&= 2^5 \int_0^\pi \sin^2(2\theta)\big(\frac{1}{3}\sin^2\theta + \cos^2\theta\big)\,\mathrm{d}\theta \\
&= 2^4 \int_0^\pi \sin^2(2\theta)\Big(\frac{1}{3}[1-\cos(2\theta)] + [1+\cos(2\theta)]\Big)\,\mathrm{d}\theta \\
&\qquad\qquad \text{since } \cos(2\theta) = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta \\
&= \frac{2^5}{3}\int_0^\pi \sin^2(2\theta)\big[2 + \cos(2\theta)\big]\,\mathrm{d}\theta \\
&= \frac{2^5}{3}\int_0^\pi \big[1 - \cos(4\theta) + \sin^2(2\theta)\cos(2\theta)\big]\,\mathrm{d}\theta \\
&= \frac{2^5}{3}\Big[\theta - \frac{1}{4}\sin(4\theta) + \frac{1}{6}\sin^3(2\theta)\Big]_0^\pi = \frac{32}{3}\pi
\end{aligned}$$

S-8: Let's use Green's theorem. The rectangle, which we shall denote $\mathcal{R}$, is

$$\mathcal{R} = \{(x,y)\}\, 1 \leqslant x \leqslant 3,\ 0 \leqslant y \leqslant 1$$



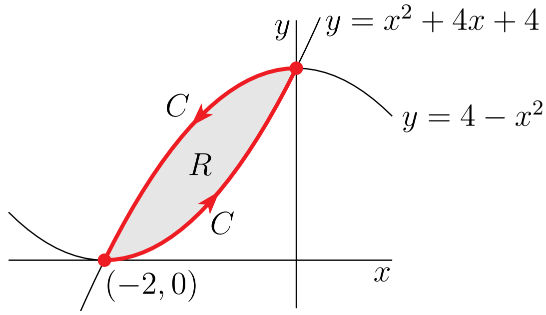
So Green's theorem gives

$$\begin{aligned}\oint_{\mathcal{C}} (3y^2 + 2xe^{y^2})\,\mathrm{d}x + (2yx^2e^{y^2})\,\mathrm{d}y &= \iint_{\mathcal{R}} \left[\frac{\partial}{\partial x}(2yx^2e^{y^2}) - \frac{\partial}{\partial y}(3y^2 + 2xe^{y^2})\right] \mathrm{d}x\mathrm{d}y \\ &= \iint_{\mathcal{R}} \left[4xye^{y^2} - 6y - 4xye^{y^2}\right] \mathrm{d}x\mathrm{d}y \\ &= -6\int_1^3 \mathrm{d}x \int_0^1 \mathrm{d}y\; y = -6\int_1^3 \mathrm{d}x\; \frac{1}{2} \\ &= -6\end{aligned}$$

S-9: (a) The curves $y = x^2 + 4x + 4$ and $y = 4 - x^2$ meet when

$$x^2 + 4x + 4 = 4 - x^2 \iff 2x^2 + 4x = 2x(x+2) = 0$$

So the curves intersect at $(0, 4)$ and $(-2, 0)$. Here is a sketch.



(b) Let

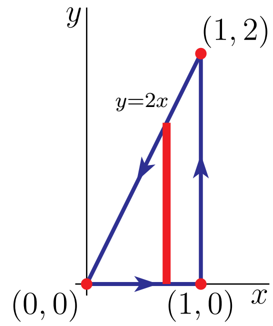
$$R = \left\{\, (x, y) \in \mathbb{R}^2 \;\middle|\; x^2 + 4x + 4 \leqslant y \leqslant 4 - x^2,\ -2 \leqslant x \leqslant 0 \,\right\}$$

By Green's theorem (Theorem 4.3.2 in the CLP-4 text)

$$\begin{aligned}\oint_C xy\,\mathrm{d}x + (e^y + x^2)\mathrm{d}y &= \iint_R \left\{\tfrac{\partial}{\partial x}(e^y + x^2) - \tfrac{\partial}{\partial y}(xy)\right\} \mathrm{d}x\mathrm{d}y \\ &= \int_{-2}^0 \mathrm{d}x \int_{x^2+4x+4}^{4-x^2} \mathrm{d}y\; x \\ &= \int_{-2}^0 \mathrm{d}x\; (-2x^2 - 4x)x \\ &= \left[-\frac{x^4}{2} - \frac{4x^3}{3}\right]_{-2}^0 \\ &= -\frac{8}{3}\end{aligned}$$

S-10: The integral that would be used for direct evaluation looks very complicated. So let's try Green's theorem. The curve C is the boundary of the triangle

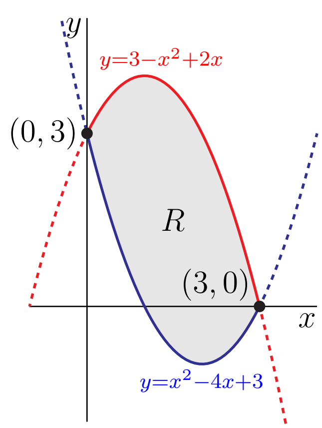
$$T = \{\ (x,y)\ \big|\ 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 2x\ \}$$



So

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C \big\{\big(y^2 - e^{-y^2} + \sin x\big)dx + \big(2xye^{-y^2} + x\big)dy\big\} \\
&= \iint_T \big\{\tfrac{\partial}{\partial x}\big(2xye^{-y^2} + x\big) - \tfrac{\partial}{\partial y}\big(y^2 - e^{-y^2} + \sin x\big)\big\}dxdy \\
&= \iint_T \big\{\big(2ye^{-y^2} + 1\big) - \big(2y + 2ye^{-y^2}\big)\big\}dxdy \\
&= \int_0^1 dx \int_0^{2x} dy\ \{1 - 2y\} \\
&= \int_0^1 dx\ \{2x - 4x^2\} \\
&= 1 - \frac{4}{3} = -\frac{1}{3}
\end{aligned}$$

S-11: Here is a sketch of the two curves in question.



Note that the curves $y = x^2 - 4x + 3$ and $y = 3 - x^2 + 2x$ intersect when $x^2 - 4x + 3 = 3 - x^2 + 2x$ or $2x^2 - 6x = 2x(x-3) = 0$ or $x = 0, 3$.

The integrand for direct evaluation looks complicated. So let's use Green's theorem with $F_1(x,y) = 2xe^y + \sqrt{2+x^2}$, $F_2(x,y) = x^2(2+e^y)$ and

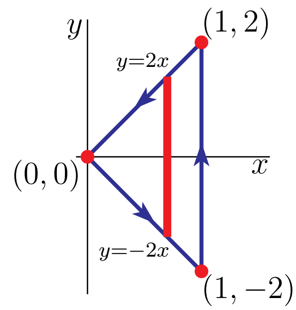
$$R = \left\{ (x,y) \mid x^2 - 4x + 3 \leqslant y \leqslant 3 - x^2 + 2x,\ 0 \leqslant x \leqslant 3 \right\}$$

By Green's theorem, which is Theorem 4.3.2 in the CLP-4 text,

$$\begin{aligned}
\int_C \left(2xe^y + \sqrt{2+x^2}\right) \mathrm{d}x + x^2(2+e^y)\,\mathrm{d}y &= \iint_R \left\{ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\} \mathrm{d}x\mathrm{d}y \\
&= \iint_R \left\{ 2x(2+e^y) - 2xe^y \right\} \mathrm{d}x\mathrm{d}y \\
&= 4\int_0^3 \mathrm{d}x \int_{x^2-4x+3}^{3-x^2+2x} \mathrm{d}y\ x \\
&= 4\int_0^3 \mathrm{d}x\ (6x - 2x^2)x \\
&= 4\left[2x^3 - \frac{1}{2}x^4\right]_0^3 \\
&= 54
\end{aligned}$$

S-12: Direct evaluation will lead to three integrals, one for each side of the triangle. The integral from $(0,0)$ and $(1,-2)$ and the integral from $(1,2)$ to $(0,0)$ will each contain six (nonconstant) terms. This does not look very efficient. So let's try Green's theorem. Denote by T, the triangle

$$T = \left\{ (x,y) \mid 0 \leqslant x \leqslant 1,\ -2x \leqslant y \leqslant 2x \right\}$$



It has boundary $\partial T = C$, oriented counterclockwise as desired. So, by Green's theorem,

$$\begin{aligned}
\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_{\partial T} \left\{ \left(\tfrac{3}{2}y^2 + e^{-y} + \sin x\right)\mathrm{d}x + \left(\tfrac{1}{2}x^2 + x - xe^{-y}\right)\mathrm{d}y \right\} \\
&= \iint_T \left\{ \tfrac{\partial}{\partial x}\left(\tfrac{1}{2}x^2 + x - xe^{-y}\right) - \tfrac{\partial}{\partial y}\left(\tfrac{3}{2}y^2 + e^{-y} + \sin x\right) \right\}\mathrm{d}x\mathrm{d}y \\
&= \iint_T \left\{ \left(x + 1 - e^{-y}\right) - \left(3y - e^{-y}\right) \right\}\mathrm{d}x\mathrm{d}y \\
&= \iint_T \left\{ x - 3y + 1 \right\}\mathrm{d}x\mathrm{d}y
\end{aligned}$$

Now

$$\iint_T \mathrm{d}x\mathrm{d}y = \mathrm{Area}(T) = \frac{1}{2}(4)(1) = 2$$

$$\iint_T y\ \mathrm{d}x\mathrm{d}y = 0 \qquad \text{since } y \text{ is odd under } y \to -y$$

$$\iint_T x\ \mathrm{d}x\mathrm{d}y = \int_0^1 \mathrm{d}x \int_{-2x}^{2x} \mathrm{d}y\ x = \int_0^1 4x^2\ \mathrm{d}x = \frac{4}{3}$$

So

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \frac{4}{3} - 3 \times 0 + 2 = \frac{10}{3}$$

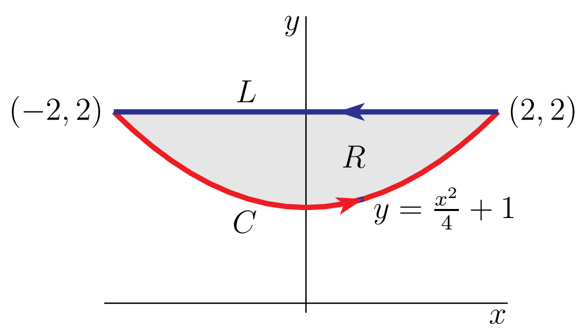
S-13: Set

$$\mathbf{F} = \frac{-y}{x^2+y^2}\hat{\boldsymbol{\imath}} + \frac{x}{x^2+y^2}\hat{\boldsymbol{\jmath}}$$

(a) Green's theorem must be applied to a curve that is closed, so that it is the boundary of a region in $\mathbb{R}^2$. The given curve C is not closed. But it is part of the boundary of

$$R = \left\{ (x,y) \,\middle|\, -2 \leqslant x \leqslant 2,\ \tfrac{x^2}{4} + 1 \leqslant y \leqslant 2 \right\}$$

Here is a sketch of R.



The boundary of R consists of two parts — C on the bottom and the line segment L from $(2,2)$ to $(-2,2)$ on the top. Note that $\mathbf{F}$ is well-defined on all of R and that

$$\begin{aligned}
\frac{\partial}{\partial x}\mathbf{F}_2 - \frac{\partial}{\partial y}\mathbf{F}_1 &= \frac{\partial}{\partial x}\frac{x}{x^2+y^2} + \frac{\partial}{\partial y}\frac{y}{x^2+y^2} \\
&= \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2} \\
&= 0
\end{aligned}$$

on all of R. So, by Green's theorem (Theorem 4.3.2 in the CLP-4 text),

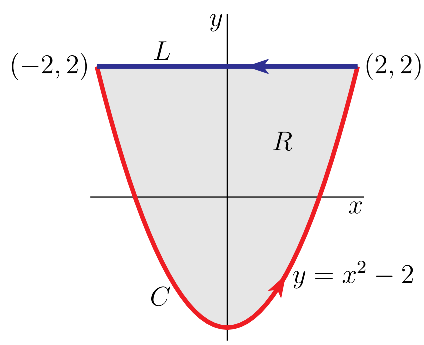
$$\begin{aligned}
\int_C \frac{-y}{x^2+y^2}\mathrm{d}x + \frac{x}{x^2+y^2}\mathrm{d}y &= \iint_R \Big(\frac{\partial}{\partial x}\mathbf{F}_2 - \frac{\partial}{\partial y}\mathbf{F}_1\Big)\mathrm{d}x\mathrm{d}y - \int_L \mathbf{F}\cdot\mathrm{d}\mathbf{r} \\
&= \int_{-L} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_{-2}^{2} \mathbf{F}_1\,\mathrm{d}x = -\int_{-2}^{2}\frac{2}{x^2+4}\,\mathrm{d}x \quad \text{since } y=2 \text{ on } L \\
&= -\int_{-1}^{1}\frac{4}{4u^2+4}\,\mathrm{d}x \qquad \text{with } x=2u,\ \mathrm{d}x=2\mathrm{d}u \\
&= -\arctan u\Big|_{-1}^{1} = -\frac{\pi}{2}
\end{aligned}$$

In the second line, we used the notation $-L$ for the line segment from $(-2,2)$ to $(2,2)$.

(b) This question looks a lot like that of part (a). But there is a critical difference. Again C is not closed and again it is part of the boundary of a simple region in the xy-plane, namely

$$R = \big\{\ (x,y)\ \big|\ -2 \leqslant x \leqslant 2,\ x^2-2 \leqslant y \leqslant 2\ \big\}$$

This R is sketched below.



We *cannot* continue as in part (a), using this R, because $\frac{\partial}{\partial x}\mathbf{F}_2 - \frac{\partial}{\partial y}\mathbf{F}_1$ is *not* zero througout R. In fact, it is *not* even defined throughout R — it is not defined at $(0,0)$, which is a point of R. We can work around this obstruction by

- choosing a number $\rho > 0$ that is small enough that the circle C_ρ parametrized by

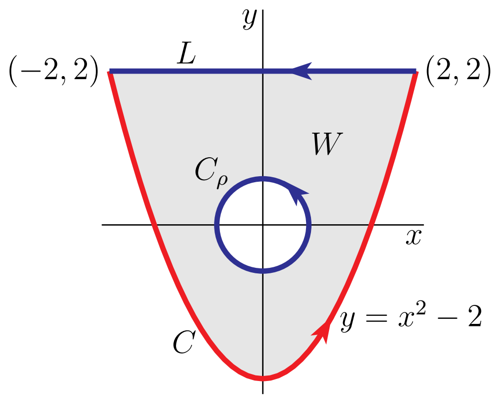
$$\mathbf{r}(\theta) = \rho\cos\theta\,\hat{\boldsymbol{\imath}} + \rho\sin\theta\,\hat{\boldsymbol{\jmath}} \qquad 0 \leqslant \theta \leqslant 2\pi$$

  is completely contained inside R (ror example, $\rho = 1$ is fine)
- and then removing from R the interior of C_ρ.

This produces the "deformed washer"

$$W = \big\{\ (x,y)\ \big|\ -2 \leqslant x \leqslant 2,\ x^2-2 \leqslant y \leqslant 2,\ x^2+y^2 \geqslant \rho^2\ \big\}$$

that is sketched below.



The boundary of W consists the three parts — the curve of interest C on the bottom, the line segment L from $(2,2)$ to $(-2,2)$ on the top, and the circle $-C_\rho$ (that is C_ρ but oriented clockwise, rather than counter-clockwise) around the hole in the middle. Now $\frac{\partial}{\partial x}\mathbf{F}_2 - \frac{\partial}{\partial y}\mathbf{F}_1$ is well-defined and zero throughout W. So, by Green's theorem (Theorem 4.3.2 in the CLP-4 text),

$$\begin{aligned}\int_C \frac{-y}{x^2+y^2}\mathrm{d}x + \frac{x}{x^2+y^2}\mathrm{d}y &= \iint_W \Big(\frac{\partial}{\partial x}\mathbf{F}_2 - \frac{\partial}{\partial y}\mathbf{F}_1\Big)\mathrm{d}x\mathrm{d}y - \int_L \mathbf{F}\cdot\mathrm{d}\mathbf{r} - \int_{-C_\rho}\mathbf{F}\cdot\mathrm{d}\mathbf{r}\\ &= \int_{-L}\mathbf{F}\cdot\mathrm{d}\mathbf{r} + \int_{C_\rho}\mathbf{F}\cdot\mathrm{d}\mathbf{r}\end{aligned}$$

We have already found, in part (a), that $\int_{-L}\mathbf{F}\cdot\mathrm{d}\mathbf{r} = -\frac{\pi}{2}$. So it remains only to use

$$\begin{aligned}\mathbf{r}(\theta) &= \rho\cos\theta\,\hat{\boldsymbol{\imath}} + \rho\sin\theta\,\hat{\boldsymbol{\jmath}}\\ \mathbf{r}'(\theta) &= -\rho\sin\theta\,\hat{\boldsymbol{\imath}} + \rho\cos\theta\,\hat{\boldsymbol{\jmath}}\end{aligned}$$

to evaluate

$$\begin{aligned}\int_{C_\rho}\mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \int_0^{2\pi}\mathbf{F}\big(\mathbf{r}(\theta)\big)\cdot\mathbf{r}'(\theta)\,\mathrm{d}\theta\\ &= \int_0^{2\pi}\Big(\overbrace{-\frac{1}{\rho}\sin\theta\,\hat{\boldsymbol{\imath}} + \frac{1}{\rho}\cos\theta\,\hat{\boldsymbol{\jmath}}}^{\mathbf{F}(\mathbf{r}(\theta))}\Big)\cdot\big(\overbrace{-\rho\sin\theta\,\hat{\boldsymbol{\imath}} + \rho\cos\theta\,\hat{\boldsymbol{\jmath}}}^{\mathbf{r}'(\theta)}\big)\,\mathrm{d}\theta = \int_0^{2\pi}\mathrm{d}\theta\\ &= 2\pi\end{aligned}$$

All together

$$\int_C \frac{-y}{x^2+y^2}\mathrm{d}x + \frac{x}{x^2+y^2}\mathrm{d}y = \int_{-L}\mathbf{F}\cdot\mathrm{d}\mathbf{r} + \int_{C_\rho}\mathbf{F}\cdot\mathrm{d}\mathbf{r} = -\frac{\pi}{2} + 2\pi = \frac{3\pi}{2}$$

(c) No, $\mathbf{F}$ is not conservative. We found, in parts (a) and (b), two different values for the integrals along two paths, both of which start at $(-2,2)$ and end at $(2,2)$. So $\mathbf{F}$ does not have the "path independence" property of Theorem 2.4.6.c in the CLP-4 text and cannot be conservative.

S-14: The given integral is of the form $\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}$ with

$$\mathbf{F} = \big(2xe^y + \sqrt{2} + x^2\big)\,\hat{\pmb{\imath}} + x^2\big(2+e^y\big)\,\hat{\pmb{\jmath}}$$

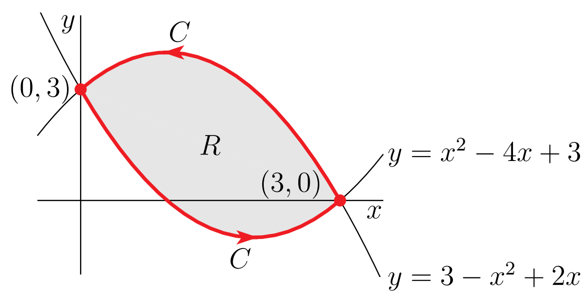
If we were to try to evaluate this integral directly, then on the $y = x^2-4x+3$ part of C, the integrand would contain $x^2e^y = x^2e^{x^2-4x+3}$. That looks hard to integrate, so let's try Green's theorem. The parabolas $y=x^2-4x+3$ and $y=3-x^2+2x$ intersect at (x,y) with

$$\begin{aligned} x^2-4x+3 = 3-x^2+2x &\iff 2x^2-6x=0 \iff 2x(x-3)=0 \\ &\iff x=0 \text{ or } x=3 \end{aligned}$$

The curve C is the boundary of

$$R = \big\{\ (x,y)\ \big|\ 0\leqslant x\leqslant 3,\ x^2-4x+3\leqslant y\leqslant 3-x^2+2x\ \big\}$$

It is sketched below.



By Green's theorem (Theorem 4.3.2 in the CLP-4 text),

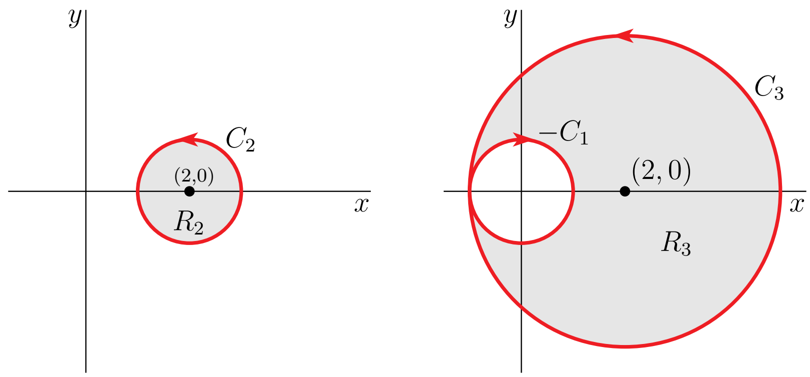
$$\begin{aligned}
\int_C \big(2xe^y+\sqrt{2}+x^2\big)\mathrm{d}x + x^2\big(2+e^y\big)\mathrm{d}y \\
&= \iint_R \Big[\frac{\partial}{\partial x}\big(x^2(2+e^y)\big) - \frac{\partial}{\partial y}\big(2xe^y+\sqrt{2}+x^2\big)\Big]\mathrm{d}x\mathrm{d}y \\
&= \iint_R \big(2x(2+e^y) - 2xe^y\big)\,\mathrm{d}x\mathrm{d}y \\
&= 4\int_0^3 \mathrm{d}x \int_{x^2-4x+3}^{3-x^2+2x}\mathrm{d}y\ x = 4\int_0^3 \mathrm{d}x\ x\big[(3-x^2+2x)-(x^2-4x+3)\big] \\
&= 4\int_0^3 \mathrm{d}x\ \big(6x^2-2x^3\big) = 4\big(2\times 3^3 - \frac{1}{2}3^4\big) = 54
\end{aligned}$$

S-15: (a) Denote by

$$R_2 = \big\{\ (x,y)\ \big|\ (x-2)^2+y^2\leqslant 1\ \big\}$$

the interior of the circle C_2. Note that $(0,0)$ is *not* in R_2. Consequently, $Q_x - P_y = 0$ *everywhere* in R_2 and, by Green's theorem (Theorem 4.3.2 in the CLP-4 text),

$$I_2 = \int_{C_2}\mathbf{F}\cdot\mathrm{d}\mathbf{r} = \iint_{R_2}\big(Q_x-P_y\big)\,\mathrm{d}x\mathrm{d}y = 0$$



(b) We cannot blindly apply Green's theorem to $I_3 = \int_{C_3} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ because $(0,0)$ *is* in the interior of C_3, so that $Q_x - P_y$ is not identically zero in the interior of C_3 — it is not even defined throughout the interior of C_3. We can work around this obstruction by considering the interior of C_3 with the interior of C_1 removed. That is, by considering

$$R_3 = \{\ (x,y)\ \big|\ x^2 + (y-2)^2 \leqslant 9,\ x^2 + y^2 \geqslant 1\ \}$$

It is sketched on the right above. The boundary of R_3 consists of two parts

- the circle C_3, oriented counterclockwise, and
- the circle $-C_1$. That is, the circle C_1 but oriented clockwise, rather than counterclockwise.

Then $Q_x - P_y$ is well-defined and zero throughout R_3 and, by Green's theorem,

$$\begin{aligned} 0 = \iint_{R_3} \big(Q_x - P_y\big)\, \mathrm{d}x\mathrm{d}y &= \int_{C_3} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{-C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\ &= \int_{C_3} \mathbf{F} \cdot \mathrm{d}\mathbf{r} - \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\ &= \int_{C_3} \mathbf{F} \cdot \mathrm{d}\mathbf{r} - \pi \end{aligned}$$

So $\int_{C_3} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \pi$.

(c) Again, we cannot blindly apply Green's theorem to $I_4 = \int_{C_4} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ because $(0,0)$ *is* in the interior of C_4. This time we cannot remove the interior of C_1 from the interior of C_4, because C_1 is not contained in the interior of C_4. Instead we pick a number $\rho > 0$ which is small enough that the positively oriented circle

$$C_\rho = \{\ (x,y)\ \big|\ x^2 + y^2 = \rho^2\ \}$$

is completely inside C_4. Then we can define

$$R_4 = \{\ (x,y)\ \big|\ (x-2)^2 + (y-2)^2 \leqslant 9,\ x^2 + y^2 \geqslant \rho^2\ \}$$

It is sketched on the left below. We can now argue as in part (b). The boundary of R_4 consists of two parts

- the circle C_4, oriented counterclockwise, and

- the circle $-C_\rho$. That is, the circle C_ρ but oriented clockwise, rather than counterclockwise.

Then $Q_x - P_y$ is well-defined and zero throughout R_4 and, by Green's theorem,

$$\begin{aligned}
0 &= \iint_{R_4} \big(Q_x - P_y\big)\, \mathrm{d}x\mathrm{d}y \\
&= \int_{C_4} \mathbf{F}\cdot \mathrm{d}\mathbf{r} + \int_{-C_\rho} \mathbf{F}\cdot \mathrm{d}\mathbf{r} \\
&= \int_{C_4} \mathbf{F}\cdot \mathrm{d}\mathbf{r} - \int_{C_\rho} \mathbf{F}\cdot \mathrm{d}\mathbf{r}
\end{aligned}$$

So $\int_{C_4} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{C_\rho} \mathbf{F}\cdot \mathrm{d}\mathbf{r}$.

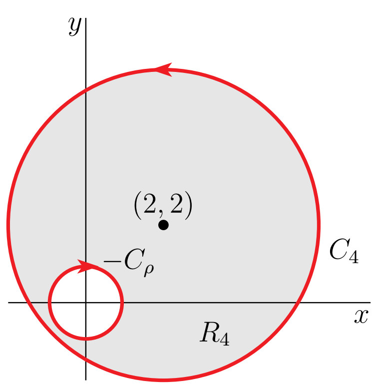
To complete our computation, we have to determine $\int_{C_\rho} \mathbf{F}\cdot \mathrm{d}\mathbf{r}$. We can do so by repeating the same "removing a small disk containing $(0,0)$" argument for the third time. Set

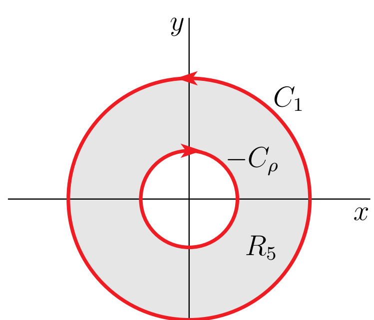
$$R_5 = \left\{ \ (x,y) \ \middle| \ x^2+y^2 \leqslant 1,\ x^2+y^2 \geqslant \rho^2 \ \right\}$$

Then the boundary of R_5 consists of C_1 and $-C_\rho$, and, as $Q_x - P_y$ is well-defined and zero throughout R_5,

$$\begin{aligned}
0 &= \iint_{R_5} \big(Q_x - P_y\big)\, \mathrm{d}x\mathrm{d}y \\
&= \int_{C_1} \mathbf{F}\cdot \mathrm{d}\mathbf{r} + \int_{-C_\rho} \mathbf{F}\cdot \mathrm{d}\mathbf{r} \\
&= \pi - \int_{C_\rho} \mathbf{F}\cdot \mathrm{d}\mathbf{r}
\end{aligned}$$

So $\int_{C_4} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{C_\rho} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \pi$.





S-16: (a) If $(x, y) \neq (0,0)$, we have

$$\begin{aligned} Q_x - P_y &= \frac{\partial}{\partial x}\Big(\frac{y-x}{x^2+y^2}\Big) - \frac{\partial}{\partial y}\Big(\frac{x+y}{x^2+y^2}\Big) \\ &= \frac{-(x^2+y^2)-(y-x)(2x)}{(x^2+y^2)^2} - \frac{(x^2+y^2)-(x+y)(2y)}{(x^2+y^2)^2} \\ &= 0 \end{aligned}$$

(b) Parametrize C_R by

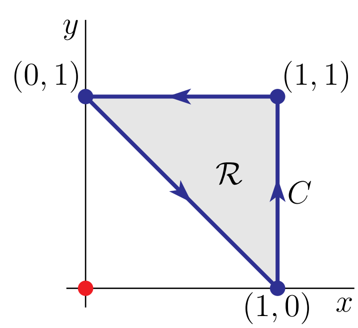
$$\mathbf{r}(\theta) = R\cos\theta\,\hat{\boldsymbol{\imath}} + R\sin\theta\,\hat{\boldsymbol{\jmath}} \qquad 0 \leqslant \theta \leqslant 2\pi$$

So

$$\begin{aligned} \int_{C_R} \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \overbrace{\frac{1}{R}\big\{\big(\cos\theta+\sin\theta\big)\hat{\boldsymbol{\imath}} + \big(\sin\theta-\cos\theta\big)\hat{\boldsymbol{\jmath}}\big\}}^{\mathbf{F}(\mathbf{r}(\theta))} \cdot \big(\overbrace{-R\sin\theta\,\hat{\boldsymbol{\imath}} + R\cos\theta\,\hat{\boldsymbol{\jmath}}}^{\mathbf{r}'(\theta)}\big)\,d\theta \\ &= \int_0^{2\pi} (-1)\,d\theta \\ &= -2\pi \end{aligned}$$

(c) If $\mathbf{F}$ were conservative, the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ would be 0 for any closed curve C, by Theorem 2.4.6.b in the CLP-4 text. So $\mathbf{F}$ is *not* conservative. Note that $\mathbf{F}$ is not defined at $(x, y) = (0,0)$ and so fails the screening test $\boldsymbol{\nabla}\times\mathbf{F} = \mathbf{0}$ at $(x, y) = (0,0)$.

(d) Denote by $\mathcal{R}$ the interior of the triangle C. It is the grey region in the figure



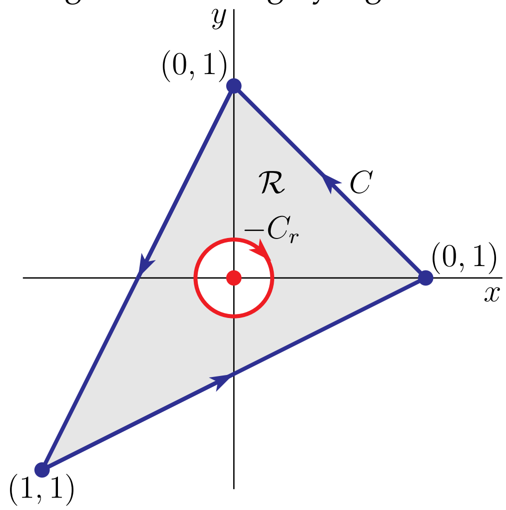
Note that $(0,0)$ is *not* in $\mathcal{R}$. So $Q_x - P_y$ is defined and zero throughout $\mathcal{R}$. So, by Green's theorem (Theorem 4.3.2 in the CLP-4 text),

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_{\mathcal{R}} \big(Q_x - P_y\big)\,dx\,dy = 0$$

(e) Note that $(0,0)$ *is* in the interior of triangle C specified for this part. So $Q_x - P_y$ is *not* defined in that interior and we cannot apply Green's theorem precisely as we did in part (d). We can work around this obstruction by

- picking a number $r > 0$ that is small enough that the circle C_r, of radius r centred on $(0,0)$, is completely contained in the interior of the triangle C.

- Then we work with the region $\mathcal{R}$ defined by removing the interior of the circle C_r from the interior of the triangle C. It is the grey region sketched below.



The boundary of $\mathcal{R}$ consists of two parts

- the triangle C, oriented counterclockwise, and
- the circle $-C_r$. That is, the circle C_r, but oriented clockwise, rather than counterclockwise.

Then $Q_x - P_y$ is well-defined and zero throughout $\mathcal{R}$ and, by Green's theorem,

$$\begin{aligned} 0 &= \iint_{\mathcal{R}} \big(Q_x - P_y\big)\, \mathrm{d}x\mathrm{d}y \\ &= \int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} + \int_{-C_r} \mathbf{F}\cdot \mathrm{d}\mathbf{r} \\ &= \int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} - \int_{C_r} \mathbf{F}\cdot \mathrm{d}\mathbf{r} \end{aligned}$$

So $\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{C_r} \mathbf{F}\cdot \mathrm{d}\mathbf{r}$. By part (b), with $R = r$, $\int_{C_r} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = -2\pi$, so $\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = -2\pi$

S-17: (a) The given integral is of the form $\int_C F_1(x,y)\,\mathrm{d}x + F_2(x,y)\,\mathrm{d}y$ with

$$F_1(x,y) = \sqrt{1+x^3} \qquad F_2(x,y) = 2xy^2 + y^2 \qquad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 2y^2$$

As C is ∂R with

$$R = \big\{\ (x,y)\ \big|\ x^2+y^2 \leqslant 1\ \big\}$$

Green's theorem (Theorem 4.3.2 in the CLP-4 text) gives

$$\begin{aligned} \int_C \sqrt{1+x^3}\,\mathrm{d}x + \big(2xy^2+y^2\big)\,\mathrm{d}y &= \int_C F_1(x,y)\,\mathrm{d}x + F_2(x,y)\,\mathrm{d}y = \iint_R \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\,\mathrm{d}x\mathrm{d}y \\ &= 2\iint_R y^2\,\mathrm{d}x\mathrm{d}y \end{aligned}$$

Switching to polar coordinates

$$\begin{aligned}\int_C \sqrt{1+x^3}\,\mathrm{d}x + \big(2xy^2+y^2\big)\,\mathrm{d}y &= 2\int_0^{2\pi}\mathrm{d}\theta\int_0^1 \mathrm{d}r\; r\big(r\sin\theta\big)^2 \\ &= 2\left[\int_0^{2\pi}\mathrm{d}\theta\;\sin^2\theta\right]\left[\int_0^1\mathrm{d}r\; r^3\right] = 2\,\pi\,\frac{1}{4} = \frac{\pi}{2}\end{aligned}$$

To do the θ integral, we have used

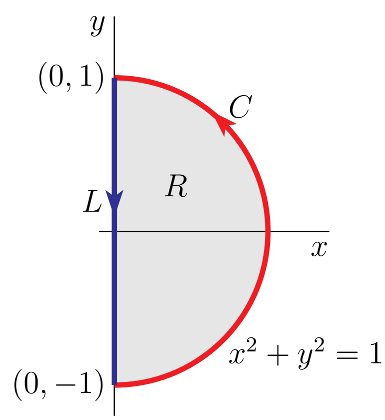
$$\int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta = \int_0^{2\pi}\Big(\frac{1-\cos(2\theta)}{2}\Big)\,\mathrm{d}\theta = \Big[\frac{\theta-\sin(2\theta)/2}{2}\Big]_0^{2\pi} = \pi$$

For an efficient, sneaky, way to evaluate $\int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta$, see Example 2.4.4 in the CLP-4 text.

(b) It is again natural to use Green's theorem. But Green's theorem must be applied to a curve that is closed, so that it is the boundary of a region in $\mathbb{R}^2$. The given curve C is not closed. But it is part of the boundary of

$$R = \big\{\ (x,y)\ \big|\ x^2+y^2\leqslant 1,\ x\geqslant 0\ \big\}$$

Here is a sketch of R.



The boundary of R consists of two parts — C on the right and the line segment L from $(0,1)$ to $(0,-1)$ on the left. Note that $\mathbf{F} = F_1\,\hat{\boldsymbol{\imath}} + F_2\,\hat{\boldsymbol{\jmath}}$ is well-defined on all of R and that we still have, from part (a),

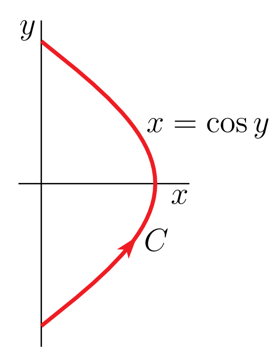
$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 2y^2$$

on all of R. So, by Green's theorem (Theorem 4.3.2 in the CLP-4 text),

$$\begin{aligned}
\int_C \sqrt{1+x^3}\,\mathrm{d}x + \big(2xy^2+y^2\big)\,\mathrm{d}y &= \iint_R \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_2}{\partial y}\Big)\mathrm{d}x\mathrm{d}y - \int_L \mathbf{F}\cdot\mathrm{d}\mathbf{r} \\
&= \iint_{\substack{x^2+y^2\leqslant 1\\ x\geqslant 0}} 2y^2\,\mathrm{d}x\mathrm{d}y + \int_{-L} F_2\,\mathrm{d}y \\
&= \iint_{x^2+y^2\leqslant 1} y^2\,\mathrm{d}x\mathrm{d}y + \int_{-1}^{1} y^2\,\mathrm{d}y \\
&\qquad \text{by symmetry for the first integral and} \\
&\qquad \text{since } x=0 \text{ and } \mathrm{d}x = 0 \text{ in the second integral} \\
&= \frac{\pi}{4} + \frac{2}{3}
\end{aligned}$$

In the second line, we used the notation $-L$ for the line segment from $(0,-1)$ to $(0,1)$.

S-18: First, here is a sketch of the curve C.



We'll evaluate this integral in three different ways.

(1) *Direct evaluation:* To evaluate the integral directly, we'll parametrize C using y as the parameter. That is, we'll make $y(t) = t$:

$$\begin{aligned}
\mathbf{r}(t) &= x(t)\,\hat{\boldsymbol{\imath}} + y(t)\,\hat{\boldsymbol{\jmath}} = \cos t\,\hat{\boldsymbol{\imath}} + t\,\hat{\boldsymbol{\jmath}} \qquad -\frac{\pi}{2} \leqslant t \leqslant \frac{\pi}{2} \\
\mathbf{r}'(t) &= x'(t)\,\hat{\boldsymbol{\imath}} + y'(t)\,\hat{\boldsymbol{\jmath}} = -\sin t\,\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}
\end{aligned}$$

So the integral is

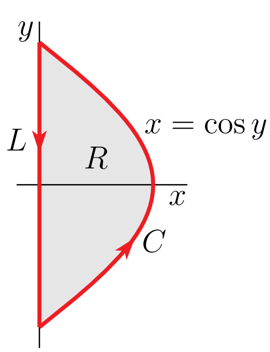
$$
\begin{aligned}
&\int_C (x^2 + ye^x)\,\mathrm{d}x + (x\cos y + e^x)\,\mathrm{d}y \\
&\qquad = \int_{-\pi/2}^{\pi/2} \Big\{ \big[x(t)^2 + y(t)e^{x(t)}\big]\frac{\mathrm{d}x}{\mathrm{d}t}(t) + \big[x(t)\cos(y(t)) + e^{x(t)}\big]\frac{\mathrm{d}y}{\mathrm{d}t}(t)\Big\}\mathrm{d}t \\
&\qquad = \int_{-\pi/2}^{\pi/2} \Big\{ -\big[\cos^2 t + te^{\cos t}\big]\sin t + \big[\cos^2 t + e^{\cos t}\big]\Big\}\mathrm{d}t \\
&\qquad = \int_{-\pi/2}^{\pi/2} \Big\{ -\cos^2 t\sin t + \cos^2 t - te^{\cos t}\sin t + e^{\cos t}\Big\}\mathrm{d}t \\
&\qquad = \int_{-\pi/2}^{\pi/2} \Big\{ \cos^2 t + \frac{\mathrm{d}}{\mathrm{d}t}\Big[\frac{\cos^3 t}{3} + te^{\cos t}\Big]\Big\}\mathrm{d}t \\
&\qquad = \int_{-\pi/2}^{\pi/2} \Big\{ \frac{\cos(2t)+1}{2} + \frac{\mathrm{d}}{\mathrm{d}t}\Big[\frac{\cos^3 t}{3} + te^{\cos t}\Big]\Big\}\mathrm{d}t \\
&\qquad = \Big[\frac{\sin(2t)}{4} + \frac{t}{2} + \frac{\cos^3 t}{3} + te^{\cos t}\Big]_{-\pi/2}^{\pi/2} \\
&\qquad = \frac{3\pi}{2}
\end{aligned}
$$

For an efficient, sneaky, way to evaluate $\int_{-\pi/2}^{\pi/2}\cos^2 t\,\mathrm{d}t$ see Example 2.4.4 in the CLP-4 text.

(2) *Green's (or Stokes') theorem:* The curve C is not closed so we cannot apply Green's theorem directly. However the boundary of the region

$$R = \big\{\ (x, y)\ \big|\ 0 \leqslant x \leqslant \cos y,\ -\pi/2 \leqslant y \leqslant \pi/2\ \big\}$$

(sketched below) consists of two parts, one of which is C. The other is the line L from $(0, \pi/2)$ to $(0, -\pi/2)$.



So Green's theorem gives

$$\begin{aligned}
&\int_C (x^2 + ye^x)\, \mathrm{d}x + (x\cos y + e^x)\, \mathrm{d}y \\
&\quad = \iint_R \Big\{\frac{\partial}{\partial x}(x\cos y + e^x) - \frac{\partial}{\partial y}(x^2 + ye^x)\Big\}\mathrm{d}x\mathrm{d}y - \int_L (x^2 + ye^x)\, \mathrm{d}x + (x\cos y + e^x)\, \mathrm{d}y \\
&\quad = \iint_R \cos y\ \mathrm{d}x\mathrm{d}y - \int_L \mathrm{d}y \qquad \text{since } x = 0 \text{ and } \mathrm{d}x = 0 \text{ on } L \\
&\quad = \int_{-\pi/2}^{\pi/2} \mathrm{d}y \int_0^{\cos y} \mathrm{d}x\ \cos y - \int_{\pi/2}^{-\pi/2} \mathrm{d}y \\
&\quad = \int_{-\pi/2}^{\pi/2} \mathrm{d}y\ \cos^2 y + \pi \\
&\quad = \int_{-\pi/2}^{\pi/2} \frac{\cos(2y) + 1}{2}\, \mathrm{d}y + \pi \\
&\quad = \Big[\frac{\sin(2y)}{4} + \frac{y}{2}\Big]_{-\pi/2}^{\pi/2} + \pi \\
&\quad = \frac{3\pi}{2}
\end{aligned}$$

(3) *(Sort of) conservative fields:* The given integral is $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ with $\mathbf{F} = (x^2 + ye^x)\,\hat{\boldsymbol{\imath}} + (x\cos y + e^x)\,\hat{\boldsymbol{\jmath}}$. The curl of this field is

$$\nabla \times \mathbf{F} = \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + ye^x & x\cos y + e^x & 0 \end{bmatrix} = \cos y\, \hat{\mathbf{k}}$$

So $\mathbf{F}$ violates our screening test and consequently is not conservative. But it violates the screening test only because of the term $x\cos y\,\hat{\boldsymbol{\jmath}}$. This suggests that we split up

$$\mathbf{F} = \mathbf{G} + \mathbf{H} \qquad \text{with} \quad \mathbf{G} = (x^2 + ye^x)\,\hat{\boldsymbol{\imath}} + e^x\,\hat{\boldsymbol{\jmath}}, \quad \mathbf{H} = x\cos y\,\hat{\boldsymbol{\jmath}}$$

Then $\mathbf{G}$ is conservative with potential $g = \frac{x^3}{3} + ye^x$ and $\mathbf{H}$ is pretty simple, so that it is not hard to evaluate $\int_C \mathbf{H} \cdot \mathrm{d}\mathbf{r}$ directly. Using the parametrization $\mathbf{r}(t) = \cos t\,\hat{\boldsymbol{\imath}} + t\,\hat{\boldsymbol{\jmath}}$,

$-\frac{\pi}{2} \leqslant t \leqslant \frac{\pi}{2}$ as above,

$$\begin{aligned}
\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_C \mathbf{G}\cdot \mathrm{d}\mathbf{r} + \int_C \mathbf{H}\cdot \mathrm{d}\mathbf{r}\\
&= \int_C \nabla g\cdot \mathrm{d}\mathbf{r} + \int_C \mathbf{H}\cdot \mathrm{d}\mathbf{r}\\
&= g\big(\mathbf{r}(\pi/2)\big) - g\big(\mathbf{r}(-\pi/2)\big) + \int_{-\pi/2}^{\pi/2} x(t)\cos(y(t))\frac{\mathrm{d}y}{\mathrm{d}t}(t)\ \mathrm{d}t\\
&= g\big(0,\pi/2\big) - g\big(0,-\pi/2\big) + \int_{-\pi/2}^{\pi/2} \cos^2 t\,\mathrm{d}t\\
&= \frac{\pi}{2} - \Big(-\frac{\pi}{2}\Big) + \int_{-\pi/2}^{\pi/2} \frac{\cos(2t)+1}{2}\mathrm{d}t\\
&= \pi + \Big[\frac{\sin(2t)}{4} + \frac{t}{2}\Big]_{-\pi/2}^{\pi/2}\\
&= \frac{3\pi}{2}
\end{aligned}$$

S-19: Call the region enclosed by the curve R. By Green's theorem, Theorem 4.3.2 in the CLP-4 text,

$$\frac{1}{2}\oint_C x\,\mathrm{d}y - y\,\mathrm{d}x = \frac{1}{2}\iint_R \Big(\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}(-y)\Big)\ \mathrm{d}x\mathrm{d}y = \frac{1}{2}\iint_R 2\ \mathrm{d}x\mathrm{d}y = A$$

as desired. The curve $x^{2/3}+y^{2/3}=1$ may be parametrized in the counterclockwise orientation by $x(\theta)=\cos^3\theta$, $y(\theta)=\sin^3\theta$, $0\leqslant\theta\leqslant 2\pi$. Then

$$\begin{aligned}
A &= \frac{1}{2}\oint_C x\,\mathrm{d}y - y\,\mathrm{d}x\\
&= \frac{1}{2}\int_0^{2\pi}\big(x(\theta)y'(\theta) - y(\theta)x'(\theta)\big)\ \mathrm{d}\theta = \frac{1}{2}\int_0^{2\pi}\big(3\cos^4\theta\sin^2\theta + 3\sin^4\theta\cos^2\theta\big)\ \mathrm{d}\theta\\
&= \frac{3}{2}\int_0^{2\pi}\sin^2\theta\cos^2\theta\ \mathrm{d}\theta = \frac{3}{8}\int_0^{2\pi}\sin^2(2\theta)\ \mathrm{d}\theta\\
&= \frac{3}{16}\int_0^{2\pi}\big(1-\cos(4\theta)\big)\ \mathrm{d}\theta = \frac{3}{16}\Big[\theta - \frac{1}{4}\sin(4\theta)\Big]_0^{2\pi} = \frac{3\pi}{8}
\end{aligned}$$

S-20: If we use D to denote the disk inside the circle C then we want

$$\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} - A\oint_C \mathbf{G}\cdot\mathrm{d}\mathbf{r} = \oint_C(\mathbf{F}-A\mathbf{G})\cdot\mathrm{d}\mathbf{r} = \iint_D \Big[\frac{\partial}{\partial x}(\mathbf{F}-A\mathbf{G})_2 - \frac{\partial}{\partial y}(\mathbf{F}-A\mathbf{G})_1\Big]\ \mathrm{d}x\mathrm{d}y$$

to vanish for all disks D. We used Green's theorem, which is Theorem 4.3.2 in the CLP-4 text, in the last step. This is the case if and only if

$$\begin{aligned}
&\frac{\partial}{\partial x}(\mathbf{F}-A\mathbf{G})_2 = \frac{\partial}{\partial y}(\mathbf{F}-A\mathbf{G})_1\\
\iff &\frac{\partial}{\partial x}[(x+y)-A(2x-3y)] = \frac{\partial}{\partial y}[(x+3y)-A(x+y)]\\
\iff &1-2A = 3-A\\
\iff &A=-2
\end{aligned}$$

S-21: (a) Parametrize the circle $\mathbf{r}(\theta) = (\cos\theta, \sin\theta)$. Then

$$\begin{aligned}
\mathbf{F}\big(\mathbf{r}(\theta)\big) &= \sin^3\theta\,\hat{\boldsymbol{\imath}} - \cos\theta\sin^2\theta\,\hat{\boldsymbol{\jmath}}\\
\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\theta}(\theta) &= -\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}\\
\mathbf{F}\big(\mathbf{r}(\theta)\big)\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\theta}(\theta) &= -\sin^4\theta - \cos^2\theta\sin^2\theta = -\sin^2\theta\\
\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \int_0^{2\pi}\mathbf{F}\big(\mathbf{r}(\theta)\big)\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\theta}(\theta)\,\mathrm{d}\theta = -\int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta = -\int_0^{2\pi}\frac{1-\cos(2\theta)}{2}\,\mathrm{d}\theta\\
&= -\Big[\frac{\theta}{2}-\frac{\sin(2\theta)}{4}\Big]_0^{2\pi} = -\pi
\end{aligned}$$

For an efficient, sneaky, way to evaluate $\int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta$ see Example 2.4.4 in the CLP-4 text.

(b) Denote by W the washer shaped region between the circle $x^2+y^2=1$ and the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$. It is sketched below. By Green's theorem

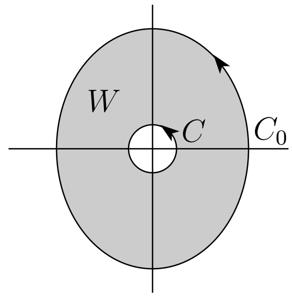
$$\oint_{C_0}\mathbf{F}\cdot\mathrm{d}\mathbf{r} - \oint_C\mathbf{F}\cdot\mathrm{d}\mathbf{r} = \iint_W\Big[\frac{\partial}{\partial x}\mathbf{F}_2 - \frac{\partial}{\partial y}\mathbf{F}_1\Big]\,\mathrm{d}x\mathrm{d}y$$

For the specified $\mathbf{F}$

$$\begin{aligned}
\frac{\partial}{\partial x}\mathbf{F}_2 - \frac{\partial}{\partial y}\mathbf{F}_1 &= -\frac{\partial}{\partial x}\frac{xy^2}{(x^2+y^2)^2} - \frac{\partial}{\partial y}\frac{y^3}{(x^2+y^2)^2}\\
&= -\frac{y^2}{(x^2+y^2)^2} + 2\frac{xy^2(2x)}{(x^2+y^2)^3} - \frac{3y^2}{(x^2+y^2)^2} + 2\frac{y^3(2y)}{(x^2+y^2)^3}\\
&= \frac{-y^2(x^2+y^2)+4x^2y^2-3y^2(x^2+y^2)+4y^4}{(x^2+y^2)^3}\\
&= 0
\end{aligned}$$

Consequently

$$\oint_{C_0}\mathbf{F}\cdot\mathrm{d}\mathbf{r} - \oint_C\mathbf{F}\cdot\mathrm{d}\mathbf{r} = 0 \implies \oint_{C_0}\mathbf{F}\cdot\mathrm{d}\mathbf{r} = \oint_C\mathbf{F}\cdot\mathrm{d}\mathbf{r} = -\pi$$

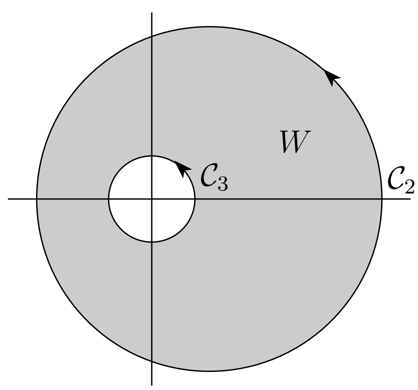


S-22: Observe that

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}\Big(\frac{x}{x^2+y^2}\Big) - \frac{\partial}{\partial y}\Big(\frac{-y}{x^2+y^2}\Big)$$
$$= \frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)-y(2y)}{(x^2+y^2)^2} = 0$$

except at $(0,0)$, where $\mathbf{F}$ is not defined. Hence by Green's theorem (Theorem 4.3.2 in the CLP-4 text), $\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = 0$ for any closed curve that does not contain $(0,0)$ in its interior. In particular, $\oint_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{r} = 0$. On the other hand, $(0,0)$ is contained in the interior of $\mathcal{C}_2$, so we cannot use Green's theorem to conclude that $\oint_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r} = 0$.

Let $\mathcal{C}_3$ be the circle of radius one centred on $(0,0)$ and denote by W the washer shaped region between the circle $\mathcal{C}_2$ and the circle $\mathcal{C}_3$. It is sketched below.



By Green's theorem (Theorem 4.3.2 in the CLP-4 text),

$$\oint_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r} - \oint_{\mathcal{C}_3} \mathbf{F}\cdot d\mathbf{r} = \iint_W \Big[\frac{\partial}{\partial x}\mathbf{F}_2 - \frac{\partial}{\partial y}\mathbf{F}_1\Big]\, dx dy = 0$$

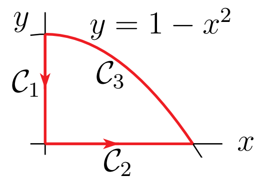
So $\oint_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r} = \oint_{\mathcal{C}_3} \mathbf{F}\cdot d\mathbf{r}$. Parameterize $\mathcal{C}_3$ by $x = \cos\theta$, $y = \sin\theta$. Then

$$\begin{aligned}
\mathbf{r}(\theta) &= \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}} \qquad 0 \leqslant \theta \leqslant 2\pi\\
\mathbf{r}'(\theta) &= -\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}\\
\mathbf{F}\big(\mathbf{r}(\theta)\big) &= -\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}\\
\mathbf{F}\big(\mathbf{r}(\theta)\big)\cdot\mathbf{r}'(\theta) &= 1
\end{aligned}$$

so that

$$\oint_{\mathcal{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_{\mathcal{C}_3} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} \mathrm{d}\theta \, 1 = 2\pi$$

S-23: (a) Let $\mathcal{C}_1$ be the line segment from $(0,1)$ to $(0,0)$, $\mathcal{C}_2$ be the line segment from $(0,0)$ to $(1,0)$ and $\mathcal{C}_3$ be the curve $y = 1 - x^2$ from $(1,0)$ to $(0,1)$.



Then

$$\int_{\mathcal{C}_1} x \, \mathrm{d}s = \int_{\mathcal{C}_1} 0 \, \mathrm{d}s = 0$$

$$\int_{\mathcal{C}_2} x \, \mathrm{d}s = \int_0^1 x \, \mathrm{d}x = \frac{1}{2}$$

On $\mathcal{C}_3$, $y = 1 - x^2$ so that $\frac{\mathrm{d}y}{\mathrm{d}x} = -2x$ and

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{1 + \Big(\frac{\mathrm{d}y}{\mathrm{d}x}\Big)^2} \, \mathrm{d}x = \sqrt{1 + 4x^2} \, \mathrm{d}x$$

and

$$\begin{aligned} \int_{\mathcal{C}_3} x \, \mathrm{d}s &= \int_0^1 x\sqrt{1 + 4x^2} \, \mathrm{d}x \\ &= \Big[\frac{1}{12}(1 + 4x^2)^{3/2}\Big]_0^1 = \frac{1}{12}\big[5^{3/2} - 1\big] \end{aligned}$$

All together

$$\int_{\mathcal{C}} x \, \mathrm{d}s = \frac{1}{2} + \frac{1}{12}\big[5^{3/2} - 1\big] \approx 1.3484$$

(b) By either Stokes' theorem or Green's theorem

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \iint_R \Big[\frac{\partial}{\partial x}\big(x^2 + \cos(y^2)\big) - \frac{\partial}{\partial y}\big(\sin(x^2) - xy\big)\Big] \, \mathrm{d}x\mathrm{d}y = \iint_R 3x \, \mathrm{d}x\mathrm{d}y \\ &= 3\int_0^1 \mathrm{d}x \int_0^{1-x^2} \mathrm{d}y \, x = 3\int_0^1 \mathrm{d}x \, (1 - x^2)x = 3\Big[\frac{1}{2} - \frac{1}{4}\Big] = \frac{3}{4} \end{aligned}$$

S-24: (a) If (x, y, z) is on the curve, it must obey both $z = x + y$ and $z = x^2 + y^2$ and hence it must also obey $x^2 + y^2 = x + y$ or $(x - 1/2)^2 + (y - 1/2)^2 = 1/2$. That's a circle. We can parametrize the curve by

$$\begin{aligned}
x(\theta) &= \frac{1}{2} + \frac{1}{\sqrt{2}}\cos\theta \\
y(\theta) &= \frac{1}{2} + \frac{1}{\sqrt{2}}\sin\theta \\
z(\theta) &= x + y = 1 + \frac{1}{\sqrt{2}}\big[\cos\theta + \sin\theta\big]
\end{aligned}$$

with $0 \leqslant \theta < 2\pi$. As θ runs from 0 to 2π, $\big(x(\theta), y(\theta)\big)$ runs once around the circle without crossing itself so that $\big(x(\theta), y(\theta), z(\theta)\big)$ runs once around the curve without crossing itself. As $\big(x(2\pi), y(2\pi), z(2\pi)\big) = \big(x(0), y(0), z(0)\big)$, C is a simple closed curve.

(b) (i) The vector field $\mathbf{F} = x^2\,\hat{\imath} + y^2\,\hat{\jmath} + 3e^z\,\hat{\mathbf{k}}$ is conservative (with potential $\frac{1}{3}x^3 + \frac{1}{3}y^3 + 3e^z$). So $\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = 0$.

(b) (ii) Note that the question did not specify the orientation of C. It should have. We'll stick with the most commonly used orientation — counterclockwise when viewed from high on the z-axis. The vector field $\mathbf{G} = 3e^z\,\hat{\mathbf{k}}$ is conservative (with potential $3e^z$). So $\oint_C \mathbf{G}\cdot \mathrm{d}\mathbf{r} = 0$ and, using the parametrization

$$\begin{aligned}
\mathbf{r}(\theta) &= \Big[\frac{1}{2} + \frac{1}{\sqrt{2}}\cos\theta\Big]\hat{\imath} + \Big[\frac{1}{2} + \frac{1}{\sqrt{2}}\sin\theta\Big]\hat{\jmath} + \Big[1 + \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta\Big]\hat{\mathbf{k}} \\
\mathbf{r}'(\theta) &= -\frac{1}{\sqrt{2}}\sin\theta\,\hat{\imath} + \frac{1}{\sqrt{2}}\cos\theta\,\hat{\jmath} + \Big[\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\Big]\hat{\mathbf{k}}
\end{aligned}$$

of part (a), we have

$$\begin{aligned}
\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \oint_C (\mathbf{F} - \mathbf{G})\cdot \mathrm{d}\mathbf{r} \\
&= \int_0^{2\pi} \big[y(\theta)^2 x'(\theta) + x(\theta)^2 y'(\theta)\big]\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \Big\{ -\Big[\frac{1}{2} + \frac{1}{\sqrt{2}}\sin\theta\Big]^2 \frac{1}{\sqrt{2}}\sin\theta + \Big[\frac{1}{2} + \frac{1}{\sqrt{2}}\cos\theta\Big]^2 \frac{1}{\sqrt{2}}\cos\theta \Big\}\mathrm{d}\theta
\end{aligned}$$

Because the integral of any odd power of $\sin\theta$ or $\cos\theta$ over $0 \leqslant \theta \leqslant 2\pi$ is zero (see Example 4.4.6 in the text),

$$\begin{aligned}
\oint_c \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_0^{2\pi} \Big\{ -\frac{1}{2}\sin^2\theta + \frac{1}{2}\cos^2\theta \Big\}\mathrm{d}\theta \\
&= 0
\end{aligned}$$

since (see Example 2.4.4 in the text)

$$\int_0^{2\pi} \cos^2\theta\ \mathrm{d}\theta = \int_0^{2\pi} \sin^2\theta\ \mathrm{d}\theta = \pi$$

S-25: By Green's Theorem

$$\oint_C (y^3 - y)\,\mathrm{d}x - 2x^3\,\mathrm{d}y = \iint_R \left[\frac{\partial}{\partial x}(-2x^3) - \frac{\partial}{\partial y}(y^3 - y)\right]\mathrm{d}x\,\mathrm{d}y = \iint_R \left[1 - 6x^2 - 3y^2\right]\mathrm{d}x\,\mathrm{d}y$$

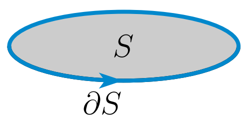
where R is the region in the xy-plane whose boundary is C. Observe that the integrand $1 - 6x^2 - 3y^2$ is positive in the elliptical region $6x^2 + 3y^2 \leqslant 1$ and negative outside of it. To maximize the integral $\iint_R \left[1 - 6x^2 - 3y^2\right]\mathrm{d}x\,\mathrm{d}y$ we should choose R to contain all points (x, y) with the integrand $1 - 6x^2 - 3y^2 \geqslant 0$ and to exclude all points (x, y) with the integrand $1 - 6x^2 - 3y^2 < 0$. So we choose

$$R = \{\, (x, y) \mid 6x^2 + 3y^2 \leqslant 1 \,\}$$

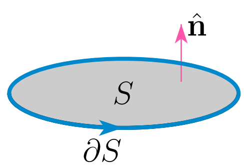
The corresponding C is $6x^2 + 3y^2 = 1$.

Solutions to Exercises 4.4 — Jump to TABLE OF CONTENTS

S-1: One approach is to first consider



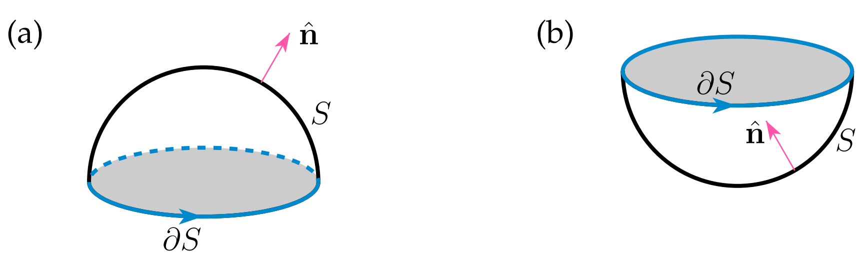
The correct normal to this surface is sketched in



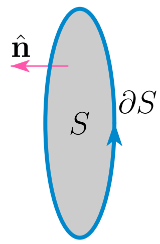
It is correct because

- if you walk along ∂S in the direction of the arrow on ∂S,
- with the vector from your feet to your head having direction $\hat{\mathbf{n}}$
- then S is on your left hand side.

Now pretend that the surface S is made of rubber and that $\hat{\mathbf{n}}$ is glued to S. We can push on this S to deform it to the S of part (a) or to the S of part (b). This gives the solutions to parts (a) and (b).

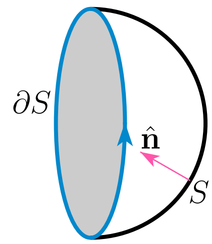


To deal with part (c), we can first rotate the flat disk that we considered above to get

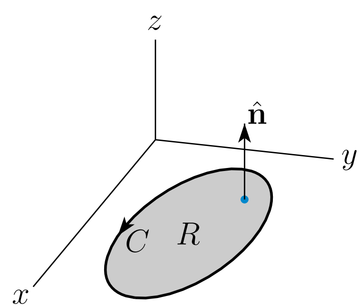


We can push on this S to deform it to the S of part (c). This gives the solution to part (c).

(c)



S-2: Think of the xy-plane as being the plane $z = 0$ in $\mathbb{R}^3$.



We are going to apply Stokes' theorem (Theorem 4.4.1 in the CLP-4 text) with S being the given region R in the xy-plane and with $\mathbf{F}(x, y, z) = F_1(x, y)\,\hat{\imath} + F_2(x, y)\,\hat{\jmath}$. Then

- the unit normal vector to S specified in Stokes theorem is $\hat{\mathbf{k}}$ (if you walk along $\partial S = C$ in the direction of the arrow on C with the vector from your feet to your head having direction $\hat{\mathbf{k}}$ then $S = R$ is on your left hand side) and
- $\mathrm{d}S = \mathrm{d}x\,\mathrm{d}y$ and
- the curl of $\mathbf{F}$ is

$$\nabla \times \mathbf{F} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1(x, y) & F_2(x, y) & 0 \end{bmatrix} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\mathbf{k}}$$

So Stokes' theorem gives

$$\oint_C \left[F_1(x, y)\,\mathrm{d}x + F_2(x, y)\,\mathrm{d}y\right] = \oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathrm{d}x\mathrm{d}y$$

S-3: We are to show that $\oint_C [\phi\nabla\psi + \psi\nabla\phi]\cdot \mathrm{d}\mathbf{r} = 0$. Suppose that $C = \partial S$. Then, by Stokes' theorem

$$\oint_C [\phi\nabla\psi + \psi\nabla\phi]\cdot \mathrm{d}\mathbf{r} = \iint_S \nabla\times[\phi\nabla\psi + \psi\nabla\phi]\cdot\hat{\mathbf{n}}\,\mathrm{d}S$$

We will show below that $\nabla\times[\phi\nabla\psi + \psi\nabla\phi] = \mathbf{0}$. This will imply that $\oint_C [\phi\nabla\psi + \psi\nabla\phi]\cdot \mathrm{d}\mathbf{r} = \mathbf{0}$. One way to see that $\nabla\times[\phi\nabla\psi + \psi\nabla\phi] = \mathbf{0}$ is

$$\begin{aligned}
\nabla\times[\phi\nabla\psi + \psi\nabla\phi] &= \nabla\times[\nabla(\phi\psi)] && \text{(by part (c) of Theorem 4.1.3 in the CLP-4 text)}\\
&= \mathbf{0} && \text{(by part (b) of Theorem 4.1.7 in the CLP-4 text)}
\end{aligned}$$

Another way to see that $\nabla\times[\phi\nabla\psi + \psi\nabla\phi] = \mathbf{0}$ is

$$\begin{aligned}
\nabla\times[\phi\nabla\psi + \psi\nabla\phi] &= \nabla\phi\times\nabla\psi + \phi\nabla\times(\nabla\psi) + \nabla\psi\times\nabla\phi + \psi\nabla\times(\nabla\phi)\\
&= \nabla\phi\times\nabla\psi + \nabla\psi\times\nabla\phi \qquad \text{since } \phi\nabla\times(\nabla\psi) = \psi\nabla\times(\nabla\phi) = \mathbf{0}\\
&= \mathbf{0}
\end{aligned}$$

S-4: (a) Observe that $x(t) = \cos t$ and $y(t) = \sin t$ obey $x(t)^2 + y(t)^2 = 1$. Then $z(t) = y(t)^2 = \sin^2 t$. So we may parametrize the curve by $\mathbf{r}(t) = (\cos t, \sin t, \sin^2 t)$ with $0 \leqslant t \leqslant 2\pi$. Then

$$\begin{aligned}
\mathbf{r}'(t) &= (-\sin t\,,\ \cos t\,,\ 2\sin t\cos t)\\
\mathbf{F}\big(\mathbf{r}(t)\big) &= \big(\cos^2 t - \sin t\,,\ \sin^2 t + \cos t\,,\ 1\big)\\
\mathbf{F}\big(\mathbf{r}(t)\big)\cdot\mathbf{r}'(t) &= -\sin t\cos^2 t + \sin^2 t + \sin^2 t\cos t + \cos^2 t + 2\sin t\cos t\\
&= 1 + \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}[\cos^3 t + \sin^3 t] + \sin(2t)\\
\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \int_0^{2\pi}\left\{1 + \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}[\cos^3 t + \sin^3 t] + \sin(2t)\right\}\,\mathrm{d}t\\
&= \left[t + \frac{1}{3}[\cos^3 t + \sin^3 t] - \frac{1}{2}\cos(2t)\right]_0^{2\pi}\\
&= 2\pi
\end{aligned}$$

(b) Let S be the surface $z = f(x,y)$ with $f(x,y) = y^2$ and $x^2 + y^2 \leqslant 1$. Since C is oriented counter clockwise when viewed from high on the z-axis, Stokes' theorem requires that we use the normal $\hat{\mathbf{n}}$ to S with positive z component. Hence

$$\begin{aligned}
\hat{\mathbf{n}}\, dS &= \Big[-\frac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} - \frac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\Big]\,\mathrm{d}x\,\mathrm{d}y = \Big[-2y\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\Big]\,\mathrm{d}x\,\mathrm{d}y \\
\nabla \times \mathbf{F} &= \det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y & y^2 + x & 1 \end{bmatrix} = 2\hat{\mathbf{k}} \\
\nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, dS &= 2\,\mathrm{d}x\,\mathrm{d}y \\
\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S = 2 \iint_{x^2+y^2 \leqslant 1} \mathrm{d}x\,\mathrm{d}y \\
&= 2\pi
\end{aligned}$$

S-5: We apply Stokes' theorem. First,

$$\nabla \times \mathbf{F} = \det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^x & x + e^x & z^2 \end{bmatrix} = \big(1 + e^x - e^x\big)\,\hat{\mathbf{k}} = \hat{\mathbf{k}}$$

Note that $\mathbf{r}(t) = x(t)\,\hat{\boldsymbol{\imath}} + y(t)\,\hat{\boldsymbol{\jmath}} + z(t)\,\hat{\mathbf{k}}$ obeys $x(t) + y(t) + z(t) = 3$, for every t, and that $x(t)\,\hat{\boldsymbol{\imath}} + y(t)\,\hat{\boldsymbol{\jmath}} = (1 + \cos t)\,\hat{\boldsymbol{\imath}} + (1 + \sin t)\,\hat{\boldsymbol{\jmath}}$ runs counterclockwise around the circle of radius 1 centered on $(1,1)$. So we choose S to be the part of the plane $G(x,y,z) = x + y + z = 3$ with $(x-1)^2 + (y-1)^2 \leqslant 1$. Then, by Stokes' Theorem,

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S = \iint_S \hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\,\mathrm{d}S$$

with

$$\hat{\mathbf{n}}\,\mathrm{d}S = \pm\frac{\nabla G}{\nabla G \cdot \hat{\mathbf{k}}}\,\mathrm{d}x\mathrm{d}y = \pm\big(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\big)\,\mathrm{d}x\mathrm{d}y$$

As $(1 + \cos t)\,\hat{\boldsymbol{\imath}} + (1 + \sin t)\,\hat{\boldsymbol{\jmath}}$ runs counterclockwise around the circle $(x-1)^2 + (y-1)^2 \leqslant 1$, Stokes' theorem specifies the plus sign and

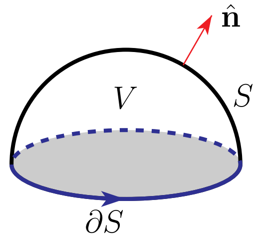
$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_{(x-1)^2+(y-1)^2 \leqslant 1} \mathrm{d}x\,\mathrm{d}y = \pi$$

S-6: The boundary of S is

$$\partial S = \big\{\ (x,y,z)\ \big|\ z = 0,\ x^2 + y^2 = 4\ \big\}$$

and can be parametrized

$$\mathbf{r}(\theta) = 2\cos\theta\,\hat{\boldsymbol{\imath}} + 2\sin\theta\,\hat{\boldsymbol{\jmath}} \qquad 0 \leqslant \theta \leqslant 2\pi$$



So, by Stokes' theorem (Theorem 4.4.1 in the CLP-4 text)

$$\begin{aligned}
\iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S &= \oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\
&= \int_0^{2\pi} \overbrace{(-2\sin\theta\,\hat{\boldsymbol{\imath}} + 2\cos\theta\,\hat{\boldsymbol{\jmath}} - 2\cos\theta\,\hat{\mathbf{k}})}^{\mathbf{F}(\mathbf{r}(\theta))} \cdot \overbrace{(-2\sin\theta\,\hat{\boldsymbol{\imath}} + 2\cos\theta\,\hat{\boldsymbol{\jmath}})}^{\mathbf{r}'(\theta)}\, \mathrm{d}\theta \\
&= 4\int_0^{2\pi} \mathrm{d}\theta \\
&= 8\pi
\end{aligned}$$

S-7: The boundary of $\mathcal{S}$ is the circle $x^2 + y^2 = 4$, $z = 0$. Let $\mathcal{C}$ be this circle, oriented by the parametrization $x(t) = 2\cos t$, $y(t) = 2\sin t$, $z(t) = 0$. By Stokes' theorem

$$\begin{aligned}
\iint_{\mathcal{S}} \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(2\cos t, 2\sin t, 0) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\, \mathrm{d}t \\
&= \int_0^{2\pi} \big[0\,\hat{\boldsymbol{\imath}} + 2\cos t(3 + 2\sin t)\,\hat{\boldsymbol{\jmath}} + 2\sin t\,\hat{\mathbf{k}}\big] \cdot \big[-2\sin t\,\hat{\boldsymbol{\imath}} + 2\cos t\,\hat{\boldsymbol{\jmath}}\big]\, \mathrm{d}t \\
&= \int_0^{2\pi} \big[12\cos^2 t + 8\cos^2 t\sin t\big]\, \mathrm{d}t \\
&= \int_0^{2\pi} \big[6 + 6\cos(2t) + 8\cos^2 t\sin t\big]\, \mathrm{d}t \\
&= \Big[6t + 3\sin(2t) - \frac{8}{3}\cos^3 t\Big]_0^{2\pi} = 12\pi
\end{aligned}$$

For an efficient, sneaky, way to evaluate $\int_0^{2\pi} \cos^2 t\, \mathrm{d}t$, see Example 2.4.4 in the CLP-4 text.

S-8: Let S be the portion of the paraboloid $z = f(x, y) = 4 - x^2 - y^2$ with $x^2 + (y-1)^2 \leqslant 1$ and let $\hat{\mathbf{n}}$ be the upward normal to S. For this surface

$$\hat{\mathbf{n}}\, \mathrm{d}S = \big(-f_x(x, y)\,\hat{\boldsymbol{\imath}} - f_y(x, y)\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\big)\mathrm{d}x\mathrm{d}y = \big(2x\,\hat{\boldsymbol{\imath}} + 2y\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\big)\, \mathrm{d}x\mathrm{d}y$$

by (3.3.2) in the CLP-4 text. As (x, y, z) runs over S, (x, y) runs over the circular disk

$$D = \big\{\, (x, y) \;\big|\; x^2 + (y-1)^2 \leqslant 1 \,\big\}$$

For the given vector field

$$\begin{aligned}\nabla \times \mathbf{F} &= \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & x & yz \end{bmatrix} \\ &= z\,\hat{\boldsymbol{\imath}} + x\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\end{aligned}$$

so that, by Stokes' theorem (Theorem 4.4.1 in the CLP-4 text),

$$\begin{aligned}\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \iint_S \nabla \times \mathbf{F}\cdot \hat{\mathbf{n}}\,\mathrm{d}S \\ &= \iint_D \Big[2x\,\overbrace{(4-x^2-y^2)}^{z=f(x,y)} + 2xy + 1\Big]\,\mathrm{d}x\mathrm{d}y\end{aligned}$$

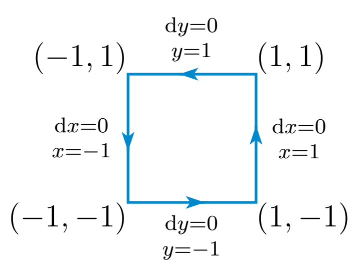
By oddness under $x \to -x$, all terms integrate to zero except for the last. So

$$\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_D \mathrm{d}x\mathrm{d}y = \mathrm{Area}(D) = \pi$$

S-9: The surface

$$\begin{aligned}S &= \big\{\,(x,y,z)\ \big|\ -1 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 1,\ z \geqslant 0,\ z = (1-x^2)(1-y^2)\,\big\} \\ &= \big\{\,(x,y,z)\ \big|\ -1 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 1,\ \ z = (1-x^2)(1-y^2)\,\big\}\end{aligned}$$

Note that when $x = 1$ or $x = -1$ or $y = 1$ or $y = -1$, we have $z = (1-x^2)(1-y^2) = 0$. So the boundary of S, call it C, is the boundary of the square $-1 \leqslant x, y \leqslant 1$, $z = 0$, oriented counterclockwise. Here is a sketch of C.



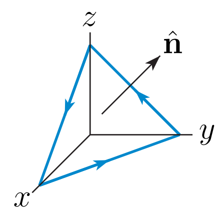
Apply Stokes' theorem. Observing that $z = 0$ on C so that $\mathbf{F} = -y\,\hat{\boldsymbol{\imath}} + x^3\,\hat{\boldsymbol{\jmath}}$,

$$\begin{aligned}\iint_S \nabla \times \mathbf{F}\cdot \hat{\mathbf{n}}\,\mathrm{d}S &= \oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \oint_C [-y\,\hat{\boldsymbol{\imath}} + x^3\hat{\boldsymbol{\jmath}}]\cdot \mathrm{d}\mathbf{r} \\ &= \underbrace{\int_{-1}^{1} -(-1)\,\mathrm{d}x}_{y=-1\text{ side}} + \underbrace{\int_{-1}^{1} (1)^3\,\mathrm{d}y}_{x=1\text{ side}} + \underbrace{\int_{1}^{-1} -(1)\,\mathrm{d}x}_{y=1\text{ side}} + \underbrace{\int_{1}^{-1} (-1)^3\,\mathrm{d}y}_{x=-1\text{ side}} \\ &= 8\end{aligned}$$

S-10: We shall apply Stokes' Theorem. The curl of $\mathbf{F}$ is

$$\boldsymbol{\nabla}\times\mathbf{F} = \det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{x^2}-yz & \sin y - yz & xz+2y \end{bmatrix} = (2+y)\,\hat{\boldsymbol{\imath}} - (z+y)\,\hat{\boldsymbol{\jmath}} + (0+z)\,\hat{\mathbf{k}}$$

The curve C is a triangle. All three vertices of the triangle obey $x+y+z=1$. So the triangle is the boundary of the surface $S = \big\{\,(x,y,z)\,\big|\, x\geqslant 0,\ y\geqslant 0,\ z=1-x-y\geqslant 0\,\big\}$.



The equation of the surface is $z=f(x,y)=1-x-y$. So, by (3.3.2) in the CLP-4 text,

$$\begin{aligned}\hat{\mathbf{n}}\,\mathrm{d}S &= \big(-f_x\,\hat{\boldsymbol{\imath}} - f_y\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\big)\,\mathrm{d}x\,\mathrm{d}y \\ &= (\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}}+\hat{\mathbf{k}})\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

Here $\hat{\mathbf{n}}$ is the upward pointing unit normal. The set of points (x,y) for which there is a corresponding (x,y,z) in S is $T=\big\{\,(x,y)\,\big|\,x\geqslant 0,\ y\geqslant 0,\ x+y\leqslant 1\,\big\}$, which is a triangle of area $\frac{1}{2}$. Since $\boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = [(2+y)\hat{\boldsymbol{\imath}} - (z+y)\hat{\boldsymbol{\jmath}} + (0+z)\hat{\mathbf{k}}]\cdot(\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}}+\hat{\mathbf{k}})\,\mathrm{d}x\,\mathrm{d}y = 2\,\mathrm{d}x\,\mathrm{d}y$,

$$\begin{aligned}\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \iint_S \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S \\ &= \iint_T 2\,\mathrm{d}x\,\mathrm{d}y = 2\,\mathrm{Area}(T) = 1\end{aligned}$$

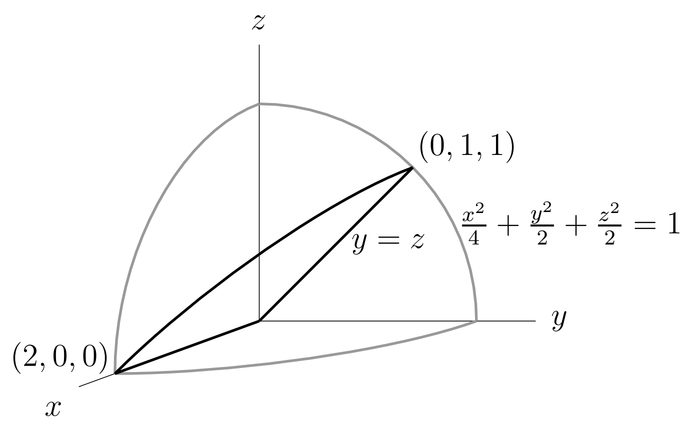
S-11: Stokes' theorem, which is Theorem 4.4.1 in the CLP-4 text, says that $\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \iint_S \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ for any surface S whose boundary is C. For the given vector field

$$\begin{aligned}\nabla\times\mathbf{F}(x,y,z) &= \det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z & x & y \end{bmatrix} \\ &= \hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\end{aligned}$$

Choose

$$\begin{aligned}S &= \big\{\,(x,y,z)\,\big|\,z=y,\ \tfrac{x^2}{4}+\tfrac{y^2}{2}+\tfrac{z^2}{2}\leqslant 1\,\big\} \\ &= \big\{\,(x,y,z)\,\big|\,z=y,\ \tfrac{x^2}{4}+y^2\leqslant 1\,\big\}\end{aligned}$$

to be the part of the plane $z=y$ bounded by the ellipsoid.



As S is part of the plane $z = f(x, y) = y$, (3.3.2) of the CLP-4 text, gives that

$$\begin{aligned}\hat{\mathbf{n}}\,\mathrm{d}S &= \pm\big(-f_x\,,\,-f_y\,,\,1\big)\mathrm{d}x\mathrm{d}y \\ &= \pm(0\,,\,-1\,,\,1)\mathrm{d}x\mathrm{d}y\end{aligned}$$

As C has the standard orientation (counter-clockwise when viewed from high on the z-axis), we want $\hat{\mathbf{n}}$ to have a positive z-component. So $\hat{\mathbf{n}}\,\mathrm{d}S = (0\,,\,-1\,,\,1)\mathrm{d}x\mathrm{d}y$. From the second form of S given above, we see that as (x, y, z) runs over S, (x, y) runs over

$$D = \left\{\,(x, y)\;\middle|\;\tfrac{x^2}{4} + y^2 \leqslant 1\,\right\}$$

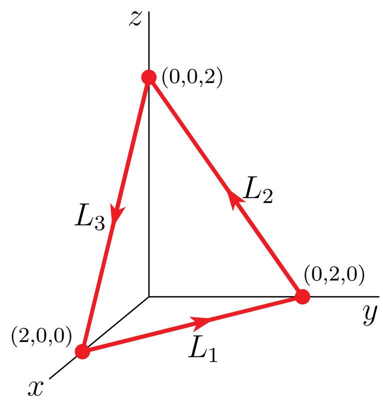
Consequently, Stokes' theorem gives that

$$\begin{aligned}\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \iint_D \overbrace{(1,-1,1)}^{\nabla\times\mathbf{F}}\cdot\overbrace{(0\,,\,-1\,,\,1)\mathrm{d}x\mathrm{d}y}^{\hat{\mathbf{n}}\,\mathrm{d}S} \\ &= 2\iint_D \mathrm{d}x\mathrm{d}y = 2\,\mathrm{Area}(D)\end{aligned}$$

The ellipse D, that is $\frac{x^2}{4} + y^2 \leqslant 1$, has semi-axes $a = 2$ and $b = 1$ and hence area $\pi ab = 2\pi$. Finally

$$\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = 2\,\mathrm{Area}(D) = 4\pi$$

S-12: Note that the curve of part (a) is a simple closed curve that lies in the plane $x + y + z = 2$ and is oriented in a counterclockwise direction as observed from the positive x-axis. The curve of part (a) encloses a triangle. Two of the sides of the triangle



are $(0,2,0) - (2,0,0) = (-2,2,0)$ and $(0,0,2) - (0,2,0) = (0,-2,2)$ so the area of the triangle is

$$\frac{1}{2}\big|(-2,2,0) \times (0,-2,2)\big| = \frac{1}{2}\det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ -2 & 2 & 0 \\ 0 & -2 & 2 \end{bmatrix} = \frac{1}{2}\big|(4,4,4)\big| = 2\sqrt{3}$$

So let's do part (b) first.

(b) We are not told explicitly what C_2 is, so we certainly can't do a direct evaluation. Instead, let's use Stokes' theorem (Theorem 4.4.1 in the CLP-4 text). The curl of $\mathbf{F}$ is

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{F} &= \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & x^2 & y^2 \end{bmatrix} \\ &= 2y\,\hat{\imath} + 2z\,\hat{\jmath} + 2x\,\hat{\mathbf{k}} \end{aligned}$$

The upward pointing unit normal to E is $\hat{\mathbf{n}} = \frac{\hat{\imath}+\hat{\jmath}+\hat{\mathbf{k}}}{\sqrt{3}}$. So, by Stokes' theorem,

$$\begin{aligned} I_2 &= \int_{C_2} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_R \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_R \big(2y\,\hat{\imath} + 2z\,\hat{\jmath} + 2x\,\hat{\mathbf{k}}\big)\cdot\frac{\hat{\imath}+\hat{\jmath}+\hat{\mathbf{k}}}{\sqrt{3}}\,\mathrm{d}S \\ &= \frac{2}{\sqrt{3}}\iint_R \big(\overbrace{y+z+x}^{=2 \text{ on } R}\big)\,\mathrm{d}S = \frac{4}{\sqrt{3}}\text{Area}(R) = 4\sqrt{3} \end{aligned}$$

(a) Denote by T the triangle enclosed by C_1. By the computation that we have just done in part (b)

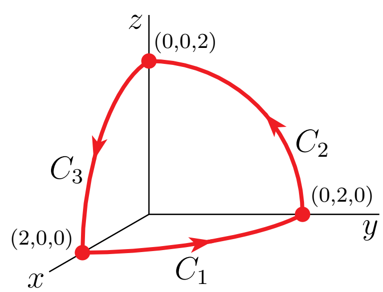
$$I_1 = \int_{C_1} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \frac{4}{\sqrt{3}}\text{Area}(T) = 8$$

S-13: (a) Observe that

- the curve C_1 is one quarter of a circle in the xy-plane, centred on the origin, of radius 2, starting at $(2,0,0)$ and ending at $(0,2,0)$ and

- the curve C_2 is one quarter of a circle in the yz-plane, centred on the origin, of radius 2, starting at $(0,2,0)$ and ending at $(0,0,2)$ and
- the curve C_3 is one quarter of a circle in the xz-plane, centred on the origin, of radius 2, starting at $(0,0,2)$ and ending at $(2,0,0)$.

Here is a sketch.



(b) C lies completely on the sphere $x^2+y^2+z^2=4$. So it is natural to choose

$$S = \left\{ (x,y,z) \,\middle|\, x^2+y^2+z^2=4,\ x\geqslant 0,\ y\geqslant 0,\ z\geqslant 0 \right\}$$

and to parametrize S using spherical coordinates

$$\mathbf{r}(\theta,\varphi) = 2\cos\theta\sin\varphi\,\hat{\imath} + 2\sin\theta\sin\varphi\,\hat{\jmath} + 2\cos\varphi\,\hat{\mathbf{k}}, \qquad 0\leqslant\theta\leqslant\frac{\pi}{2},\ 0\leqslant\varphi\leqslant\frac{\pi}{2}$$

Since

$$\begin{aligned}\frac{\partial\mathbf{r}}{\partial\theta} &= -2\sin\theta\sin\varphi\,\hat{\imath} + 2\cos\theta\sin\varphi\,\hat{\jmath}\\ \frac{\partial\mathbf{r}}{\partial\varphi} &= 2\cos\theta\cos\varphi\,\hat{\imath} + 2\sin\theta\cos\varphi\,\hat{\jmath} - 2\sin\varphi\,\hat{\mathbf{k}}\end{aligned}$$

so that

$$\begin{aligned}\frac{\partial\mathbf{r}}{\partial\theta}\times\frac{\partial\mathbf{r}}{\partial\varphi} &= \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ -2\sin\theta\sin\varphi & 2\cos\theta\sin\varphi & 0\\ 2\cos\theta\cos\varphi & 2\sin\theta\cos\varphi & -2\sin\varphi\end{bmatrix}\\ &= -4\cos\theta\sin^2\varphi\,\hat{\imath} - 4\sin\theta\sin^2\varphi\,\hat{\jmath} - 4\sin\varphi\cos\varphi\,\hat{\mathbf{k}}\end{aligned}$$

(3.3.1) in the CLP-4 text gives

$$\hat{\mathbf{n}}\,\mathrm{d}S = \pm\frac{\partial\mathbf{r}}{\partial\theta}\times\frac{\partial\mathbf{r}}{\partial\varphi}\,\mathrm{d}\theta\mathrm{d}\varphi = \mp 4\big(\cos\theta\sin\varphi\,\hat{\imath} + \sin\theta\sin\varphi\,\hat{\jmath} + \cos\varphi\,\hat{\mathbf{k}}\big)\sin\varphi\,\mathrm{d}\theta\mathrm{d}\varphi$$

We want $\hat{\mathbf{n}}$ to point outward, for compatibility with the orientation of C. So we choose the $+$ sign.

$$\hat{\mathbf{n}}\,\mathrm{d}S = 4\big(\cos\theta\sin\varphi\,\hat{\imath} + \sin\theta\sin\varphi\,\hat{\jmath} + \cos\varphi\,\hat{\mathbf{k}}\big)\sin\varphi\,\mathrm{d}\theta\mathrm{d}\varphi = 2\,\mathbf{r}(\theta,\varphi)\sin\varphi\,\mathrm{d}\theta\mathrm{d}\varphi$$

(c) The vector field $\mathbf{F}$ looks too complicated for a direct evaluation of the line integral. So, in preparation for an application of Stokes' theorem, we compute

$$\begin{aligned}
\nabla \times \mathbf{F} &= \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+\sin(x^2) & z-3x+\ln(1+y^2) & y+e^{z^2} \end{bmatrix} \\
&= -4\,\hat{\mathbf{k}}
\end{aligned}$$

So, by Stokes' theorem (Theorem 4.4.1 in the CLP-4 text),

$$\begin{aligned}
\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \iint_S \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S \\
&= \int_0^{\pi/2}\mathrm{d}\varphi\int_0^{\pi/2}\mathrm{d}\theta\ \big(-4\hat{\mathbf{k}}\big)\cdot\big(\cos\theta\sin\varphi\,\hat{\boldsymbol{\imath}}+\sin\theta\sin\varphi\,\hat{\boldsymbol{\jmath}}+\cos\varphi\,\hat{\mathbf{k}}\big)\,4\sin\varphi \\
&= -16\int_0^{\pi/2}\mathrm{d}\varphi\int_0^{\pi/2}\mathrm{d}\theta\ \cos\varphi\sin\varphi = 16\,\frac{\pi}{2}\,\frac{\cos^2\varphi}{2}\bigg|_0^{\pi/2} \\
&= -4\pi
\end{aligned}$$

S-14: (a) The boundary, ∂S_1, of S_1 as specified in Stokes' theorem (Theorem 4.4.1 in the CLP-4 text) is the circle $\sqrt{x^2+y^2}=4$, $z=4$ oriented *clockwise* when viewed from high on the z-axis. That is, we can parametrize ∂S_1 by

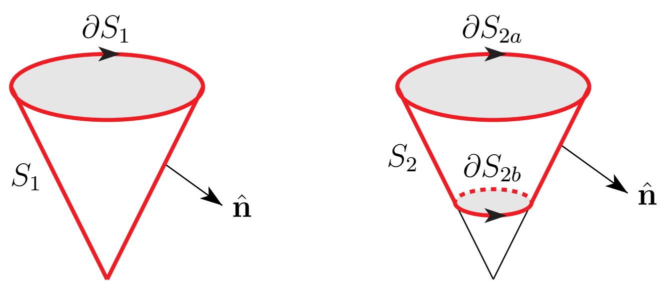
$$\mathbf{r}(t) = 4\cos t\,\hat{\boldsymbol{\imath}} - 4\sin t\,\hat{\boldsymbol{\jmath}} + 4\,\hat{\mathbf{k}}, \qquad 0\leqslant t\leqslant 2\pi$$

So

$$\begin{aligned}
\mathbf{F}\big(\mathbf{r}(t)\big)\cdot\mathrm{d}\mathbf{r} &= \big(16\sin t\,,\,16\cos t\,,\,-16\sin t\cos t\cos(-16\sin t)\big)\cdot\big(-4\sin t\,,\,-4\cos t\,,\,0\big)\,\mathrm{d}t \\
&= -64\,\mathrm{d}t
\end{aligned}$$

and, by Stokes' theorem,

$$\iint_{S_1}\nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \oint_{\partial S_1}\mathbf{F}\big(\mathbf{r}(t)\big)\cdot\mathrm{d}\mathbf{r} = -64\int_0^{2\pi}\mathrm{d}t = -128\pi$$



(b) The boundary, ∂S_2, of S_2 consists of two parts, a circle in the plane $z = 4$ and a circle in the plane $z = 1$. We'll call the first part ∂S_{2a}. It is the same as ∂S_1. We'll call the second part ∂S_{2b}. It is the circle $\sqrt{x^2 + y^2} = 1$, $z = 1$ oriented *counterclockwise* when viewed from high on the z-axis. We can parametrize it

$$\mathbf{r}(t) = \cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}, \qquad 0 \leqslant t \leqslant 2\pi$$

So, on ∂S_{2b},

$$\begin{aligned}\mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathrm{d}\mathbf{r} &= \big(-\sin t\,,\ \cos t\,,\ \sin t \cos t \cos(\sin t)\big) \cdot \big(-\sin t\,,\ \cos t\,,\ 0\big)\,\mathrm{d}t \\ &= \mathrm{d}t\end{aligned}$$

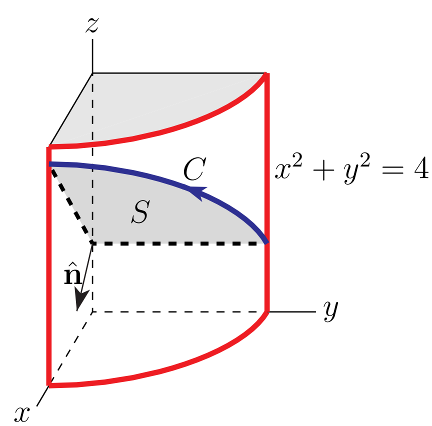
and, by Stokes' theorem,

$$\begin{aligned}\iint_{S_2} \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S &= \oint_{\partial S_{2a}} \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathrm{d}\mathbf{r} + \oint_{\partial S_{2b}} \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathrm{d}\mathbf{r} = -128\pi + \int_0^{2\pi} \mathrm{d}t \\ &= -126\pi\end{aligned}$$

S-15: Denote by

$$S = \big\{\ (x, y, z)\ \big|\ z = x + 4,\ x^2 + y^2 \leqslant 4\ \big\}$$

the part of the plane $z = x + 4$ that is contained in the cylinder $x^2 + y^2 = 4$. Orient S by the downward pointing normal $\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}(1, 0, -1)$. Then C is the boundary of S. The part of C and S that are in the first octant are sketched below.



We may parametrize S by

$$\mathbf{r}(x, y) = (x, y, x + 4) \qquad \text{with} \quad x^2 + y^2 \leqslant 4$$

So,

$$\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \big(-1, 0, 1\big)$$

and, by (3.3.1) in the CLP-4 text,

$$\hat{\mathbf{n}}\,\mathrm{d}S = -\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y}\,\mathrm{d}x\mathrm{d}y = (1, 0, -1)\,\mathrm{d}x\mathrm{d}y$$

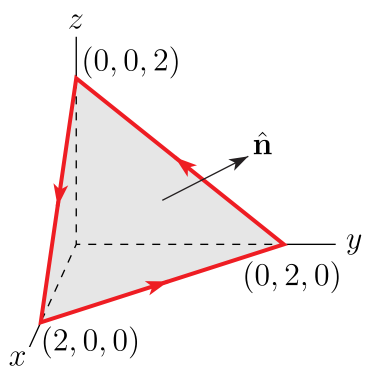
We have chosen to "$-$" sign in $\hat{\mathbf{n}}\,\mathrm{d}S = \pm\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y}\,\mathrm{d}x\mathrm{d}y$ to give the downward pointing normal. As the curl of $\mathbf{F}$ is

$$\begin{aligned}\nabla \times \mathbf{F} &= \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 + 2y & \sin(y) + z & x + \sin(z^2) \end{bmatrix} \\ &= -\hat{\imath} - \hat{\jmath} - 2\hat{\mathbf{k}}\end{aligned}$$

Stokes' theorem (Theorem 4.4.1 in the CLP-4 text) gives

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S = \iint_S (-1, -1, -2) \cdot (1, 0, -1)\,\mathrm{d}x\mathrm{d}y = \iint_S \mathrm{d}x\mathrm{d}y = 4\pi$$

S-16: (a) Note that all three vertices, $(2, 0, 0)$, $(0, 2, 0)$ and $0, 0, 2)$, lie in the plane $x + y + z = 2$. So the entire path lies in that plane too. In part (b) we will need to



evaluate a line integral that clearly cannot be computed directly — we will need to use Stokes' theorem. So let's use Stokes's theorem in part (a) too. First, we find

$$\nabla \times \mathbf{F} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & x^2 & y^2 \end{bmatrix} = 2y\,\hat{\imath} + 2z\,\hat{\jmath} + 2x\,\hat{\imath}$$

Let S be the triangular surface that is contained in the plane $x + y + z = 2$ and is bounded by L_1, L_2 and L_3. Orient S by the normal vector $\hat{\mathbf{n}} = \frac{1}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{\mathbf{k}})$. Then,

$$\nabla \times \mathbf{F} \cdot \hat{\mathbf{n}} = \frac{1}{\sqrt{3}}\big(2y\,\hat{\imath} + 2z\,\hat{\jmath} + 2x\,\hat{\imath}\big) \cdot \big(\hat{\imath} + \hat{\jmath} + \hat{\mathbf{k}}\big) = \frac{2}{\sqrt{3}}(x + y + z)$$

and, by Stokes' theorem,

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S = \frac{2}{\sqrt{3}} \iint_S (x + y + z)\,\mathrm{d}S = \frac{4}{\sqrt{3}} \iint_S \mathrm{d}S = \frac{4}{\sqrt{3}}\mathrm{Area}(S)$$

The triangle S is half of the prallelogram with sides $(0,2,0)-(2,0,0)=(-2,2,0)$ and $(0,0,2)-(2,0,0)=(-2,0,2)$. The area of the parallelogram is

$$\big|(-2,2,0)\times(-2,0,2)\big| = \big|(4,4,4)\big| = 4\sqrt{3}$$

So

$$\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \frac{4}{\sqrt{3}} 2\sqrt{3} = 8$$

(b) Let $\tilde{S}$ be the specified surface. Then, as in part (a),

$$\begin{aligned}\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \iint_{\tilde{S}} \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \frac{2}{\sqrt{3}}\iint_{\tilde{S}} (x+y+z)\,\mathrm{d}S = \frac{4}{\sqrt{3}}\iint_{\tilde{S}} \mathrm{d}S = \frac{4}{\sqrt{3}}\text{Area}(\tilde{S})\\ &= 4\sqrt{3}\end{aligned}$$

S-17: Let's try Stokes' theorem with

$$\mathbf{F} = \left(z+\frac{1}{1+z}\right)\hat{\boldsymbol{\imath}} + xz\,\hat{\boldsymbol{\jmath}} + \left(3xy - \frac{x}{(z+1)^2}\right)\hat{\mathbf{k}}$$

The curl of $\mathbf{F}$ is

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{F} &= \det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ z+\frac{1}{1+z} & xz & 3xy-\frac{x}{(z+1)^2}\end{bmatrix}\\ &= (3x-x)\hat{\boldsymbol{\imath}} - \left(3y - \frac{1}{(z+1)^2} - 1 + \frac{1}{(1+z)^2}\right)\hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}}\\ &= 2x\,\hat{\boldsymbol{\imath}} + (1-3y)\,\hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}}\end{aligned}$$

Write

$$S = \left\{\ (x,y,z)\ \middle|\ z=f(x,y)=1-x^2y,\ x^2+y^2\leqslant 1\ \right\}$$

For S, with the upward pointing normal, by (3.3.2) of the CLP-4 text,

$$\begin{aligned}\hat{\mathbf{n}}\,\mathrm{d}S &= \big(-f_x\,,\,-f_y\,,\,1\big)\mathrm{d}x\mathrm{d}y\\ &= \big(2xy\,,\,x^2\,,\,1\big)\mathrm{d}x\mathrm{d}y\end{aligned}$$

so that

$$\boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \big\{4x^2y + (x^2-3x^2y) + \overbrace{(1-x^2y)}^{z}\big\}\,\mathrm{d}x\mathrm{d}y$$

and, by Stokes' theorem,

$$\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \iint_S \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_{x^2+y^2\leqslant 1} \big\{4x^2y + x^2 - 3x^2y + 1 - x^2y\big\}\,\mathrm{d}x\mathrm{d}y$$

So

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint\limits_{x^2+y^2\leqslant 1} \left\{x^2+1\right\} \mathrm{d}x\mathrm{d}y = \pi + \iint\limits_{x^2+y^2\leqslant 1} x^2\,\mathrm{d}x\mathrm{d}y$$

To evaluate the final remaining integral, let's switch to polar coordinates.

$$\begin{aligned}\iint\limits_{x^2+y^2\leqslant 1} x^2\,\mathrm{d}x\mathrm{d}y &= \int_0^1 \mathrm{d}r\; r\int_0^{2\pi} \mathrm{d}\theta\; \big(r\cos\theta\big)^2 \\ &= \int_0^1 \mathrm{d}r\; r^3\int_0^{2\pi} \mathrm{d}\theta\;\cos^2\theta\end{aligned}$$

Since

$$\int_0^{2\pi} \cos^2\theta\,\mathrm{d}\theta = \int_0^{2\pi} \frac{1+\cos(2\theta)}{2}\,\mathrm{d}\theta = \left[\frac{\theta}{2}+\frac{\sin(2\theta)}{4}\right]_0^{2\pi} = \pi$$

we finally have $\int_0^1 \mathrm{d}r\; r^3\int_0^{2\pi}\mathrm{d}\theta\;\cos^2\theta = \frac{\pi}{4}$ and

$$\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$$

For an efficient, sneaky, way to evaluate $\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta$, see Example 2.4.4 in the CLP-4 text.

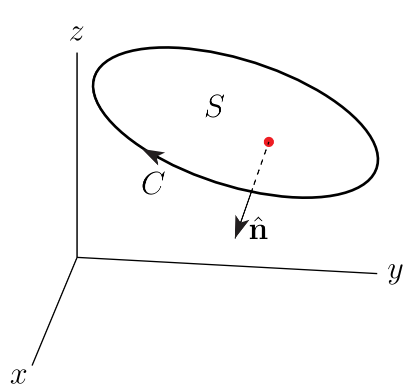
S-18: We are to evaluate a line integral around a curve C. We are told that C is the boundary of a surface S that is contained in the plane $x+y+z=1$, but we are not told precisely what C is. So we are going to have to use Stokes' theorem. The curl of $\mathbf{F}$ is

$$\boldsymbol{\nabla}\times\mathbf{F} = \det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & x^2 & y^2\end{bmatrix} = 2y\,\hat{\boldsymbol{\imath}} + 2z\,\hat{\boldsymbol{\jmath}} + 2x\,\hat{\mathbf{k}}$$

and, by (3.3.3) of the CLP-4 text with $G(x,y,z) = x+y+z$,

$$\begin{aligned}\mathrm{d}S &= \left|\frac{\boldsymbol{\nabla}\mathbf{G}}{\boldsymbol{\nabla}\mathbf{G}\cdot\hat{\mathbf{k}}}\right|\mathrm{d}x\mathrm{d}y = \sqrt{3}\,\mathrm{d}x\mathrm{d}y \\ \hat{\mathbf{n}}\,\mathrm{d}S &= \pm\frac{\boldsymbol{\nabla}\mathbf{G}}{\boldsymbol{\nabla}\mathbf{G}\cdot\hat{\mathbf{k}}}\mathrm{d}x\mathrm{d}y = \pm\big(\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}}+\hat{\mathbf{k}}\big)\,\mathrm{d}x\mathrm{d}y = \pm\frac{1}{\sqrt{3}}\big(\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}}+\hat{\mathbf{k}}\big)\,\mathrm{d}S\end{aligned}$$

Because C is oriented in a clockwise direction as observed from the positive z-axis



looking down at the plane, $\hat{\mathbf{n}}$ is to point downwards, so that

$$\hat{\mathbf{n}}\,\mathrm{d}S = -\frac{1}{\sqrt{3}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}})\,\mathrm{d}S$$

On S we have $x + y + z = 1$, so that Stokes' theorem gives

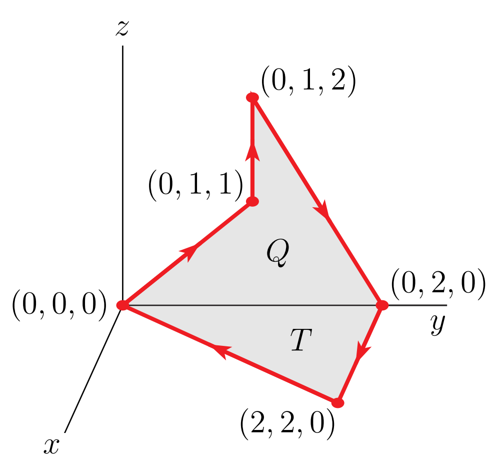
$$\begin{aligned}\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \iint_S (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_S 2(y\,\hat{\boldsymbol{\imath}} + z\,\hat{\boldsymbol{\jmath}} + x\,\hat{\mathbf{k}})\cdot\left(-\frac{1}{\sqrt{3}}\right)(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}})\,\mathrm{d}S \\ &= -\frac{2}{\sqrt{3}}\iint_S (y + z + x)\,\mathrm{d}S = -\frac{2}{\sqrt{3}}\iint_S \mathrm{d}S = -\frac{10}{\sqrt{3}}\end{aligned}$$

since S has area 5.

S-19: We are to evaluate the line integral of a complicated vector field around a relatively complicated closed curve. That certainly suggests that we should not try to evaluate the integral directly. To see if Stokes' theorem looks promising, let's compute the curl

$$\nabla\times\mathbf{F} = \det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y + e^x \sin x & y^4 & \sqrt{z}\tan z \end{bmatrix} = \hat{\mathbf{k}}$$

That's suggestive. Next we need to find a surface whose boundary is C. First, here is a sketch of C. We can choose the surface S to be the union of two flat parts:



- the quadralateral Q in the yz-plane with vertices $(0,0,0)$, $(0,1,1)$, $(0,1,2)$ and $(0,2,0)$ and
- the triangle T in the xy-plane with vertices $(0,0,0)$, $(0,2,0)$ and $(2,2,0)$.

The normal to Q is $-\hat{\imath}$ and the normal to T is $-\hat{\mathbf{k}}$. Then Stokes' theorem gives

$$\begin{aligned}
\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \iint_S \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S \\
&= \iint_Q \hat{\mathbf{k}}\cdot(-\hat{\imath})\,\mathrm{d}S + \iint_T \hat{\mathbf{k}}\cdot(-\hat{\mathbf{k}})\,\mathrm{d}S \\
&= -\iint_T \mathrm{d}S \\
&= -\text{Area}(T) \\
&= -\tfrac{1}{2}\overbrace{2}^{\text{base}}\overbrace{2}^{\text{height}} \\
&= -2
\end{aligned}$$

S-20: The integral looks messy. Let's compute the curl of

$$\mathbf{F} = (z+\sin z)\,\hat{\imath} + (x^3 - x^2 y)\,\hat{\jmath} + (x\cos z - y)\,\hat{\mathbf{k}}$$

to help gauge if Stokes' theorem would be easier.

$$\nabla\times\mathbf{F} = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z+\sin z & x^3 - x^2 y & x\cos z - y \end{bmatrix} = -\hat{\imath} + \hat{\jmath} + (3x^2 - 2xy)\,\hat{\mathbf{k}}$$

That's a lot simpler than $\mathbf{F}$. For the surface $z = f(x,y) = xy^2$, with downward pointing normal (since C is traversed clockwise)

$$\hat{\mathbf{n}}\,\mathrm{d}S = -\big(-f_x, -f_y, 1\big)\,\mathrm{d}x\mathrm{d}y = \big(y^2, 2xy, -1\big)\,\mathrm{d}x\mathrm{d}y$$

by (3.3.2) of the CLP-4 text, So, writing

$$\begin{aligned}
S &= \big\{\ (x,y,z)\ \big|\ z = xy^2,\ x^2+y^2 \leqslant 1\ \big\} \\
D &= \big\{\ (x,y)\ \big|\ x^2+y^2 \leqslant 1\ \big\}
\end{aligned}$$

Stoke's theorem gives

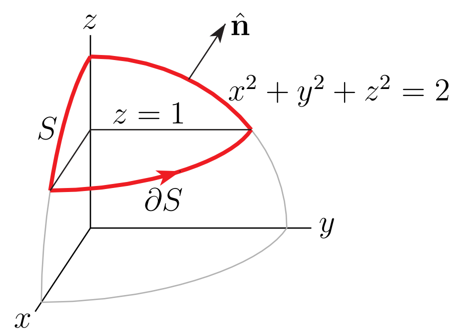
$$\begin{aligned}
\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \iint_S \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iint_D \big\{-y^2 + 2xy - 3x^2 + 2xy\big\}\,\mathrm{d}x\mathrm{d}y \\
&= -\iint_{x^2+y^2\leqslant 1} \big\{3x^2 + y^2 - 4xy\big\}\,\mathrm{d}x\mathrm{d}y
\end{aligned}$$

To evaluate this integral, switch to polar coordinates.

$$\begin{aligned}
\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= -\int_0^1 \mathrm{d}r\ r\int_0^{2\pi}\mathrm{d}\theta\ \big\{3r^2\cos^2\theta + r^2\sin^2\theta - 4r^2\sin\theta\cos\theta\big\} \\
&= -4\pi\int_0^1 \mathrm{d}r\ r^3 = -\pi
\end{aligned}$$

since $\int_0^{2\pi} \sin\theta\cos\theta\,\mathrm{d}\theta = \frac{1}{2}\int_0^{2\pi} \sin(2\theta)\,\mathrm{d}\theta = 0$ and $\int_0^{2\pi} \sin^2\theta\,\mathrm{d}\theta = \int_0^{2\pi} \cos^2\theta\,\mathrm{d}\theta = \pi$. (See Example 2.4.4 in the CLP-4 text.)

S-21: Here is a sketch of the part of S in the first octant.



The boundary, ∂S, of S is the circle $x^2 + y^2 = 1$, $z = 1$, oriented counterclockwise when viewed from above. It is parametrized by

$$\mathbf{r}(\theta) = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} + \hat{\mathbf{k}} \qquad 0 \leqslant \theta \leqslant 2\pi$$

So Stokes' theorem gives

$$\begin{aligned}
\iint_S \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \oint_{\partial S} \mathbf{F}\cdot\mathrm{d}\mathbf{r} \\
&= \int_0^{2\pi} \Big(\overbrace{-\sin^2\theta\,\hat{\imath} + \cos^3\theta\,\hat{\jmath} + (\text{mess})\hat{\mathbf{k}}}^{\mathbf{F}(\mathbf{r}(t))}\Big)\cdot\big(\overbrace{-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}}^{\mathbf{r}'(t)}\big)\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \big(\sin^3\theta + \cos^4\theta\big)\,\mathrm{d}\theta
\end{aligned}$$

The integral of any odd power of $\sin\theta$ or $\cos\theta$ over $0 \leqslant \theta \leqslant 2\pi$ is zero. (See Example 4.4.6 in the CLP-4 text.) In particular, $\int_0^{2\pi} \sin^3\theta\,\mathrm{d}\theta = 0$. To integrate $\cos^4\theta$ we use the trig identity

$$\begin{aligned}
\cos^2\theta &= \frac{\cos(2\theta)+1}{2} \\
\implies \cos^4\theta &= \frac{\cos^2(2\theta) + 2\cos(2\theta) + 1}{4} \\
&= \frac{1}{4}\,\frac{\cos(4\theta)+1}{2} + \frac{\cos(2\theta)}{2} + \frac{1}{4} \\
&= \frac{3}{8} + \frac{\cos(4\theta)}{8} + \frac{\cos(2\theta)}{2}
\end{aligned}$$

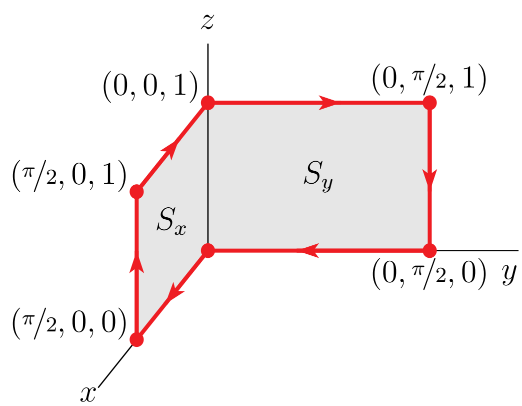
Finally

$$\iint_S \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_0^{2\pi} \Big(\frac{3}{8} + \frac{\cos(4\theta)}{8} + \frac{\cos(2\theta)}{2}\Big)\,\mathrm{d}\theta = \frac{3\pi}{4}$$

S-22: We are to evaluate the line integral of a complicated vector field around a relatively complicated closed curve. That certainly suggests that we should not try to evaluate the integral directly. As we are to use Stokes' theorem, let's compute the curl

$$\nabla \times \mathbf{F} = \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x \sin y & -y \sin x & (x-y)z^2 \end{bmatrix} = -z^2\, \hat{\boldsymbol{\imath}} - z^2\, \hat{\boldsymbol{\jmath}} - (y \cos x + x \cos y)\hat{\mathbf{k}}$$

Next we need to find a surface whose boundary is C. First, here is a sketch of C. We can



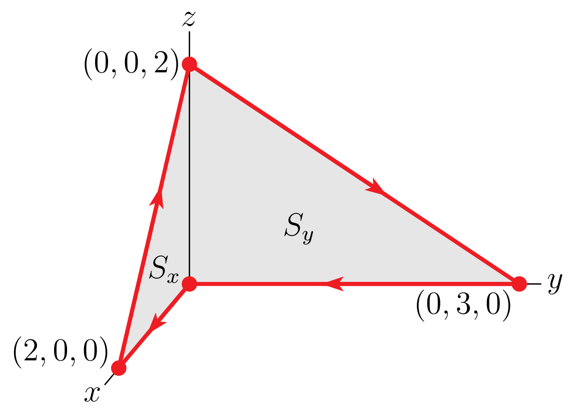
choose the surface S to be the union of two flat parts:

- the rectangle S_x in the xz-plane with vertices $(0,0,0)$, $(\pi/2,0,0)$, $(\pi/2,0,1)$ and $(0,0,1)$ and
- the rectangle S_y in the yz-plane with vertices $(0,0,0)$, $(0,0,1)$, $(0,\pi/2,1)$ and $(0,\pi/2,0)$

The normal to S_x is $-\hat{\boldsymbol{\jmath}}$ and the normal to S_y is $-\hat{\boldsymbol{\imath}}$. Then Stokes' theorem gives

$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S \\
&= \iint_{S_x} \nabla \times \mathbf{F} \cdot (-\hat{\boldsymbol{\jmath}})\, \mathrm{d}S + \iint_{S_y} \nabla \times \mathbf{F} \cdot (-\hat{\boldsymbol{\imath}})\, \mathrm{d}S \\
&= \int_0^{\pi/2} \mathrm{d}x \int_0^1 \mathrm{d}z\, z^2 + \int_0^{\pi/2} \mathrm{d}y \int_0^1 \mathrm{d}z\, z^2 \\
&= \int_0^{\pi/2} \mathrm{d}x\, \frac{1}{3} + \int_0^{\pi/2} \mathrm{d}y\, \frac{1}{3} \\
&= \frac{\pi}{3}
\end{aligned}$$

S-23: (a) Here is a sketch.



(b) We are to evaluate the line integral of a complicated vector field around a relatively complicated closed curve. That certainly suggests that we should not try to evaluate the integral directly. Let's try Stokes' theorem. First, we compute the curl

$$\begin{aligned}\nabla \times \mathbf{F} &= \det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{2z}{1+y} + \sin(x^2) & \frac{3z}{1+x} + \sin(y^2) & 5(x+1)(y+2) \end{bmatrix} \\ &= \Big(5(x+1) - \frac{3}{1+x}\Big)\,\hat{\boldsymbol{\imath}} - \Big(5(y+2) - \frac{2}{1+y}\Big)\,\hat{\boldsymbol{\jmath}} + \Big(-\frac{3z}{(1+x)^2} + \frac{2z}{(1+y)^2}\Big)\,\hat{\mathbf{k}}\end{aligned}$$

Next we need to find a surface S whose boundary is C. We can choose the surface S to be the union of two flat parts:

- the triangle S_x in the xz-plane with vertices $(0,0,0)$, $(2,0,0)$, and $(0,0,2)$ and
- the triangle S_y in the yz-plane with vertices $(0,0,0)$, $(0,0,2)$, and $(0,3,0)$

Note that

- The normal to S_x specified by Stokes' theorem is $-\hat{\boldsymbol{\jmath}}$. On S_x we have $y = 0$, so that $\nabla \times \mathbf{F} \cdot \hat{\boldsymbol{\jmath}}$ simplifies to $-\big(5(0+2) - \frac{2}{1+0}\big) = -8$.
- The normal to S_y specified by Stokes' theorem is $-\hat{\boldsymbol{\imath}}$. On S_y we have $x = 0$, so that $\nabla \times \mathbf{F} \cdot \hat{\boldsymbol{\imath}}$ simplifies to $\big(5(0+1) - \frac{3}{1+0}\big) = 2$.

So Stokes' theorem gives

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S = \iint_{S_x} \overbrace{\nabla \times \mathbf{F} \cdot (-\hat{\boldsymbol{\jmath}})}^{8}\,\mathrm{d}S + \iint_{S_y} \overbrace{\nabla \times \mathbf{F} \cdot (-\hat{\boldsymbol{\imath}})}^{-2}\,\mathrm{d}S \\ &= 8\,\text{Area}(S_x) - 2\,\text{Area}(S_y) = 8\frac{1}{2}(2)(2) - 2\frac{1}{2}(3)(2) \\ &= 10\end{aligned}$$

S-24: The boundary, ∂S, of S is the circle $x^2 + y^2 = 1$ oriented counter clockwise as usual.

It may be parametrized by $\mathbf{r}(\theta) = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$, $0 \leqslant \theta \leqslant 2\pi$. By Stokes' theorem

$$\begin{aligned}
\iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S &= \oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} \mathbf{F}\big(\mathbf{r}(\theta)\big) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\theta}(\theta)\, \mathrm{d}\theta \\
&= \int_0^{2\pi} (\sin\theta, 0, 3\cos\theta) \cdot (-\sin\theta, \cos\theta, 0)\, \mathrm{d}\theta \\
&= -\int_0^{2\pi} \sin^2\theta\, \mathrm{d}\theta = -\int_0^{2\pi} \mathrm{d}\theta\, \frac{1-\cos(2\theta)}{2} = -\left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right]_0^{2\pi} \\
&= -\pi
\end{aligned}$$

For an efficient, sneaky, way to evaluate $\int_0^{2\pi} \mathrm{d}\theta\ \sin^2\theta$, see Example 2.4.4 in the CLP-4 text.

S-25: The given surface is an ellipsoid centred at $(x, y, z) = (0, 0, 1)$. It caps a curve $\mathcal{C}$ in the plane $z = 0$, given by $x^2 + y^2 = 4$. This is a circle of radius 2 centred at the origin, oriented counterclockwise when viewed from the positive z-axis.

Method I — double Stokes': Let $\mathcal{D}$ denote the plane disk $x^2 + y^2 \leqslant 4$, $z = 0$. Using Stokes' theorem twice gives

$$\iint_{\mathcal{S}} \mathbf{G} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iint_{\mathcal{S}} \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \oint_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_{\mathcal{D}} \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iint_{\mathcal{D}} \mathbf{G} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$$

Now in $\mathcal{D}$ we have $\hat{\mathbf{n}} = \hat{\mathbf{k}}$ and $z = 0$, so on this surface,

$$\begin{aligned}
\mathbf{G} \cdot \hat{\mathbf{n}} = (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{k}} &= \det \begin{bmatrix} 0 & 0 & 1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (xz - y^3\cos z) & x^3 e^z & xyze^{x^2+y^2+z^2} \end{bmatrix}_{z=0} \\
&= \big[3x^2 e^z + 3y^2 \cos z\big]_{z=0} = 3(x^2 + y^2)
\end{aligned}$$

Hence, using polar coordinates,

$$\iint_{\mathcal{S}} \mathbf{G} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iint_{\mathcal{D}} 3(x^2 + y^2)\, \mathrm{d}x\mathrm{d}y = 3\int_{\theta=0}^{2\pi} \int_{r=0}^{2} (r^2)\, r\, \mathrm{d}r\, \mathrm{d}\theta = 3(2\pi)(4) = 24\pi$$

Method II — single Stokes': By Stokes' theorem

$$\iint_{\mathcal{S}} \mathbf{G} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iint_{\mathcal{S}} \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \oint_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

Parametrize the circle $\mathcal{C}$ using

$$\mathbf{r}(\theta) = 2\cos\theta\,\hat{\imath} + 2\sin\theta\,\hat{\jmath}, \qquad 0 \leqslant \theta \leqslant 2\pi$$

to obtain

$$\mathrm{d}\mathbf{r} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\theta}\,\mathrm{d}\theta = \left(-2\sin\theta\,\hat{\boldsymbol{\imath}} + 2\cos\theta\,\hat{\boldsymbol{\jmath}}\right)\mathrm{d}\theta.$$

Then since $z = 0$ on $\mathcal{C}$,

$$\begin{aligned}\iint\limits_{\mathcal{S}} \mathbf{G}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \int_0^{2\pi} \big(\overbrace{-(2\sin\theta)^3\,\hat{\boldsymbol{\imath}} + (2\cos\theta)^3\,\hat{\boldsymbol{\jmath}}}^{\mathbf{F}\left(\mathbf{r}(\theta)\right)}\big)\cdot\big(\overbrace{-2\sin\theta\,\hat{\boldsymbol{\imath}} + 2\cos\theta\,\hat{\boldsymbol{\jmath}}}^{\mathbf{r}'(\theta)}\big)\,\mathrm{d}\theta \\ &= 16\int_0^{2\pi}\left(\sin^4\theta + \cos^4\theta\right)\mathrm{d}\theta\end{aligned}$$

By the double angle trig identities

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2} \qquad \sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

we have

$$\begin{aligned}\sin^4\theta + \cos^4\theta &= \frac{\left[1-\cos(2\theta)\right]^2}{4} + \frac{\left[1+\cos(2\theta)\right]^2}{4} \\ &= \frac{1}{2} + \frac{\cos^2(2\theta)}{2} = \frac{1}{2} + \frac{1+\cos(4\theta)}{4}\end{aligned}$$

So

$$\iint\limits_{\mathcal{S}} \mathbf{G}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 16\int_0^{2\pi}\left(\frac{3}{4} + \frac{1}{4}\cos(4\theta)\right)\mathrm{d}\theta = 16\times\frac{3}{4}\times(2\pi) = 24\pi$$

S-26: Note that

$$\boldsymbol{\nabla}\times\mathbf{F} = \det\begin{vmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & x^2 & y^2\end{vmatrix} = 2y\,\hat{\boldsymbol{\imath}} + 2z\,\hat{\boldsymbol{\jmath}} + 2x\,\hat{\mathbf{k}}$$

Let D be the disk in the plane $x+y+z=3$ whose boundary is C and let $\hat{\mathbf{n}} = \frac{1}{\sqrt{3}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}})$ be the upward unit normal to D. If the circle is oriented counterclockwise, when viewed from above, then, by Stokes' theorem (Theorem 4.4.1 in the CLP-4 text),

$$\begin{aligned}\oint\limits_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \iint\limits_D \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \frac{1}{\sqrt{3}}\iint\limits_D \left(2y\,\hat{\boldsymbol{\imath}} + 2z\,\hat{\boldsymbol{\jmath}} + 2x\,\hat{\mathbf{k}}\right)\cdot(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}})\,\mathrm{d}S \\ &= \frac{1}{\sqrt{3}}\iint\limits_D 2\,\overbrace{(x+y+z)}^{=3\text{ on }D}\,\mathrm{d}S = 2\sqrt{3}\iint\limits_D \mathrm{d}S = 2\sqrt{3}\pi R^2\end{aligned}$$

S-27: *Solution 1*: Let S' be the bottom surface of the cube, oriented with normal $\hat{\mathbf{k}}$. Then, by Stokes' theorem, since $\partial S = \partial S'$,

$$\iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_{\partial S'} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_{S'} \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$$

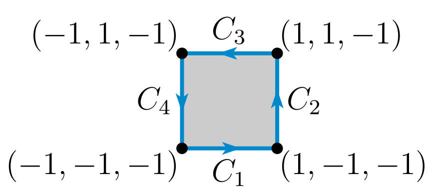
Since

$$\nabla \times \mathbf{F} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & xy^2 & x^2yz \end{bmatrix} = (\cdots, \cdots, y^2 - xz)$$

and $\hat{\mathbf{n}} = \hat{\mathbf{k}}$ on S' and $z = -1$ on S'

$$\begin{aligned} \iint_{S'} \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S &= \int_{-1}^{1} \mathrm{d}x \int_{-1}^{1} \mathrm{d}y\ (\cdots, \cdots, y^2 - xz) \cdot \hat{\mathbf{k}}\Big|_{z=-1} = \int_{-1}^{1} \mathrm{d}x \int_{-1}^{1} \mathrm{d}y\ (y^2 + x) \\ &= \int_{-1}^{1} \mathrm{d}x \int_{-1}^{1} \mathrm{d}y\ y^2 = 2 \times 2 \int_0^1 \mathrm{d}y\ y^2 = \frac{4}{3} \end{aligned}$$

Solution 2: The boundary of S is the square C, with sides $C_1, \cdots, C_4$, in the sketch



By Stokes' theorem,

$$\iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

Parametrize C_1 by x. That is, $\mathbf{r}(x) = x\hat{\imath} - \hat{\jmath} - \hat{\mathbf{k}}$, $-1 \leqslant x \leqslant 1$. Since $\mathbf{r}'(x) = \hat{\imath}$, and $y = z = -1$ on C_1,

$$\begin{aligned} \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_{-1}^{1} \mathbf{F}\big(\mathbf{r}(x)\big) \cdot \mathbf{r}'(x)\, \mathrm{d}x = \int_{-1}^{1} \mathbf{F}\big(\mathbf{r}(x)\big) \cdot \hat{\imath}\, \mathrm{d}x = \int_{-1}^{1} \overbrace{x(-1)(-1)}^{xyz}\, \mathrm{d}x \\ &= 0 \qquad \text{(since } x \text{ is odd)} \end{aligned}$$

Parametrize C_2 by y. That is, $\mathbf{r}(y) = \hat{\imath} + y\hat{\jmath} - \hat{\mathbf{k}}$, $-1 \leqslant y \leqslant 1$. Since $\mathbf{r}'(y) = \hat{\jmath}$, and $x = 1$ on C_2,

$$\int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{-1}^{1} \mathbf{F}\big(\mathbf{r}(y)\big) \cdot \hat{\jmath}\, \mathrm{d}y = \int_{-1}^{1} \overbrace{y^2}^{xy^2}\, \mathrm{d}y = \left[\frac{y^3}{3}\right]_{-1}^{1} = \frac{2}{3}$$

Parametrize C_3 by x. That is, $\mathbf{r}(x) = x\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \hat{\mathbf{k}}$ with x running from 1 to -1. (If you're nervous about this, parametrize by $t = -x$. That is $\mathbf{r}(t) = -t\,\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \hat{\mathbf{k}}$, $-1 \leqslant t \leqslant 1$.) Since $\mathbf{r}'(x) = \hat{\boldsymbol{\imath}}$, and $y = 1$, $z = -1$ on C_3,

$$\int_{C_3} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_1^{-1} \mathbf{F}\big(\mathbf{r}(x)\big) \cdot \hat{\boldsymbol{\imath}}\,\mathrm{d}x = \int_1^{-1} \overbrace{x\,(1)(-1)}^{xyz}\ \mathrm{d}x = 0 \qquad \text{(since } x \text{ is odd)}$$

Parametrize C_4 by y. That is, $\mathbf{r}(y) = -\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} - \hat{\mathbf{k}}$, with y running from 1 to -1. Since $\mathbf{r}'(y) = \hat{\boldsymbol{\jmath}}$, and $x = -1$ on C_4,

$$\int_{C_4} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_1^{-1} \mathbf{F}\big(\mathbf{r}(y)\big) \cdot \hat{\boldsymbol{\jmath}}\,\mathrm{d}y = \int_1^{-1} \overbrace{(-1)y^2}^{xy^2}\ \mathrm{d}y = -\left[\frac{y^3}{3}\right]_1^{-1} = \frac{2}{3}$$

All together

$$\iint_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_3} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_4} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \frac{4}{3}$$

S-28: Let's try Stokes' Theorem. Call $\mathbf{F} = y\,\hat{\boldsymbol{\imath}} - x\,\hat{\boldsymbol{\jmath}} + xy\,\hat{\mathbf{k}}$. Then

$$\boldsymbol{\nabla} \times \mathbf{F} = \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & xy \end{bmatrix} = x\,\hat{\boldsymbol{\imath}} - y\,\hat{\boldsymbol{\jmath}} - 2\,\hat{\mathbf{k}}$$

Now compute $\hat{\mathbf{n}}\,\mathrm{d}S$ in the (u, v)-parametrization.

$$\begin{aligned}
\mathbf{r}(u,v) &= \big(u\cos v, u\sin v, v\big) \\
\frac{\partial \mathbf{r}}{\partial u}(u,v) &= \big(\cos v, \sin v, 0\big) \\
\frac{\partial \mathbf{r}}{\partial v}(u,v) &= \big(-u\sin v, u\cos v, 1\big) \\
\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} &= \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{bmatrix} = \big(\sin v, -\cos v, u\big) \\
\hat{\mathbf{n}}\,\mathrm{d}S &= \big(\sin v, -\cos v, u\big)\,\mathrm{d}u\,\mathrm{d}v
\end{aligned}$$

Since $u \geqslant 0$, we do indeed have the upward pointing normal. So, Stokes' theorem tells us

$$\begin{aligned}
\int_C y\,\mathrm{d}x - x\,\mathrm{d}y + xy\,\mathrm{d}z = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \iint_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S \\
&= \int_0^1 \mathrm{d}u \int_0^{2\pi} \mathrm{d}v\ \big(u\cos v, -u\sin v, -2\big) \cdot \big(\sin v, -\cos v, u\big) \\
&= \int_0^1 \mathrm{d}u \int_0^{2\pi} \mathrm{d}v\ \big(2u\sin v\cos v - 2u\big) = \left[\int_0^1 \mathrm{d}u\ u\right]\left[\int_0^{2\pi} \mathrm{d}v\ \big(\sin 2v - 2\big)\right] \\
&= \tfrac{1}{2}(-4\pi) = -2\pi
\end{aligned}$$

S-29: Given the form of $\mathbf{F}$, direct evaluation looks hard. So let's try Stokes' theorem, using as S the part of the plane $G(x,y,z) = x + 2y - z = 7$ that is inside $x^2 - 2x + 4y^2 = 15$. Then

$$\hat{\mathbf{n}}\,\mathrm{d}S = \pm\frac{\boldsymbol{\nabla} G}{\boldsymbol{\nabla} G\cdot\hat{\mathbf{k}}}\,\mathrm{d}x\,\mathrm{d}y = \pm\big(\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} - \hat{\mathbf{k}}\big)\,\mathrm{d}x\,\mathrm{d}y$$

As C is oriented counterclockwise when viewed from high, Stokes' theorem specifies the upward pointing normal so that $\hat{\mathbf{n}}\,\mathrm{d}S = -\big(\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} - \hat{\mathbf{k}}\big)\,\mathrm{d}x\,\mathrm{d}y$.

From the observations that

$$\boldsymbol{\nabla}\times\mathbf{F} = \det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{x^2} + yz & \cos(y^2) - x^2 & \sin(z^2) + xy \end{bmatrix} = x\,\hat{\boldsymbol{\imath}} - (z + 2x)\,\hat{\mathbf{k}}$$

and that we can rewrite $x^2 - 2x + 4y^2 = 15$ as $(x-1)^2 + 4y^2 = 16$, we have

$$\begin{aligned}
\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \iint_S \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S \\
&= \iint_{(x-1)^2+4y^2\leqslant 16} \big[x\hat{\boldsymbol{\imath}} - (z+2x)\hat{\mathbf{k}}\big]\Big|_{z=-7+x+2y}\cdot(-1,-2,1)\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_{(x-1)^2+4y^2\leqslant 16} \big[-x - (-7 + x + 2y + 2x)\big]\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_{(x-1)^2+4y^2\leqslant 16} [7 - 4x - 2y]\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

To evaluate the integrals of x and y we use that, for any region R in the xy–plane,

$$\bar{x} = \frac{\iint_R x\,\mathrm{d}x\,\mathrm{d}y}{\mathrm{Area}(R)} \qquad \bar{y} = \frac{\iint_R y\,\mathrm{d}x\,\mathrm{d}y}{\mathrm{Area}(R)}$$

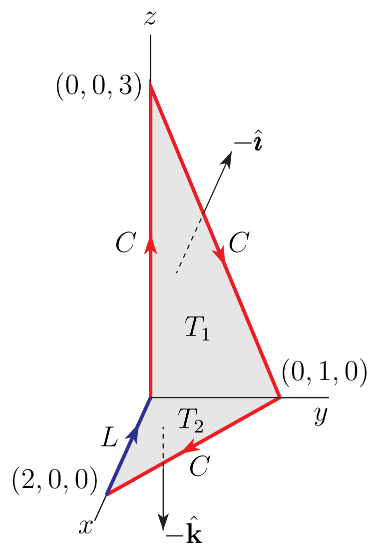
Our ellipse is $\frac{(x-1)^2}{4^2} + \frac{y^2}{2^2} = 1$ and so has area $\pi ab = \pi\times 4\times 2 = 8\pi$ and centroid $(\bar{x},\bar{y}) = (1,0)$. So, using $R = \left\{ (x,y) \mid \frac{(x-1)^2}{4^2} + \frac{y^2}{2^2} \leqslant 1 \right\}$,

$$\begin{aligned}
\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \iint_R [7 - 4x - 2y]\,\mathrm{d}x\,\mathrm{d}y \\
&= \mathrm{Area}(R)\{7 - 4\bar{x} - 2\bar{y}\} \\
&= 8\pi[7 - 4\times 1 - 2\times 0] = 24\pi
\end{aligned}$$

S-30: (a) The curl is

$$\nabla \times \mathbf{F} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2 + x^2 + z & 0 & 3 + x^2 z \end{bmatrix} = (1 - 2xz)\,\hat{\jmath}$$

(b) We are going to use Stokes' theorem. The specified curve C is not closed and so is not the boundary of a surface. So we extend C to a closed curve $\tilde{C}$ by appending to C the line segment L from $(2,0,0)$ to $(0,0,0)$. In the figure below, C is the red curve and $\tilde{C}$ is C plus the blue line segment. The closed curve $\tilde{C}$ is boundary of the surface S that is the union



of

- the triangle T_1 in the yz-plane with vertices $(0,0,0)$, $(0,0,3)$ and $(0,1,0)$ and with normal vector $-\hat{\imath}$ and
- the triangle T_2 in the xy-plane with vertices $(0,0,0)$, $(0,1,0)$ and $(2,0,0)$ and with normal vector $-\hat{\mathbf{k}}$.

So, by Stokes' theorem

$$\begin{aligned} \int_C \mathbf{F}\cdot d\mathbf{r} + \int_L \mathbf{F}\cdot d\mathbf{r} = \int_{\partial S} \mathbf{F}\cdot d\mathbf{r} &= \iint_S \nabla \times \mathbf{F}\cdot \hat{\mathbf{n}}\, dS \\ &= \iint_{T_1} \nabla \times \mathbf{F}\cdot(-\hat{\imath})\, dS + \iint_{T_2} \nabla \times \mathbf{F}\cdot(-\hat{\mathbf{k}})\, dS \\ &= \iint_{T_1} (1-2xz)\,\hat{\jmath}\cdot(-\hat{\imath})\, dS + \iint_{T_2} (1-2xz)\,\hat{\jmath}\cdot(-\hat{\mathbf{k}})\, dS \\ &= 0 \end{aligned}$$

Consequently the integral of interest

$$\begin{aligned}\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= -\int_L \mathbf{F}\cdot \mathrm{d}\mathbf{r} = -\int_2^0 (2+x^2)\,\mathrm{d}x \qquad \text{since } \mathrm{d}y=\mathrm{d}z=z=0 \text{ on } L\\ &= \int_0^2 (2+x^2)\,\mathrm{d}x = \left[2x+\frac{x^3}{3}\right]_0^2 = \frac{20}{3}\end{aligned}$$

S-31: (a) *by direct evaluation:* The curl of $\mathbf{G}$ is

$$\nabla\times\mathbf{G} = \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & -z & y\end{bmatrix} = 2\hat{\imath}$$

The part of S in the first octant is sketched in the figure on the left below. S consists of two parts — the cylindrical surface

$$S_1 = \left\{\ (x,y,z)\ \middle|\ y^2+z^2=9,\ 0\leqslant x\leqslant 5\ \right\}$$

and the disc

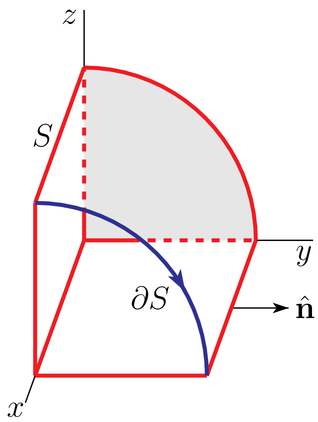
$$S_2 = \left\{\ (x,y,z)\ \middle|\ x=0,\ y^2+z^2\leqslant 9\ \right\}$$

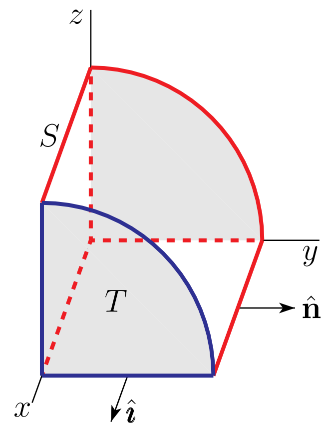
The normal $\hat{\mathbf{n}}$ to S_1 always points radially outward from the cylinder and so always has $\hat{\imath}$ component zero. The normal of S_2 is $-\hat{\imath}$. So the flux is

$$\begin{aligned}\iint_S \nabla\times\mathbf{G}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iint_{S_1} 2\hat{\imath}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \iint_{S_2} 2\hat{\imath}\cdot(-\hat{\imath})\,\mathrm{d}S\\ &= -2\iint_{S_2}\mathrm{d}S\\ &= -2\big(\pi 3^2\big) = -18\pi\end{aligned}$$

(a) *using Stokes' theorem:* Let's use Stokes' theorem. The boundary ∂S of S is the cirlce $y^2+z^2=9$, $x=5$, oriented clockwise when viewed from far down the x-axis. We'll parametrize it by $\mathbf{r}(\theta)=\big(5,3\cos\theta,-3\sin\theta\big)$. Then Stokes' theorem gives

$$\begin{aligned}\iint_S \nabla\times\mathbf{G}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \oint_{\partial S}\mathbf{G}\cdot\mathrm{d}\mathbf{r}\\ &= \int_0^{2\pi}\big(5\,,\,3\sin\theta, 3\cos\theta\big)\cdot\big(0\,,\,-3\sin\theta\,,\,-3\cos\theta\big)\,\mathrm{d}\theta\\ &= \int_0^{2\pi}\big(-9\sin^2\theta-9\cos^2\theta\big)\,\mathrm{d}\theta\\ &= -18\pi\end{aligned}$$





(b) This time we'll use the divergence theorem. The surface S is not closed. So we'll use the auxilary surface formed by "topping S off" with the cap $T = \{ (5, y, z) \mid y^2 + z^2 \leqslant 9 \}$. If we give T the normal vector $\hat{\boldsymbol{\imath}}$, this auxiliary surface, the union of S and T, is the boundary of $V = \{ (x, y, z) \mid y^2 + z^2 \leqslant 9,\ 0 \leqslant x \leqslant 5 \}$. So the divergence theorem gives

$$\begin{aligned} \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS + \iint_T \mathbf{F} \cdot \hat{\mathbf{n}}\, dS &= \iint_{\partial V} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS \\ &= \iiint_V \boldsymbol{\nabla} \cdot \mathbf{F}\, dV \\ &= 0 \end{aligned}$$

since $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$. Thus the flux of interest is

$$\begin{aligned} \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = -\iint_T \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = -\iint_T \mathbf{F} \cdot \hat{\boldsymbol{\imath}}\, dS &= -\iint_T (2 + z)\, dS \\ &= -2 \iint_T dS \qquad \text{since} \iint_T z\, dS = 0, \text{ because } z \text{ is odd} \\ &= -18\pi \qquad \text{since } T \text{ has area } 9\pi \end{aligned}$$

S-32: (a) Since

- $\frac{y}{x}$ is defined when $x \neq 0$ and
- $x^{1+x^2} = e^{(1+x^2)\ln x}$ is defined when $\ln x$ is defined, which is when $x > 0$ (assuming that we are not allowed to use complex numbers) and
- $y^{1+y^2} = e^{(1+y^2)\ln y}$ is defined when $\ln y$ is defined, which is when $y > 0$ and
- $\cos^5(\ln z)$ is defined when $\ln z$ is defined, which when $z > 0$

the domain of $\mathbf{F}$ is

$$D = \{ (x, y, z) \mid x > 0,\ y > 0,\ z > 0 \}$$

(b) The domain D is both connected (any two points in D can be joined by a curve that lies completely in D) and simply connected (any simple closed curve in D can be shrunk to a point continuously in D).

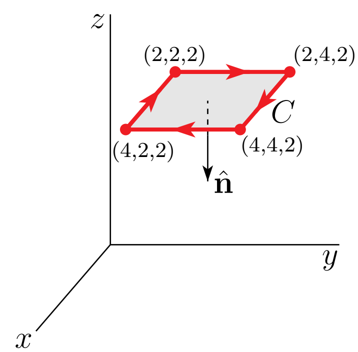
(c) The curl of $\mathbf{F}$ is

$$\nabla \times \mathbf{F} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{x} + x^{1+x^2} & x^2 - y^{1+y^2} & \cos^5(\ln z) \end{bmatrix} = \big(2x - {}^1\!/_x\big)\hat{\mathbf{k}}$$

(d) The integrand for direct evaluation looks very complicated. On the other hand $\nabla \times \mathbf{F}$ is quite simple. So let's try Stokes' thoerem. Denote

$$S = \{\, (x,y,z) \,\big|\, 2 \leqslant x \leqslant 4,\ 2 \leqslant y \leqslant 4,\ z = 2 \,\}$$

The boundary of S is C. Because of the clockwise orientation of C, we assign the normal vector $-\hat{\mathbf{k}}$ to S. See the sketch below



Then, by Stokes' theorem,

$$\begin{aligned} \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iint_S \nabla \times \mathbf{F} \cdot (-\hat{\mathbf{k}})\, \mathrm{d}S = -\iint_S \big(2x - \tfrac{1}{x}\big)\, \mathrm{d}S \\ &= -\int_2^4 \mathrm{d}x \int_2^4 \mathrm{d}y\ \big(2x - \tfrac{1}{x}\big) = -\int_2^4 \mathrm{d}x\, 2\big(2x - \tfrac{1}{x}\big) = -2\Big[x^2 - \ln x\Big]_2^4 \\ &= -2\big[12 - \ln 2\big] = 2\ln 2 - 24 \end{aligned}$$

(e) Since $\nabla \times \mathbf{F}$ is not $\mathbf{0}$, $\mathbf{F}$ cannot be conservative.

S-33: (a) By the vector identity $\nabla \cdot \big(\nabla \times \mathbf{G}\big) = 0$ (Theorem 4.1.7.a of the CLP-4 text). So we must have

$$\begin{aligned} 0 = \nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}\big(xz\big) + \frac{\partial}{\partial y}\big(axe^y z + byz\big) + \frac{\partial}{\partial z}\big(y^2 - xe^y z^2\big) \\ &= z + \big(axe^y z + bz\big) + \big(-2xe^y z\big) \\ &= (1+b)z + (a-2)xe^y z \end{aligned}$$

So we need $a = 2$ and $b = -1$.

(b) Note that the boundary, ∂S, is the circle $x^2 + y^2 = 1$, $z = 0$, oriented counter-clockwise. Also note that, if we knew what $\mathbf{G}$ was, we would be able to use

Stokes' theorem to give

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_S (\boldsymbol{\nabla}\times\mathbf{G})\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \oint_{\partial S} \mathbf{G}\cdot\mathrm{d}\mathbf{r}$$

So let's find a vector potential $\mathbf{G}$. That is, let's try and find a vector field $\mathbf{G} = G_1\,\hat{\imath} + G_2\,\hat{\jmath} + G_3\,\hat{\mathbf{k}}$ that obeys $\boldsymbol{\nabla}\times\mathbf{G} = \mathbf{F}$, or equivalently,

$$\begin{aligned}
\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} &= F_1 = xz \\
-\frac{\partial G_3}{\partial x} + \frac{\partial G_1}{\partial z} &= F_2 = 2xe^y z - yz \\
\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} &= F_3 = y^2 - xe^y z^2
\end{aligned}$$

Let's also require that $G_3 = 0$. (If this is mysterious to you, review §4.1.2 in the CLP-4 text.) Then the equations above simplify to

$$\begin{aligned}
-\frac{\partial G_2}{\partial z} &= xz \\
\frac{\partial G_1}{\partial z} &= 2xe^y z - yz \\
\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} &= y^2 - xe^y z^2
\end{aligned}$$

Now the first equation contains only a single unknown, namely G_2 and we can find all G_2's that obey the first equation simply by integrating with respect to z:

$$G_2 = -\frac{xz^2}{2} + N(x,y)$$

Note that, because $\frac{\partial}{\partial z}$ treats x and y as constants, the constant of integration N is allowed to depend on x and y.

Similarly, the second equation contains only a single unknown, G_1, and is easily solved by integrating with respect to z. The second equation is satisfied if and only if

$$G_1 = xe^y z^2 - \frac{1}{2}yz^2 + M(x,y)$$

for some function M.

Finally, the third equation is also satisfied if and only if $M(x,y)$ and $N(x,y)$ obey

$$\frac{\partial}{\partial x}\Big(-\frac{xz^2}{2} + N(x,y)\Big) - \frac{\partial}{\partial y}\Big(xe^y z^2 - \frac{yz^2}{2} + M(x,y)\Big) = y^2 - xe^y z^2$$

which simplifies to

$$\frac{\partial N}{\partial x}(x,y) - \frac{\partial M}{\partial y}(x,y) = y^2$$

This is one linear equation in two unknowns, M and N. Typically, we can easily solve one linear equation in one unknown. So we are free to eliminate one of the unknowns by setting, for example, $M = 0$, and then choosing any N that obeys

$$\frac{\partial N}{\partial x}(x, y) = y^2$$

Integrating with respect to x gives, as one possible choice, $N(x, y) = xy^2$. So we have found a vector potential. Namely

$$\mathbf{G} = \Big(xe^y z^2 - \frac{1}{2}yz^2\Big)\,\hat{\boldsymbol{\imath}} + \Big(xy^2 - \frac{xz^2}{2}\Big)\,\hat{\boldsymbol{\jmath}}$$

We can now evaluate the flux. Parametrize ∂S by

$$\begin{aligned}\mathbf{r}(\theta) &= \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}\\ \mathbf{r}'(\theta) &= -\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}\end{aligned}$$

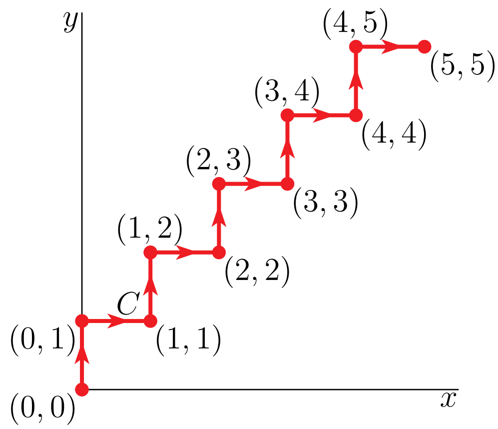
with $0 \leqslant \theta \leqslant 2\pi$. So

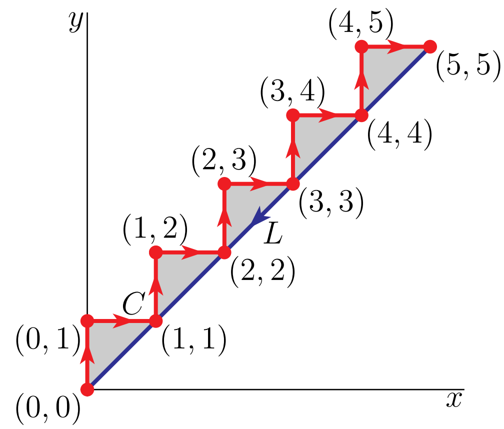
$$\begin{aligned}\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \oint_{\partial S} \mathbf{G}\cdot\mathrm{d}\mathbf{r}\\
&= \int_0^{2\pi} \Big(\overbrace{\cos\theta\sin^2\theta\,\hat{\boldsymbol{\jmath}}}^{\mathbf{G}(\mathbf{r}(\theta))}\Big)\cdot\overbrace{\big(-\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}\big)}^{\mathbf{r}'(\theta)}\ \mathrm{d}\theta\\
&= \int_0^{2\pi} \sin^2\theta\cos^2\theta\ \mathrm{d}\theta\\
&= \int_0^{2\pi} \frac{1-\cos(2\theta)}{2}\ \frac{1+\cos(2\theta)}{2}\ \mathrm{d}\theta\\
&= \frac{1}{4}\int_0^{2\pi} \big\{1-\cos^2(2\theta)\big\}\ \mathrm{d}\theta\\
&= \frac{1}{4}\int_0^{2\pi} \Big\{1-\frac{1+\cos(4\theta)}{2}\Big\}\ \mathrm{d}\theta\\
&= \frac{1}{4}\,\frac{1}{2}\,2\pi \qquad \text{since} \int_0^{2\pi}\cos(4\theta)\ \mathrm{d}\theta = 0\\
&= \frac{\pi}{4}\end{aligned}$$

S-34: Considering that there are ten line segments in C, it is probably not very efficient to use direct evaluation. Two other possible methods come to mind. If $\mathbf{F}$ is conservative, we can use $\mathbf{F}$'s potential. Even if $\mathbf{F}$ is not conservative, it may be possible to efficiently use Stokes' (or Green's) theorem. So let's compute

$$\boldsymbol{\nabla}\times\mathbf{F} = \det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y & 2x-10 & 0\end{bmatrix} = \hat{\mathbf{k}}$$

As $\nabla \times \mathbf{F} \neq \mathbf{0}$, the vector field $\mathbf{F}$ is not conservative. As $\nabla \times \mathbf{F} \neq \mathbf{0}$ is very simple, it looks like Stokes' theorem could provide an efficient way to compute the integral. The left figure below contains a sketch of C.





The curve C is not closed, and so is not the boundary of a surface, so we cannot apply Stokes' theorem directly. But we can easily come up with a surface whose boundary contains C. Let R be the shaded region in the figure on the right above. The boundary ∂R of R consists of two parts — C and the line segment L. The normal of R for $-\hat{\mathbf{k}}$ (since ∂R is oriented clockwise). So Stokes' theorem gives

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_L \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot (-\hat{\mathbf{k}})\, \mathrm{d}S = \iint_R (\hat{\mathbf{k}}) \cdot (-\hat{\mathbf{k}})\, \mathrm{d}S = -\mathrm{Area}(R)$$

R is the union of 5 triangles, each of height 1 and base 1. So

$$\mathrm{Area}(R) = 5 \times \frac{1}{2} \times 1 \times 1 = \frac{5}{2}$$

If we denote by $-L$ the line segment from $(0, 0)$ to $(5, 5)$, we can parametrize $-L$ by $\mathbf{r}(t) = t(5, 5)$, $0 \leqslant t \leqslant 1$ and

$$\int_{-L} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^1 \overbrace{\big(5t\,\hat{\imath} + (10t - 10)\,\hat{\jmath}\big)}^{\mathbf{F}(\mathbf{r}(t))} \cdot \overbrace{\big(5\,\hat{\imath} + 5\,\hat{\jmath}\big)}^{\mathbf{r}'(t)}\, \mathrm{d}t = \int_0^1 5(15t - 10)\mathrm{d}t = 25\left(\frac{3}{2} - 2\right)$$
$$= -\frac{25}{2}$$

All together

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\mathrm{Area}(R) - \int_L \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\mathrm{Area}(R) + \int_{-L} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\frac{5}{2} - \frac{25}{2} = -15$$

S-35: If we parametrize the curve as

$$x = 2\cos\theta \qquad y = 2\sin\theta \qquad z = x^2 = 4\cos^2\theta \qquad 0 \leqslant \theta \leqslant 2\pi$$

then the term $\sin x(\theta)^2\, x'(\theta)$ in the integral will be $\sin\big(4\cos^2\theta\big)\,(-2\sin\theta)$. That looks hard to integrate. So let's try Stokes' theorem. The curl of $\mathbf{F}$ is

$$\boldsymbol{\nabla}\times\mathbf{F} = \det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x^2 & xz & z^2 \end{bmatrix} = -x\,\hat{\boldsymbol{\imath}} + z\,\hat{\mathbf{k}}$$

The curve C is the boundary of the surface

$$S = \big\{\ (x,y,z)\ \big|\ x^2+y^2 \leqslant 4,\ z = x^2\ \big\}$$

with upward pointing normal. For the surface $z = f(x,y) = x^2$, (3.3.2) in the CLP-4 text gives

$$\begin{aligned} \hat{\mathbf{n}}\,\mathrm{d}S &= \pm\big[-f_x(x,y)\,\hat{\boldsymbol{\imath}} - f_y(x,y)\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\big]\,\mathrm{d}x\mathrm{d}y \\ &= \pm\big[-2x\,\hat{\boldsymbol{\imath}} + \hat{\mathbf{k}}\big]\,\mathrm{d}x\mathrm{d}y \end{aligned}$$

Since we want the upward pointing normal

$$\hat{\mathbf{n}}\,\mathrm{d}S = \big[-2x\,\hat{\boldsymbol{\imath}} + \hat{\mathbf{k}}\big]\,\mathrm{d}x\mathrm{d}y$$

So by Stokes' theorem (Theorem 4.4.1 in the CLP-4 text)

$$\begin{aligned} \oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \iint_S \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_{x^2+y^2\leqslant 4} \big(-x\,\hat{\boldsymbol{\imath}} + \overbrace{x^2}^{z}\,\hat{\mathbf{k}}\big)\cdot\big[-2x\,\hat{\boldsymbol{\imath}} + \hat{\mathbf{k}}\big]\,\mathrm{d}x\mathrm{d}y \\ &= 3\iint_{x^2+y^2\leqslant 4} x^2\,\mathrm{d}x\mathrm{d}y \end{aligned}$$

Switching to polar coordinates

$$\begin{aligned} \oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= 3\int_0^2 \mathrm{d}r\,r\int_0^{2\pi}\mathrm{d}\theta\,r^2\cos^2\theta \\ &= 3\left[\int_0^2 r^3\,\mathrm{d}r\right]\left[\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta\right] \\ &= 3\,\frac{2^4}{4}\left[\int_0^{2\pi}\frac{\cos(2\theta)+1}{2}\,\mathrm{d}\theta\right] \\ &= 12\pi \end{aligned}$$

For an efficient, sneaky, way to evaluate $\int_0^{2\pi}\cos^2 t\,\mathrm{d}t$ see Example 2.4.4 in the CLP-4 text.

S-36: By Stokes' Theorem,

$$\oint_C \mathbf{E}\cdot\mathrm{d}\mathbf{r} = \iint_S (\boldsymbol{\nabla}\times\mathbf{E})\cdot\hat{\mathbf{n}}\,\mathrm{d}S$$

so Faraday's law becomes

$$\iint_S \left(\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}\right) \cdot \hat{\mathbf{n}}\, \mathrm{d}S = 0$$

This is true for all surfaces S. So the integrand, assuming that it is continuous, must be zero.

To see this, let $\mathbf{G} = \left(\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}\right)$. Suppose that $\mathbf{G}(\mathbf{x}_0) \neq 0$. Pick a unit vector $\hat{\mathbf{n}}$ in the direction of $\mathbf{G}(\mathbf{x}_0)$. Let S be a very small flat disk centered on $\mathbf{x}_0$ with normal $\hat{\mathbf{n}}$ (the vector we picked). Then $\mathbf{G}(\mathbf{x}_0) \cdot \hat{\mathbf{n}} > 0$ and, by continuity, $\mathbf{G}(\mathbf{x}) \cdot \hat{\mathbf{n}} > 0$ for all $\mathbf{x}$ on S, if we have picked S small enough. Then $\iint_S \left(\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}\right) \cdot \hat{\mathbf{n}}\, \mathrm{d}S > 0$, which is a contradiction. So $\mathbf{G} = \mathbf{0}$ everywhere and we conclude that

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = 0$$

S-37: The curl of the specified vector field is

$$\begin{aligned}
\nabla \times \mathbf{F} &= \nabla \times \left(z\,\hat{\boldsymbol{\imath}} + x\,\hat{\boldsymbol{\jmath}} + y^3z^3\,\hat{\mathbf{k}}\right) \\
&= \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y^3z^3 \end{bmatrix} \\
&= 3y^2z^3\,\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}
\end{aligned}$$

For every t, we have $x(t) = z(t)$ and $x(t)^2 + y(t)^2 + z(t)^2 = 2$. So the specified curve is the intersection of the plane $x = z$ and the sphere $x^2 + y^2 + z^2 = 2$. This curve is the boundary of the circular disk

$$D = \left\{ (x,y,z) \,\middle|\, x = z,\ x^2 + y^2 + z^2 \leqslant 2 \right\}$$

The curve is oriented so that $\big(x(t), y(t)\big) = \big(\cos t, \sqrt{2}\sin t\big)$ runs in the standard (counterclockwise) direction. So the unit normal to D used in Stokes' theorem has positive $\hat{\mathbf{k}}$ component. Since the plane $x - z = 0$ has unit normal $\pm\frac{1}{\sqrt{2}}(1, 0, -1)$, the unit normal used in Stokes' theorem is $\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}(-1, 0, 1)$. By Stokes' theorem

$$\begin{aligned}
\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \iint_D \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \frac{1}{\sqrt{2}} \iint_D (3y^2z^3, 1, 1) \cdot (-1, 0, 1)\, \mathrm{d}S \\
&= \frac{1}{\sqrt{2}} \iint_D (1 - 3y^2z^3)\, \mathrm{d}S
\end{aligned}$$

The disk D is invariant under the reflection $(x, y, z) \to (-x, y, -z)$. Since y^2z^3 is odd under this reflection, $\iint_D y^2z^3\, \mathrm{d}s = 0$ and

$$\oint \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \frac{1}{\sqrt{2}} \iint_D \mathrm{d}S = \frac{1}{\sqrt{2}}\mathrm{Area}(D)$$

Because the centre of the ball $x^2 + y^2 + z^2 \leqslant 2$ (namely $(0,0,0)$) is contained in the plane $x = z$, the radius of the disk D is the same as the radius of the sphere $x^2 + y^2 + z^2 = 2$. So D has radius $\sqrt{2}$ and

$$\oint \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \frac{1}{\sqrt{2}}\text{Area}(D) = \frac{1}{\sqrt{2}}\pi\big(\sqrt{2}\big)^2 = \sqrt{2}\pi$$

S-38: The curl of the vector field $\mathbf{F} = z\,\hat{\boldsymbol{\imath}} + x\,\hat{\boldsymbol{\jmath}} - y\,\hat{\mathbf{k}}$ is

$$\boldsymbol{\nabla} \times \mathbf{F} = -\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}$$

The unit normal to the plane $x + y + z = 1$, with positive $\hat{\mathbf{k}}$ component as required by Stokes' theorem in this case, is $\hat{\mathbf{n}} = \frac{1}{\sqrt{3}}(1,1,1)$. If we denote by D the circular disk $x + y + z = 1$, $x^2 + y^2 + z^2 \leqslant 1$, then Stokes' theorem (Theorem 4.4.1 in the CLP-4 text) says

$$\begin{aligned}\oint_C z\,\mathrm{d}x + x\,\mathrm{d}y - y\,\mathrm{d}z = \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \iint_D \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S = \iint_D (-1,1,1) \cdot \frac{1}{\sqrt{3}}(1,1,1)\,\mathrm{d}S \\ &= \frac{1}{\sqrt{3}}\text{Area}(D)\end{aligned}$$

A reasonable guess for the centre of the disk is $\frac{1}{3}(1,1,1)$. (This guess is just based on symmetry.) To check this we just need to observe that it is indeed on the plane $x + y + z = 1$ and that the distance from $\frac{1}{3}(1,1,1)$ to any point (x,y,z) obeying $x + y + z = 1$ and $x^2 + y^2 + z^2 = 1$, namely

$$\begin{aligned}\sqrt{\Big(x - \frac{1}{3}\Big)^2 + \Big(y - \frac{1}{3}\Big)^2 + \Big(z - \frac{1}{3}\Big)^2} &= \sqrt{x^2 + y^2 + z^2 - \frac{2}{3}(x + y + z) + \frac{3}{9}} = \sqrt{1 - \frac{2}{3} + \frac{1}{3}} \\ &= \sqrt{\frac{2}{3}}\end{aligned}$$

is the same. This also tells us that D has radius $\sqrt{\frac{2}{3}}$ and hence area $\frac{2}{3}\pi$. So the specified line integral is $\frac{2\pi}{3\sqrt{3}}$.

S-39: (a) We parametrize S in cylindrical coordinates:

$$\mathbf{r}(r,\theta) = r\cos\theta\,\hat{\boldsymbol{\imath}} + r\sin\theta\,\hat{\boldsymbol{\jmath}} + r\,\hat{\mathbf{k}} \qquad \text{with } 0 \leqslant r \leqslant 1,\ 0 \leqslant \theta \leqslant \pi$$

(b) We compute

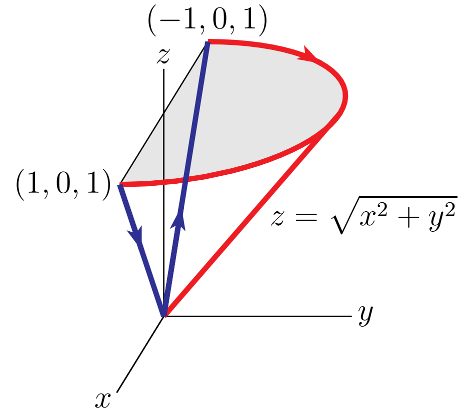
$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial r} &= \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}} \\ \frac{\partial \mathbf{r}}{\partial \theta} &= -r\sin\theta\,\hat{\boldsymbol{\imath}} + r\cos\theta\,\hat{\boldsymbol{\jmath}} \\ \hat{\mathbf{n}}\,\mathrm{d}S &= \pm\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta}\,\mathrm{d}r\mathrm{d}\theta = \pm\big(-r\cos\theta\,\hat{\boldsymbol{\imath}} - r\sin\theta\,\hat{\boldsymbol{\jmath}} + r\,\hat{\mathbf{k}}\big)\,\mathrm{d}r\mathrm{d}\theta\end{aligned}$$

To calculate the downward flux, we use the minus sign. We find

$$\iint_S \mathbf{v}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_0^\pi \mathrm{d}\theta \int_0^1 \mathrm{d}r\ (r\cos\theta, r\sin\theta, -2r)\cdot(r\cos\theta, r\sin\theta, -r)$$

$$= \int_0^\pi \mathrm{d}\theta \int_0^1 \mathrm{d}r\ 3r^2 = \pi r^3\Big|_{r=0}^{1} = \pi$$

(c) *Solution 1*: Let $\mathcal{P}$ be the path along line segments from $(1,0,1)$ to $(0,0,0)$ and from $(0,0,0)$ to $(-1,0,1)$. Here is a sketch. $\mathcal{P}$ is in blue.



Then

$$\int_{\mathcal{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} + \int_{\mathcal{P}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \iint_S \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$$

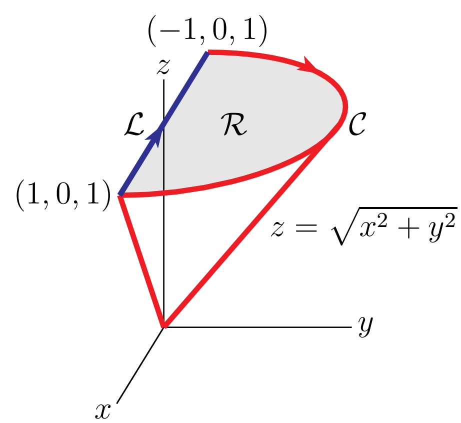
by Stokes' Theorem. Along $\mathcal{P}$, the vector field $\mathbf{F}$ is orthogonal to the curve so that $\int_{\mathcal{P}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = 0$. Note that $\nabla\times\mathbf{F}$ is the vector field $\mathbf{v}$ from part (b). Thus

$$\int_{\mathcal{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \iint_S \mathbf{v}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \pi$$

(c) *Solution 2:* Let $\mathcal{L}$ be the line segment from $(1,0,1)$ to $(-1,0,1)$ and let

$$\mathcal{R} = \{(x,y,z)\}\, x^2+y^2 \leqslant 1, y \geqslant 0, z = 1$$

Here is a sketch. $\mathcal{L}$ is in blue and $\mathcal{R}$ is shaded.



Then

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{\mathcal{L}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_{\mathcal{R}} \boldsymbol{\nabla} \times \mathbf{F} \cdot (-\hat{\mathbf{k}})\, \mathrm{d}S$$

by Stokes' Theorem. Along $\mathcal{L}$, the vector field $\mathbf{F} = \hat{\boldsymbol{\jmath}}$ is orthogonal to the curve (which has direction $-\hat{\boldsymbol{\imath}}$ so that $\int_{\mathcal{L}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$. Note that $\nabla \times \mathbf{F}$ is the vector field $\mathbf{v}$ from part (b). Thus

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\iint_{\mathcal{R}} \mathbf{v} \cdot \hat{\mathbf{k}}\, \mathrm{d}S = \iint_{\mathcal{R}} 2z\, \mathrm{d}S = 2\iint_{\mathcal{R}} \mathrm{d}S = 2\,\mathrm{Area}(\mathcal{R}) = \pi$$

S-40: Let S' be the portion of $x+y+z=1$ that is inside the sphere $x^2+y^2+z^2=1$. Then $\partial S = \partial S'$, so, by Stokes' theorem, (with $\hat{\mathbf{n}}$ always the upward pointing normal)

$$\iint_{S'} (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \oint_{\partial S'} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_{S} (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, \mathrm{d}S$$

As

$$\boldsymbol{\nabla} \times \mathbf{F} = \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z & z-x & x-y \end{bmatrix} = -2(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}})$$

and, on S', $\hat{\mathbf{n}} = \frac{1}{\sqrt{3}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}})$

$$\iint_{S'} (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iint_{S'} \big(-2\sqrt{3}\big)\, \mathrm{d}S = -2\sqrt{3} \times \mathrm{Area}(S')$$

S' is the intersection of a plane with a sphere and so is a circular disk. It's center (x_c, y_c, z_c) has to obey $x_c + y_c + z_c = 1$. By symmetry, $x_c = y_c = z_c$, so $x_c = y_c = z_c = \frac{1}{3}$. Any point, (x, y, z), which satisfies both $x+y+z=1$ and $x^2+y^2+z^2=1$, obeys

$$\left(x-\frac{1}{3}\right)^2 + \left(y-\frac{1}{3}\right)^2 + \left(z-\frac{1}{3}\right)^2 = x^2+y^2+z^2 - \frac{2}{3}(x+y+z) + 3\frac{1}{9} = 1 - \frac{2}{3} + \frac{1}{3} = \frac{2}{3}$$

That is, any point on the boundary of S' is a distance $\sqrt{\frac{2}{3}}$ from $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. So the radius of S' is $\sqrt{\frac{2}{3}}$, the area of S' is $\frac{2}{3}\pi$ and

$$\iint_{S'} (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, \mathrm{d}S = -2\sqrt{3} \times \mathrm{Area}(S') = -\frac{4}{\sqrt{3}}\pi$$

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S-1: (a) True. For any constant vector $\mathbf{a} = (a_1, a_2, a_3)$,

$$\mathbf{a} \times \mathbf{r} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ x & y & z \end{bmatrix} = (a_2 z - a_3 y)\hat{\imath} - (a_1 z - a_3 x)\hat{\jmath} + (a_1 y - a_2 x)\hat{\jmath}$$

This vector field does indeed have divergence 0.

(b) True. This is our conservative field screening condition Theorem 4.1.7.b.

(c) True. This is one of our vector identities, namely Theorem 4.1.4.c.

(d) False. The trap here is that $\mathbf{F}$ need not be defined at the origin. We saw, in Example 3.4.2 of the CLP-4 text, that the point source $\mathbf{F}_S = \frac{m\mathbf{r}}{|\mathbf{r}|^3}$ had flux $4\pi m$ through every sphere centred on the origin. We also saw, in Example 4.2.7 of the CLP-4 text, that the divergence $\nabla \cdot \mathbf{F}_S = 0$ everywhere except at the origin (where it is not defined). So if we choose m to be a very big negative number (say -10^{100}) and add in a very small vector field with positive divergence (say $10^{-100}(x\hat{\imath} + y\hat{\jmath} + z\hat{\mathbf{k}})$), we will get the vector field $\mathbf{F} = -10^{100}\frac{\mathbf{r}}{|\mathbf{r}|^3} + 10^{-100}(x\hat{\imath} + y\hat{\jmath} + z\hat{\mathbf{k}})$ which has divergence $\nabla \cdot \mathbf{F} = 3 \times 10^{-100} > 0$ everywhere except at the origin. The flux of this field through the specified sphere will be $-4\pi \times 10^{100}$ plus a very small positive number.

(e) True. The statement that "the flux out of one hemisphere is equal to the flux into the opposite hemisphere" is equivalent to the statement that "the flux out of the sphere is equal to zero". Since $\nabla \cdot \mathbf{F} = 0$ everywhere, that is true by the divergence theorem.

(f) That depends.

If $\kappa = 0$, then $\frac{d\hat{\mathbf{T}}}{ds} = 0$, so that $\frac{d\mathbf{r}}{ds} = \hat{\mathbf{T}}$ is a constant. So $\mathbf{r}(s) = s\hat{\mathbf{T}} + \mathbf{r}(0)$ is part of a straight line.

If $\kappa > 0$, then, because the curve is in a plane, the torsion τ is zero and the Frenet-Serret formulae reduce to

$$\frac{d\hat{\mathbf{T}}}{ds} = \kappa\hat{\mathbf{N}} \qquad \frac{d\hat{\mathbf{N}}}{ds} = -\kappa\hat{\mathbf{T}}$$

Now consider the centre of curvature $\mathbf{c}(s) = \mathbf{r}(s) + \frac{1}{\kappa}\hat{\mathbf{N}}(s)$. Since

$$\frac{d\mathbf{c}}{ds} = \frac{d\mathbf{r}}{ds} + \frac{1}{\kappa}\frac{d\hat{\mathbf{N}}}{ds} = \hat{\mathbf{T}}(s) + \frac{1}{\kappa}\big(-\kappa\hat{\mathbf{T}}(s)\big) = \mathbf{0}$$

$\mathbf{c}(s)$ is a constant and

$$|\mathbf{r}(s) - \mathbf{c}| = \frac{1}{\kappa}$$

which says that the curve is part of the circle of radius $\frac{1}{\kappa}$ centred on $\mathbf{c}$.

(g) False. We saw in Examples 2.3.14 and 4.3.8 of the CLP-4 text that the given vector field is not conservative.

(h) False. For example, if $P = -y$, then $\oint_C \mathbf{F} \cdot d\mathbf{r} = -\oint_C y\, dx$ is the area inside C. See Corollary 4.3.5 in the CLP-4 text.

(i) False.

If $\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{a}$ is a constant, then $\mathbf{v}(t) = \mathbf{a}\, t + \mathbf{v}_0$. Integrating a second time, $\mathbf{r}(t) = \frac{1}{2}\mathbf{a}\, t^2 + \mathbf{v}_0\, t + \mathbf{r}_0$. This is not a spiral, whether or not the speed is constant. (In fact, for the speed $|\mathbf{v}(t)| = |\mathbf{a}\, t + \mathbf{v}_0|$ to be constant, $\mathbf{a}$ has to be $\mathbf{0}$, so that $\mathbf{r}(t) = \mathbf{v}_0\, t + \mathbf{r}_0$ is a straight line.)

Another way to come to the same conclusion uses

$$\mathbf{a}(t) = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}(t)\, \hat{\mathbf{T}}(t) + \kappa(t)\Big(\frac{\mathrm{d}s}{\mathrm{d}t}(t)\Big)^2 \hat{\mathbf{N}}(t)$$

As the speed $\frac{\mathrm{d}s}{\mathrm{d}t}$ is a constant, it reduces to

$$\mathbf{a}(t) = \kappa(t)\Big(\frac{\mathrm{d}s}{\mathrm{d}t}(t)\Big)^2 \hat{\mathbf{N}}(t)$$

As $\mathbf{a}(t)$ is a constant, its direction, $\hat{\mathbf{N}}(t)$, is also a constant. The normal vector to a spiral is not constant.

S-2: (a) False. For any constant vector $\mathbf{a} = (a_1, a_2, a_3)$,

$$\mathbf{a} \times \mathbf{r} = \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ x & y & z \end{bmatrix} = (a_2 z - a_3 y)\hat{\boldsymbol{\imath}} - (a_1 z - a_3 x)\hat{\boldsymbol{\jmath}} + (a_1 y - a_2 x)\hat{\boldsymbol{\jmath}}$$

So

$$\boldsymbol{\nabla} \times (\mathbf{a} \times \mathbf{r}) = \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & -a_1 z + a_3 x & a_1 y - a_2 x \end{bmatrix} = 2a_1\hat{\boldsymbol{\imath}} + 2a_2\hat{\boldsymbol{\jmath}} + 2a_3\hat{\mathbf{k}}$$

is nonzero, unless the constant vector $\mathbf{a} = \mathbf{0}$.

(b) False. For example, if $f(x) = x^2$, then

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f) = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} x^2) = \boldsymbol{\nabla} \cdot (2x\hat{\boldsymbol{\imath}}) = 2$$

(c) False. For example, if $\mathbf{F} = x^2\hat{\boldsymbol{\imath}}$, then

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{F}) = \boldsymbol{\nabla}\big(\boldsymbol{\nabla} \cdot (x^2\hat{\boldsymbol{\imath}})\big) = \boldsymbol{\nabla}(2x) = 2\hat{\boldsymbol{\imath}}$$

(d) False. The trap here is that $\mathbf{F}$ need not be defined at the origin. We saw, in Example 3.4.2 of the CLP-4 text, that the point source $\mathbf{F} = \frac{m\mathbf{r}}{|\mathbf{r}|^3}$ had flux $4\pi m$ through every sphere centred on the origin. We also saw, in Example 4.2.7 of the CLP-4 text, that the divergence $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$ everywhere except at the origin (where it is not defined).

(e) True. Any simple, smooth, closed curve in $\mathbb{R}^3$ that avoids the origin is the boundary of a surface S that also avoids the origin. Then, by Stokes' theorem,

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = 0$$

(f) True. Let $S = \{\, \mathbf{r} \mid |\mathbf{r}-\mathbf{c}| = R \,\}$ be a sphere. Denote by $V = \{\, \mathbf{r} \mid |\mathbf{r}-\mathbf{c}| \leqslant R \,\}$ the ball whose boundary is S. Let H be one hemisphere of S with outward pointing normal and let H' be the other hemisphere of S with inward point normal. Then the boundary of V, with outward pointing normal, can be viewed as consisting of two parts, namely H and $-H'$, where by $-H'$ we mean H' but with outward pointing normal. Then, by the divergence theorem

$$\begin{aligned}\iint_H \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S - \iint_{H'} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iint_{\partial V} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S \\ &= \iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V > 0\end{aligned}$$

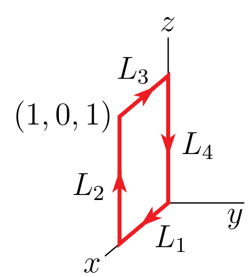
which implies that $\iint_H \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S > \iint_{H'} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$.

(g) False. The trap here is that the curve is in $\mathbb{R}^3$, not $\mathbb{R}^2$. As we saw in Example 1.4.4 of the CLP-4 text, a helix has constant curvature, but does not lie in a plane and so is not part of a circle.

(h) False. Even if we restrict $\mathbf{F}$ to the xy-plane (i.e. to $z=0$), this vector field is not conservative. We saw that in Examples 2.3.14 and 4.3.8 of the CLP-4 text.

(i) False. For example, the vector field $\mathbf{F} = x\,\hat{\mathbf{k}}$ is always parallel to the z-axis. So its flow lines are also all parallel to the z-axis. But if the closed curve C consists of the line segments

- L_1 from $(0,0,0)$ to $(1,0,0)$, followed by
- L_2 from $(1,0,0)$ to $(1,0,1)$, followed by
- L_3 from $(1,0,1)$ to $(0,0,1)$, followed by
- L_4 from $(0,0,1)$ back to $(0,0,0)$,



then

- $\int_{L_1} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_0^1 (x\hat{\mathbf{k}})\cdot\hat{\boldsymbol{\imath}}\,\mathrm{d}x = 0$ since $\hat{\mathbf{k}} \perp \hat{\boldsymbol{\imath}}$, $\mathrm{d}\mathbf{r} = \hat{\boldsymbol{\imath}}\,\mathrm{d}x$ on L_1 and
- $\int_{L_2} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_0^1 (1\hat{\mathbf{k}})\cdot\hat{\mathbf{k}}\,\mathrm{d}z = 1$ since $x=1$ and $\mathrm{d}\mathbf{r} = \hat{\mathbf{k}}\,\mathrm{d}z$ on L_2 and
- $\int_{L_3} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = -\int_0^1 (x\hat{\mathbf{k}})\cdot\hat{\boldsymbol{\imath}}\,\mathrm{d}x = 0$ since $\hat{\mathbf{k}} \perp \hat{\boldsymbol{\imath}}$ and
- $\int_{L_4} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = -\int_0^1 (0\hat{\mathbf{k}})\cdot\hat{\mathbf{k}}\,\mathrm{d}z = 0$ since $x=0$ on L_4.

All together

$$\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_{L_1} \mathbf{F}\cdot\mathrm{d}\mathbf{r} + \int_{L_2} \mathbf{F}\cdot\mathrm{d}\mathbf{r} + \int_{L_3} \mathbf{F}\cdot\mathrm{d}\mathbf{r} + \int_{L_4} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = 1$$

(j) True. If the speed $|\mathbf{v}|$ is constant then

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{v}|^2 = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}\cdot\mathbf{v}) = 2\mathbf{v}\cdot\mathbf{a}$$

S-3: (a) False. $\mathbf{r}''(t)$ is the full acceleration. So $|\mathbf{r}''(t)|$ is the magnitude of the full acceleration, not just the tangential component of acceleration. For example, if $\mathbf{r}(t) = \cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}}$ (i.e. the particle is just going around in circles), the acceleration $\mathbf{r}''(t) = -\cos t\,\hat{\boldsymbol{\imath}} - \sin t\,\hat{\boldsymbol{\jmath}}$ is perpendicular to the direction of motion. So the tangential component of acceleration is zero, while $|\mathbf{r}''(t)| = 1$.

(b) $\hat{\mathbf{T}}(t)$ is the tangent vector to the curve at $\mathbf{r}(t)$. $\hat{\mathbf{N}}(t)$ and $\hat{\mathbf{B}}(t)$ are both perpendicular to $\hat{\mathbf{T}}(t)$ (and to each other) and so span the plane normal to the curve at $\mathbf{r}(t)$.

(c) True. This is (half of) Theorem 2.4.7 in the CLP-4 text.

(d) False. The statement $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \cdot F) = 0$ is just plain gibberish, because $\boldsymbol{\nabla} \cdot F$ is a scalar valued function and there is no such thing as the curl of a scalar valued function.

(e) False. For example if $\mathbf{F} = \hat{\boldsymbol{\imath}}$, then, by the divergence theorem,

$$\iint\limits_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iiint\limits_V \boldsymbol{\nabla} \cdot \mathbf{F}\, \mathrm{d}V = 0$$

Here $V = \left\{\, x, y, z \;\middle|\; x^2 + y^2 + z^2 \leqslant 1 \,\right\}$ is the inside of the sphere.

(f) True. If S is the boundary of the solid region E, then we can orient S by always choosing the normal vector that points into E.

S-4: (a) The helix is approximately a bunch of circles stacked one on top of each other. The radius of the circles increase as z increases. So the curvature *decreases* as z increases.

(b) Here are two arguments both of which conclude that $f(x)$ is D.

- If C were the graph $y = f(x)$, then $f'(x)$ would have two points of discontinuity. The curvature $\kappa(x)$ would not the defined at those two points. The function whose graph is D is defined everywhere and so cannot be the curvature of the function whose graph is C.
- The function whose graph is D has two inflection points. So its curvature is zero at two points. The function whose graph is C is indeed zero at two points (that in fact correspond to the inflection points of D). So D is the graph of $f(x)$ and C is the graph of $\kappa(x)$.

(c) For any fixed y, $x^2 + z^2 = 1$ is a circle of radius 1. So we can parametrize it by $x(\theta) = \cos\theta$, $z(\theta) = \sin\theta$, $0 \leqslant \theta < 2\pi$. The y-coordinate of any point on the intersection is determined by $y = xz$. So we can use

$$\mathbf{r}(\theta) = \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\mathbf{k}} + \sin\theta\,\cos\theta\,\hat{\boldsymbol{\jmath}} \qquad 0 \leqslant \theta < 2\pi$$

(d) We are told that the helical ramp starts starts with the y-axis when $z = 0$.

- In the cases of parametrisations (a) and (c), $z = 0$ forces $u = 0$ and $u = 0$ forces $x = y = 0$. That is only the origin, not the y-axis. So we can rule out (a) and(c).
- In the case of parametrisation (b), $z = 0$ forces $v = 0$ and $v = 0$ forces $y = 0$ and $x = u$. As u varies that sweeps out the x-axis, not the y-axis. So we can rule out (b).

- In the case of parametrisation (d), $z = 0$ forces $v = 0$ and $v = 0$ forces $x = 0$ and $y = u$. As u varies that sweeps out the y-axis, which is what we want.

Furthermore

- we are told that $z = v$ runs from 0 to 5 and that
- $x^2 + y^2 = u^2 \geqslant 4$

So we want parametrisation (d) with domain $|u| \geqslant 2$, $0 \leqslant v \leqslant 5$.

(e) Straight lines have curvature 0. So one acceptable parametrized curve is $\mathbf{r}(t) = t\,\hat{\boldsymbol{\imath}}$, $0 \leqslant t \leqslant 1$.

(f) The cube S has six sides. So the outward flux through ∂S is 6 and, by the divergence theorem,

$$6 = \iint_{\partial S} \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iiint_{S} \boldsymbol{\nabla} \cdot \mathbf{F}\, \mathrm{d}V = \iiint_{S} C\, \mathrm{d}V = C$$

since S has volume one. So $C = 6$.

(g) For the vector field $\mathbf{F}$ to be conservative, we need

$$\begin{aligned} & & \frac{\partial \mathbf{F}_1}{\partial y} &= \frac{\partial \mathbf{F}_2}{\partial x} \\ &\iff & \frac{\partial}{\partial y}(ax + by) &= \frac{\partial}{\partial x}(cx + dy) \\ &\iff & b &= c \end{aligned}$$

When $b = c$, an allowed potential is $\frac{a}{2}x^2 + bxy + \frac{d}{2}y^2$. The specified set is

$$\big\{\ (a, b, c, d)\ \big|\ a, b, c, d \text{ all real and } b = c\ \big\}$$

(h) By the definition of arclength parametrisation, the arclength along the curve between $\mathbf{r}(0)$ and $\mathbf{r}(s)$ is s. In particular, the arclength between $\mathbf{r}(0)$ and $\mathbf{r}(3)$ is 3 and the arclength between $\mathbf{r}(0)$ and $\mathbf{r}(5)$, which is the same as the arclength between $\mathbf{r}(0)$ and $\mathbf{r}(3)$ plus the arclength between $\mathbf{r}(3)$ and $\mathbf{r}(5)$, is 5. So the arclength between $\mathbf{r}(3)$ and $\mathbf{r}(5)$ is $5 - 3 = 2$.

(i) In this solution, we'll use, for example $-T$ to refer to the curve T, but with the arrow pointing in the opposite direction to that of the arrow on T.

In parts (2), (3) and (4) we will choose $\mathbf{F}$ to be the vector field

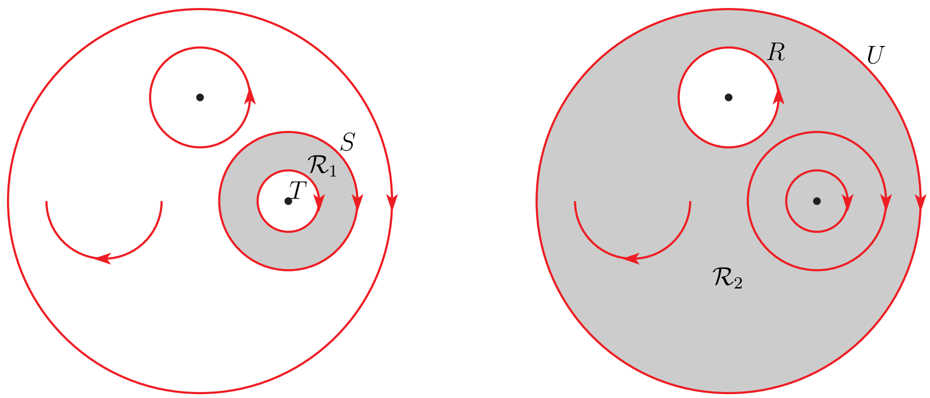
$$\mathbf{G}(x, y) = -\frac{y}{x^2 + y^2}\hat{\boldsymbol{\imath}} + \frac{x}{x^2 + y^2}\hat{\boldsymbol{\jmath}}$$

We saw, in Example 2.3.14 of the CLP-4 text, that $\boldsymbol{\nabla} \times \mathbf{G} = \mathbf{0}$ except at the origin where it is not defined. We also saw, in Example 4.3.8 of the CLP-4 text, that $\oint_C \mathbf{G} \cdot \mathrm{d}\mathbf{r} = 2\pi$ for any counterclockwise oriented circle centred on the origin.

(1) Let $\mathcal{R}_1$ be the region between S and T. It is the shaded region in the figure on the left below. Note that $\mathcal{R}_1$ is contained in the domain of $\mathbf{F}$, so that $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$ on all of $\mathcal{R}_1$. The boundary of $\mathcal{R}_1$ is $S - T$, meaning that the boundary consists of two parts, with one part being S and the other part being $-T$. So, by Stokes' theorem

$$\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{r} - \int_T \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\partial \mathcal{R}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_{\mathcal{R}_1} \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{k}} \, \mathrm{d}S = 0$$

and (1) is true.



(2) False. Choose a coordinate system so that Q is at the origin and choose $\mathbf{F} = \mathbf{G}$. We saw, in Examples 2.3.14 and 4.3.8 of the CLP-4 text, that the curl of $\mathbf{G}$ vanished everywhere except at the origin, where it was not defined, but that $\int_R \mathbf{G} \cdot \mathrm{d}\mathbf{r} \neq 0$.

(3),(4) False. Here is a counterexample that shows that both (3) and (4) are false. Choose a coordinate system so that Q is at the origin and choose $\mathbf{F} = \mathbf{G}$. By Stokes' theorem

$$\int_S \mathbf{G} \cdot \mathrm{d}\mathbf{r} = \int_T \mathbf{G} \cdot \mathrm{d}\mathbf{r} = 0$$

because $\boldsymbol{\nabla} \times \mathbf{G} = \mathbf{0}$ everywhere inside S, including at P. So now both parts (3) and (4) reduce to the claim that $\int_U \mathbf{G} \cdot \mathrm{d}\mathbf{r} = \int_R \mathbf{G} \cdot \mathrm{d}\mathbf{r}$.
We saw, in Example 4.3.8 of the CLP-4 text, that $\int_R \mathbf{G} \cdot \mathrm{d}\mathbf{r} = 2\pi$.
To finish off the counterexample, we'll now show that $\int_U \mathbf{G} \cdot \mathrm{d}\mathbf{r} = -2\pi$. Let $\mathcal{R}_2$ be the region between U and R. It is the shaded region in the figure on the right above. Note that $\boldsymbol{\nabla} \times \mathbf{G} = \mathbf{0}$ on all of $\mathcal{R}_2$. including at P. The boundary of $\mathcal{R}_2$ is $-U - R$, meaning that the boundary consists of two parts, with one part being $-U$ and the other part being $-R$. So, by Stokes' theorem

$$-\int_U \mathbf{G} \cdot \mathrm{d}\mathbf{r} - \int_R \mathbf{G} \cdot \mathrm{d}\mathbf{r} = \int_{\partial \mathcal{R}_2} \mathbf{G} \cdot \mathrm{d}\mathbf{r} = \iint_{\mathcal{R}_2} \boldsymbol{\nabla} \times \mathbf{G} \cdot \hat{\mathbf{k}} \, \mathrm{d}S = 0$$

and $\int_U \mathbf{G} \cdot \mathrm{d}\mathbf{r} = -\int_R \mathbf{G} \cdot \mathrm{d}\mathbf{r} = -2\pi$

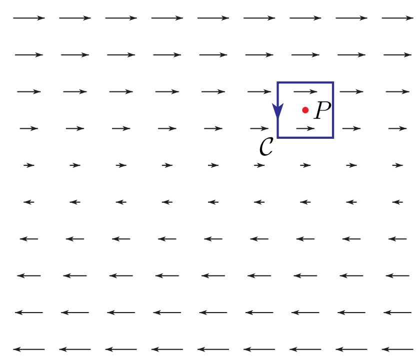
(5) False. For any conservative vector field $\mathbf{F}$, with potential f, $\int_V \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is just the difference of the values of f at the two end points of V. It is easy to choose an f for which those two values are different. For example $f(x, y) = x$ does the job.

(j) Let S be any closed surface and denote by V the volume that it encloses. Presumably the question assumes that S is oriented so that $S = \partial V$. Then by the divergence theorem

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_{\partial V} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V$$

This is exactly the volume of V if $\boldsymbol{\nabla}\cdot\mathbf{F} = 1$ everywhere. One vector field $\mathbf{F}$ with $\boldsymbol{\nabla}\cdot\mathbf{F} = 1$ everywhere is $\mathbf{F} = x\,\hat{\boldsymbol{\imath}}$.

(k) Let $\mathcal{C}$ be the counterclockwise boundary of a small square centred on P, like the blue curve in the figure below, but much smaller. Call the square (the inside of $\mathcal{C}$) S.



By Stokes' theorem

$$\iint_S \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{k}}\,\mathrm{d}S = \oint_{\mathcal{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r}$$

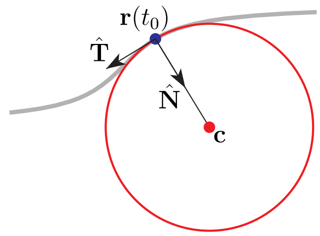
- The contribution to $\oint_{\mathcal{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ coming from the left and right sides of $\mathcal{C}$ will be zero, because $\mathbf{F}$ is perpendicular to $\mathrm{d}\mathbf{r}$ there.
- The contribution to $\oint_{\mathcal{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ coming from the top of $\mathcal{C}$ will be negative, because there $\mathbf{F}$ is a positive number times $\hat{\boldsymbol{\imath}}$ and $\mathrm{d}\mathbf{r}$ is a negative number times $\hat{\boldsymbol{\imath}}$.
- The contribution to $\oint_{\mathcal{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ coming from the bottom of $\mathcal{C}$ will be positive, because there $\mathbf{F}$ is a positive number times $\hat{\boldsymbol{\imath}}$ and $\mathrm{d}\mathbf{r}$ is a positive number times $\hat{\boldsymbol{\imath}}$.
- The magnitude of the contribution from the top of $\mathcal{C}$ will be larger than the magnitude of the contribution from the bottom of $\mathcal{C}$, because $|\mathbf{F}|$ is larger on the top than on the bottom.

So, all together, $\oint_{\mathcal{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} < 0$, and consequently (taking a limit as the square size tends to zero) $\boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{k}}$ is negative at P.

S-5: (a) False. We could have, for example, $\boldsymbol{\nabla}\cdot\mathbf{F}$ zero at one point and strictly positive elsewhere. One example would be $\mathbf{F} = x^3\,\hat{\boldsymbol{\imath}} + y^3\,\hat{\boldsymbol{\jmath}} + z^3\,\hat{\mathbf{k}}$, with S_1 and S_2 being the upward oriented top and bottom hemispheres, respectively, of the unit sphere $x^2+y^2+z^2=1$.

(b) False. The conditions that (1) $\boldsymbol{\nabla}\times\mathbf{F} = 0$ and (2) the domain of $\mathbf{F}$ is simply-connected, are sufficient, but not necessary, to imply that $\mathbf{F}$ is conservative. For example the vector field $\mathbf{F} = \mathbf{0}$, with any domain at all, is conservative with potential 0. Another example (which does not depend on choosing a domain that is smaller than the largest possible domain) is $\mathbf{F} = \boldsymbol{\nabla}\frac{1}{x^2+y^2}$ with domain $\{\ (x,y,z)\ \big|\ (x,y)\neq(0,0)\ \}$. That is, the domain is $\mathbb{R}^3$ with the z-axis removed.

(c) That's true. Consider any point $\mathbf{r}(t_0)$ on a parametrized curve $\mathbf{r}(t)$. That's the blue point in the figure below. The centre of curvature for the curve at $\mathbf{r}(t_0)$ is



$\mathbf{c} = \mathbf{r}(t_0) + \rho(t_0)\hat{\mathbf{N}}(t_0)$. It is the red dot in the figure.

- The radius of the osculating circle is the distance from its centre, $\mathbf{c}$, to any point of the circle, like $\mathbf{r}(t_0)$. That's $|\mathbf{r}(t_0) - \mathbf{c}| = |\rho(t_0)\hat{\mathbf{N}}(t_0)| = \rho(t_0)$. The curvature of the osculating circle is one over its radius. So its curvature is $\frac{1}{\rho(t_0)} = \kappa(t_0)$.
- The unit normal to the osculating circle at $\mathbf{r}(t_0)$ is a unit vector in the opposite direction to the radius vector from the centre $\mathbf{c}$ to $\mathbf{r}(t_0)$. The radius vector is $\mathbf{r}(t_0) - \mathbf{c}_0 = -\rho(t_0)\hat{\mathbf{N}}(t_0)$, so the unit normal is $\hat{\mathbf{N}}(t_0)$.
- The osculating circle lies in the plane that best fits the curve near $\mathbf{r}(t_0)$. (See the beginning of §1.4 in the CLP-4 text.) So the unit tangents to the osculating circle at $\mathbf{r}(t_0)$ are perpendicular to both $\hat{\mathbf{N}}(t_0)$ and $\hat{\mathbf{B}}(t_0)$ and so are either $\hat{\mathbf{T}}(t_0)$ or $-\hat{\mathbf{T}}(t_0)$, depending on how we orient the osculating circle.

(d) False. Kepler's third law is that a planet orbiting a sun has the *square* of the period proportional to the cube of the major axis of the orbit.

(e) True. That's part (a) of Theorem 4.1.7 in the CLP-4 text.

(f) True. Every domain contains closed surfaces. This has nothing to do with vector fields.

(g) True. We saw this in Example 2.3.4 in the CLP-4 text.

(h) False. Let $\mathbf{F}$ be an everywhere defined conservative vector field with potential φ. Then $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$ everywhere. If P and Q are two points and if $\varphi(P) - \varphi(Q) = 3$ and if C is a curve from Q to P, then $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 3$. One example would be $\varphi(x, y, z) = x$, $\mathbf{F} = \hat{\boldsymbol{\imath}}$, $P = (3, 0, 0)$, $Q = (0, 0, 0)$.

(i) False. The normal component of acceleration depends on speed, as well as curvature.

(j) False. The curve $\mathbf{r}_1$ contains only points in the xy-plane. Every $\mathbf{r}_2(t)$ with $t \neq 0$ has a nonzero z-coordinate.

S-6: (a) False. Changing the orientation of a surface does not change $\mathrm{d}S$ at all. (It changes $\hat{\mathbf{n}}\mathrm{d}S$ by a factor of (-1).) So

$$\iint_S f\,\mathrm{d}S = +\iint_{-S} f\,\mathrm{d}S$$

which is not $-\iint_{-S} f\,\mathrm{d}S$, unless the integral is zero.

(b) False. For every vector field with two continuous partial derivatives, $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) = 0$

(see Theorem 4.1.7.a in the CLP-4 text), so the divergence theorem gives

$$\iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iiint_V \nabla \cdot (\nabla \times \mathbf{F})\, \mathrm{d}V = 0$$

whether or not $\mathbf{F}$ is conservative.

(c) True. Define the vector field $\mathbf{F} = f\,\hat{\imath}$. Then, by Stokes' theorem,

$$\int_C f\, \mathrm{d}x = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iint_S \left(\frac{\partial f}{\partial z}\hat{\jmath} - \frac{\partial f}{\partial y}\hat{\mathbf{k}} \right) \cdot \hat{\mathbf{n}}\, \mathrm{d}S$$

(d) True. The left hand side, $(\nabla f) \times (\nabla f)$, is zero because (∇f) is parallel to itself and the right hand side $\nabla \times (\nabla f)$ is zero by Theorem 4.1.7.b (the screening test for conservative fields) of the CLP-4 text.

(e) True. The curve $\mathbf{r}(t) = \big(2\,,\,0\,,\,1\big) + t^3\big(4\,,\,-1\,,\,-2\big)$ is a straight line. Straight lines have curvature 0.

(f) True. In general $|\mathbf{r}'(t)| = \frac{\mathrm{d}s}{\mathrm{d}t}$. Under arc length parametrization $t = s$ so that $\frac{\mathrm{d}s}{\mathrm{d}t} = 1$.

(g) True. If $\mathbf{F}$ is a constant vector fleld, then, by the divergence thoerem,

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iiint_V \nabla \cdot \mathbf{F}\, \mathrm{d}V = \iiint_V 0\, \mathrm{d}V = 0$$

(h) False. The statement $\nabla \times \mathbf{F} = (x, y, z)$ means that $\mathbf{F}$ is a vector potential for the vector field $\mathbf{G} = (x, y, z)$. But $\mathbf{G}$ fails the screening test $\nabla \cdot \mathbf{G} = 0$ for vector potentials.

S-7: (a) P is the x-component of $\mathbf{F}$. As we travel vertically upward through A, that x-component decreases. Hence $P_y < 0$ at A.

(b) Q is the y-component of $\mathbf{F}$. As we travel horizontally to the right through A, that y-component increases. Hence $Q_x > 0$ at A.

(c) $\nabla \times \mathbf{F} = (Q_x - P_y)\hat{\mathbf{k}}$ and $Q_x - P_y > 0$ at A, so that the curl of $\mathbf{F}$ at A is in the direction of $+\hat{\mathbf{k}}$.

(d) Along the curve $\mathcal{C}_1$ the magnitude of the angle between $\mathbf{F}$ and $\mathrm{d}\mathbf{r}$ is less than $90°$, so that $\mathbf{F} \cdot \mathrm{d}\mathbf{r} > 0$ and $\int_{\mathcal{C}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} > 0$.

(e) Along the curve $\mathcal{C}_2$ the magnitude of the angle between $\mathbf{F}$ and $\mathrm{d}\mathbf{r}$ is greater than $90°$, so that $\mathbf{F} \cdot \mathrm{d}\mathbf{r} < 0$ and $\int_{\mathcal{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} < 0$.

(f) If $\mathbf{F}$ were conservative, we would have $\int_{\mathcal{C}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$. As these two integrals have opposite signs $\mathbf{F}$ is not conservative. (Since $\mathbf{F}$ is not conservative, it is not the gradient of some function. At A, $P_x > 0$ and $Q_y > 0$. So $\mathbf{F}$ is not divergence free and is not the curl of a vector potential.)

S-8: (a) False. The curve $\mathbf{r}_1$ contains only points with $z \geqslant 0$. Every $\mathbf{r}_2(t)$ with $t < 0$ has $z < 0$.

(b) True. $\mathbf{r}_2(t^2) = \mathbf{r}_1(t)$ and t^2 runs from 0 to 1 as t runs from 0 to 1.

(c) True. In general $|\mathbf{r}'(t)| = \frac{\mathrm{d}s}{\mathrm{d}t}$. When $t = s$, $\frac{\mathrm{d}s}{\mathrm{d}t} = 1$.

(d) False. The curve need not even lie in a plane. For example, as we saw in Example 1.4.4 of the CLP-4 text, the helix $\mathbf{r}(t) = a\cos t\,\hat{\boldsymbol{\imath}} + b\sin t\,\hat{\boldsymbol{\jmath}} + bt\,\hat{\mathbf{k}}$ has constant curvature $\kappa = \frac{a}{a^2+b^2}$ but is not a circle.

(e) True. If the speed $|\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}}$ of a moving object is constant, then

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}\cdot\mathbf{v}) = 2\mathbf{v}\cdot\mathbf{a}$$

(f) False. If the vector field $\mathbf{F}(x,y,z) = \frac{-y}{x^2+y^2}\hat{\boldsymbol{\imath}} + \frac{x}{x^2+y^2}\hat{\boldsymbol{\jmath}} + z\hat{\mathbf{k}}$ were conservative, its restriction, $\frac{-y}{x^2+y^2}\hat{\boldsymbol{\imath}} + \frac{x}{x^2+y^2}\hat{\boldsymbol{\jmath}}$, to the xy-plane would also be conservative. But we saw in Examples 2.3.14 and 4.3.8 of the CLP-4 text that the vector field $\frac{-y}{x^2+y^2}\hat{\boldsymbol{\imath}} + \frac{x}{x^2+y^2}\hat{\boldsymbol{\jmath}}$ is not conservative.

(g) False. The vector field of part (f), with domain $\{\, (x,y,z) \mid x^2+y^2 > 1 \,\}$, provides a counterexample.

(h) False. The curve $x^2 + y^2 = 2$ can not be shrunk to a point continuously in $\{\, (x,y) \mid x^2+y^2 > 1 \,\}$.

(i) True. Any curve in $\{\, (x,y) \mid y > x^2 \,\}$ can be shrunk to a point continuously in $\{\, (x,y) \mid y > x^2 \,\}$.

(j) True. By the divergence theorem,

$$\iint_S \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint_E \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{F})\,\mathrm{d}V = 0$$

since $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{F}) = 0$ by the vector identity of Theorem 4.1.7.a in the CLP-4 text.

S-9:

(a) True. In general $|\mathbf{r}'(t)| = \frac{\mathrm{d}s}{\mathrm{d}t}$. When $t = s$, $\frac{\mathrm{d}s}{\mathrm{d}t} = 1$.

(b) False. The curve need not even lie in a plane. For example, as we saw in Example 1.4.4 of the CLP-4 text, the helix $\mathbf{r}(t) = a\cos t\,\hat{\boldsymbol{\imath}} + b\sin t\,\hat{\boldsymbol{\jmath}} + bt\,\hat{\mathbf{k}}$ has constant curvature $\kappa = \frac{a}{a^2+b^2}$ but is not a circle.

(c) True. See Theorem 2.4.6 of the CLP-4 text.

(d) False. The vector field $\mathbf{F}(x,y,z) = \frac{-y}{x^2+y^2}\hat{\boldsymbol{\imath}} + \frac{x}{x^2+y^2}\hat{\boldsymbol{\jmath}}$, with domain

$$\{\, (x,y,z) \mid x^2+y^2 > 1 \,\}$$

provides a counterexample.

(e) False. The curve $\mathbf{r}_1$ contains only points with $z \geqslant 0$. Every $\mathbf{r}_2(t)$ with $t < 0$ has $z < 0$.

(f) True. $\mathbf{r}_2(t^2) = \mathbf{r}_1(t)$ and t^2 runs from 0 to 1 as t runs from 0 to 1.

(g) True. $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) = 0$ by the vector identity of Theorem 4.1.7.a in the CLP-4 text.

(h) False. A counterexample is $f(x, y, z) = x^2$. It has $\boldsymbol{\nabla} f = 2x\,\hat{\imath}$ and hence $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f) = 2$.

(i) False. The curve $x^2 + y^2 = 2$ can not be shrunk to a point continuously in $\{\, (x, y) \mid x^2 + y^2 > 1 \,\}$.

(j) True. Any curve in $\{\, (x, y) \mid y > x^2 \,\}$ can be shrunk to a point continuously in $\{\, (x, y) \mid y > x^2 \,\}$.

S-10: (a) False. $\boldsymbol{\nabla} f = \mathbf{0}$ if and only if f is constant. But if f is the constant K, then $\int_C f\, \mathrm{d}s$ is K times the length of C, which need not be zero.

(b) False. Any curve which lies in a plane has constant binormal. For example, the circle $\mathbf{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + 0\,\hat{\mathbf{k}}$ has constant binormal $\hat{\mathbf{B}} = \hat{\mathbf{k}}$, but is not a straight line.

(c) True. If $\mathbf{r}(t)$ has constant speed, the $\left(\frac{\mathrm{d}s}{\mathrm{d}t}(t)\right)^2 = \mathbf{r}'(t) \cdot \mathbf{r}'(t)$ is constant and

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{r}'(t) \cdot \mathbf{r}'(t)\big) = 2\mathbf{r}'(t) \cdot \mathbf{r}''(t)$$

(d) False. For the line integral $\int_C \big(\mathbf{F} \times \mathbf{G}\big) \cdot \mathrm{d}\mathbf{r}$ to be independent of the path C, the vector field $\mathbf{F} \times \mathbf{G}$ has to be conservative and so has to obey $\boldsymbol{\nabla} \times (\mathbf{F} \times \mathbf{G}) = \mathbf{0}$. But

- Not all vector fields are conservative. For example, the vector field $\mathbf{H} = x\,\hat{\jmath}$ obeys $\boldsymbol{\nabla} \times \mathbf{H} = \hat{\mathbf{k}}$ and so is not conservative.
- We can make $\mathbf{F} \times \mathbf{G}$ be *any* vector field through judicious choices of $\mathbf{F}$ and $\mathbf{G}$. For example, if $\mathbf{F} = x\,\hat{\mathbf{k}}$ and $\mathbf{G} = \hat{\imath}$, then $\mathbf{F} \times \mathbf{G} = x\,\hat{\mathbf{k}} \times \hat{\imath} = x\,\hat{\jmath} = \mathbf{H}$.

(e) True. The contribution to $\int_C f\, \mathrm{d}s$ from an "infinitesmal piece of C" is the value of f on the piece times the *length* of the piece. That does not depend on the orientation of the piece.

(f) False. The two vectors in the cross product $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial u}$ are identical. So the cross product is $\mathbf{0}$.

(g) False. The integral is completely independent of $x(u, v)$ and $y(u, v)$. In particular if, for example, $x(u, v) = 157u$, $y(u, v) = 157v$, $z(u, v) = 0$ then $\iint_D \left(1 + \left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right)^{1/2} \mathrm{d}u\mathrm{d}v$ is always exactly the area of D, while the area of S is 157^2 times the area of D.

(h) True. If the fluid is incompressible then its flow preserves volumes and consequently $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$.

(i) Not only False, but Ridiculous. The left had side is scalar valued while the right hand side is vector valued.

S-11: (a) True. That $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ is the vector identity of Theorem 4.1.7.a. That identity is the basis of the vector potential screening test.

(b) False. If $\mathbf{F}$ is not conservative, then $\int_C \mathbf{F} \cdot d\mathbf{r}$ will depend on the endpoints of C.

(c) True. If $\nabla f = 0$, then

$$\begin{array}{ccccc}
\frac{\partial f}{\partial x}(x,y,z) = 0 & \Longrightarrow & f(x,y,z) = g(y,z) & & \\
\frac{\partial f}{\partial y}(x,y,z) = 0 & \Longrightarrow & \frac{\partial g}{\partial y}(y,z) = 0 & \Longrightarrow & g(y,z) = h(z) \\
\frac{\partial f}{\partial z}(x,y,z) = 0 & \Longrightarrow & h'(z) = 0 & \Longrightarrow & h(z) = C
\end{array}$$

for some functions $g(y,z)$, $h(z)$ and constant C.

(d) False. The curl $\nabla \times \mathbf{F}$ is zero for every conservative vector fields $\mathbf{F}$. There are many nonconstant conservative vector fields, like $\mathbf{F}(x,y,z) = x\,\hat{\boldsymbol{\imath}}$.

(e) True. As S is closed, it is the boundary of a solid region V. Then, by the divergence theorem,

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iiint_V \nabla \cdot \mathbf{F}\, dV = 0$$

(f) True. If $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C, then $\mathbf{F}$ is conservative by Theorem 2.4.6 in the CLP-4 text. Consequently, $\nabla \times \mathbf{F} = 0$ by Theorem 2.3.9.

(g) True. If the speed $|\mathbf{v}|$ is constant then

$$0 = \frac{d}{dt}|\mathbf{v}|^2 = \frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) = 2\mathbf{v} \cdot \mathbf{a}$$

Since $\hat{\mathbf{T}} = \frac{\mathbf{v}}{|\mathbf{v}|}$, $\hat{\mathbf{T}} \cdot \mathbf{a} = 0$ too. Here, we have assumed that the constant $|\mathbf{v}|$ is not zero. If the constant $|\mathbf{v}|$ is zero, then $\hat{\mathbf{T}}$ is not defined at all (and $\mathbf{a} = 0$).

(h) False. The trap here is that the curve is in $\mathbb{R}^3$, not $\mathbb{R}^2$. As we saw in Example 1.4.4 of the CLP-4 text, a helix has constant curvature, but does not lie in a plane and so is not part of a circle.

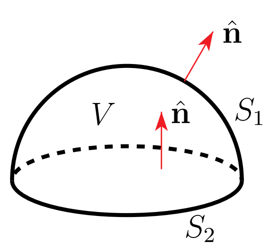
(i) False. The trap here is that we are told nothing about $\nabla \cdot \mathbf{F}$. As an example, let S_1 be the hemisphere

$$S_1 = \{\, (x,y,z) \mid x^2 + y^2 + z^2 = 1,\ z \geqslant 0 \,\}$$

with upward pointing normal and S_2 be the disk

$$S_2 = \{\, (x,y,0) \mid x^2 + y^2 \leqslant 1 \,\}$$

also with upward pointing normal.



Set

$$V = \left\{ (x,y,z) \mid 0 \leqslant z \leqslant \sqrt{x^2+y^2},\ x^2+y^2 \leqslant 1 \right\}$$

Then the boundary, ∂V, of V consists of two parts, namely S_1 (with normal pointing upwards) and S_2 (but with normal pointing downwards). The divergence theorem (Theorem 4.2.2 of the CLP-4 text) gives

$$\iint\limits_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S - \iint\limits_{S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint\limits_{V} \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V$$

If $\boldsymbol{\nabla}\cdot\mathbf{F} > 0$ (as is the case, for example, if $\mathbf{F} = x\,\hat{\boldsymbol{\imath}}$) then $\iint_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S - \iint_{S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ is definitely nonzero.

(j) True. This is one of Kepler's laws. See §1.9 in the CLP-4 text.

S-12: It's (b). (a) is gibberish — the left hand side is a scalar while the right hand side is a vector. (c) is also gibberish — the left hand side is a vector while the right hand side is a scalar. (b) is the vector identity of Theorem 4.1.4.c in the CLP-4 text.

S-13: (a) False. For example, if $f(x,y,z) = x^2$, then $\boldsymbol{\nabla} f = 2x\,\hat{\boldsymbol{\imath}}$ and $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla} f = 2$.

(b) Not only false, but ridiculous. The left hand side is a vector while the right hand side is a scalar.

(c) Not only false, but ridiculous. The right hand side is a vector while the left hand side is a scalar.

(d) True. That's the screening test for conservative fields, Theorem 4.1.7.b in the CLP-4 text.

(e) Not only false, but ridiculous. The curl of a scalar function is not defined.

(f) True. That's the screening test for vector potentials, Theorem 4.1.7.a in the CLP-4 text.

(g) False.

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\frac{\mathbf{r}}{|\mathbf{r}|^2} &= \frac{\partial}{\partial x}\frac{x}{x^2+y^2+z^2} + \frac{\partial}{\partial y}\frac{y}{x^2+y^2+z^2} + \frac{\partial}{\partial z}\frac{z}{x^2+y^2+z^2} \\
&= \frac{1}{x^2+y^2+z^2} - \frac{2x^2}{[x^2+y^2+z^2]^2} + \frac{1}{x^2+y^2+z^2} - \frac{2y^2}{[x^2+y^2+z^2]^2} \\
&\qquad\qquad + \frac{1}{x^2+y^2+z^2} - \frac{2z^2}{[x^2+y^2+z^2]^2} \\
&= \frac{3[x^2+y^2+z^2] - 2x^2 - 2y^2 - 2z^2}{[x^2+y^2+z^2]^2} \\
&= \frac{1}{x^2+y^2+z^2}
\end{aligned}$$

(h) False. For any constant vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$,

$$\boldsymbol{\omega} \times \mathbf{r} = \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{bmatrix} = (\omega_2 z - \omega_3 y)\hat{\boldsymbol{\imath}} - (\omega_1 z - \omega_3 x)\hat{\boldsymbol{\jmath}} + (\omega_1 y - \omega_2 x)\hat{\boldsymbol{\jmath}}$$

So

$$\boldsymbol{\nabla} \times (\boldsymbol{\omega} \times \mathbf{r}) = \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & -\omega_1 z + \omega_3 x & \omega_1 y - \omega_2 x \end{bmatrix} = 2\omega_1 \hat{\boldsymbol{\imath}} + 2\omega_2 \hat{\boldsymbol{\jmath}} + 2\omega_3 \hat{\mathbf{k}}$$

is nonzero, unless the constant vector $\boldsymbol{\omega} = \mathbf{0}$.

(i) True. The given equation is equivalent (by the vector identity Theorem 4.1.4.c in the CLP-4 text) to

$$\iiint_\Omega \boldsymbol{\nabla} \cdot \big(f\mathbf{F}\big)\, \mathrm{d}V = \iint_{\partial\Omega} f\mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$$

which is true by the divergence theorem.

(j) False. One of the variants of the divergence theorem given in Theorem 4.2.9 of the CLP-4 text is

$$\iint_{\partial\Omega} f\hat{\mathbf{n}}\, \mathrm{d}S = \iiint_\Omega \boldsymbol{\nabla} f\, \mathrm{d}V$$

Note that the sign on the right hand side is "$+$", not "$-$". In order for the equation given in part (j) to be true, it would be necessary that $\iiint_\Omega \boldsymbol{\nabla} f\, \mathrm{d}V = 0$ for all smooth scalar functions f. That's silly. One counterexample is

$$f(x) = x \qquad \Omega = \big\{\ (x,y,z)\ \big|\ x^2 + y^2 + z^2 \leqslant 1\ \big\}$$

Then

$$\iint_{\partial\Omega} f\hat{\mathbf{n}}\, \mathrm{d}S = \iint_{\partial\Omega} x(\overbrace{x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}}}^{\hat{\mathbf{n}}})\, \mathrm{d}S = \hat{\boldsymbol{\imath}} \iint_{\partial\Omega} x^2\, \mathrm{d}S$$
$$-\iiint_\Omega \boldsymbol{\nabla} f\, \mathrm{d}V = -\iiint_\Omega \hat{\boldsymbol{\imath}}\, \mathrm{d}V = -\hat{\boldsymbol{\imath}} \iiint_\Omega \mathrm{d}V$$

The coefficient of $\hat{\boldsymbol{\imath}}$ is obviously strictly positive in the upper integral and strictly negative in the lower integral.

S-14: (a) True. If the vector field is $\mathbf{F} = a\,\hat{\boldsymbol{\imath}} + b\,\hat{\boldsymbol{\jmath}} + c\,\hat{\mathbf{k}}$, then $f(x,y,z) = ax + by + cz$ obeys $\mathbf{F} = \boldsymbol{\nabla} f$ and so is a potential for $\mathbf{F}$.

(b) False. For example the vector field $\mathbf{F} = x\,\hat{\boldsymbol{\imath}} - y\,\hat{\boldsymbol{\jmath}}$ obeys $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$ but is not a constant vector field.

(c) True, assuming that $\mathbf{r}(t)$ is not indentically $\mathbf{0}$. If $\mathbf{r}(t)$ and $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$ are orthogonal at all points of the curve C, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{r}(t)\cdot\mathbf{r}(t)\big) = 2\mathbf{r}(t)\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t) = 0$$

So $x(t)^2 + y(t)^2 + z(t)^2 = \mathbf{r}(t)\cdot\mathbf{r}(t)$ is a constant. If $\mathbf{r}(t)$ is not indentically $\mathbf{0}$, that constant must be strictly positive. That is $x(t)^2 + y(t)^2 + z(t)^2 = a^2$ for some constant $a > 0$.

(d) False. The curvature (see §1.5 in the CLP-4 text) is

$$\kappa(t) = \frac{\left|\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}t}(t)\right|}{|\mathbf{r}'(t)|}$$

Changing the orientation of the curve amounts to replacing t by $-t$. This changes the signs of $\hat{\mathbf{T}}$ and $\mathbf{r}'$, but does not change κ, because the absolute values eliminate the signs.

(e) False. For example, the vector field $\mathbf{F} = \mathbf{0}$, with domain $\left\{\,(x,y,z)\;\middle|\; x^2+y^2>0\,\right\}$ is a conservative vector field (with potential 0) whose domain is not simply connected. As a less nitpicky example, let $\mathbf{F} = \boldsymbol{\nabla} f$ with $f = \frac{1}{x^2+y^2}$. The biggest possible domain for this vector field is also $\left\{\,(x,y,z)\;\middle|\; x^2+y^2>0\,\right\}$.

S-15: (a) We are to compute the divergence of $x^2y\,\hat{\boldsymbol{\imath}} + e^y\sin x\,\hat{\boldsymbol{\jmath}} + e^{zx}\,\hat{\mathbf{k}}$. Since

$$\begin{aligned}
\frac{\partial}{\partial x}\big(x^2y\big) &= 2xy\\
\frac{\partial}{\partial y}\big(e^y\sin x\big) &= e^y\sin x\\
\frac{\partial}{\partial z}\big(e^{zx}\big) &= xe^{xz}
\end{aligned}$$

the specified divergence is

$$\boldsymbol{\nabla}\cdot\big(x^2y\,\hat{\boldsymbol{\imath}} + e^y\sin x\,\hat{\boldsymbol{\jmath}} + e^{zx}\,\hat{\mathbf{k}}\big) = 2xy + e^y\sin x + xe^{xz}$$

(b) The specified curl is

$$\boldsymbol{\nabla}\times\big(\cos x^2\,\hat{\boldsymbol{\imath}} - y^3z\,\hat{\boldsymbol{\jmath}} + xz\,\hat{\mathbf{k}}\big) = \det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \cos x^2 & -y^3z & xz\end{bmatrix} = y^3\,\hat{\boldsymbol{\imath}} - z\,\hat{\boldsymbol{\jmath}}$$

(c) In principle, the domain could be any subset of $\left\{\,(x,y,z)\;\middle|\; x^2+y^2>0\,\right\}$. We are not told which subset to use, so, by default, D is the maximal domain

$$D = \left\{\,(x,y,z)\;\middle|\; x^2+y^2>0\,\right\} = \left\{\,(x,y,z)\;\middle|\; (x,y)\neq(0,0)\,\right\}$$

This D is connected (any two points in D can be joined by a curve that lies completely in D) but is not simply connected (the simple closed curve $\mathbf{r}(\theta) = \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}$, $0\leqslant\theta\leqslant 2\pi$ lies in D but cannot be shrunk to a point continuously in D). So (I) and (IV) are true. That's (iii).

(d) False. If the position of the particle at time t is $\mathbf{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath}$, then its speed is the constant 1 but its acceleration is $-\cos t\,\hat{\imath} - \sin t\,\hat{\jmath}$, which is nonzero.

S-16: (a) True. By the vector identity of Theorem 4.1.5.c in the CLP-4 text,

$$\boldsymbol{\nabla} \times (f\boldsymbol{\nabla} f) = (\boldsymbol{\nabla} f) \times (\boldsymbol{\nabla} f) + f\,\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = \mathbf{0}$$

The second term vanished because of the screening test vector identity of Theorem 4.1.7.b in the CLP-4 text.

(b) True. That's the vector identity of Theorem 4.1.4.c in the CLP-4 text.

(c) True. To have constant curvature 0 the curve must have unit tangent vector $\hat{\mathbf{T}}(s)$ obeying

$$\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s}(s) = \mathbf{0}$$

(See §1.5 in the CLP-4 text.) So $\mathbf{r}'(s) = \hat{\mathbf{T}}(s)$ must be a constant vector. Call it $\hat{\mathbf{T}}_0$. Integrating gives

$$\mathbf{r}(s) = s\hat{\mathbf{T}}_0 + \mathbf{r}_0$$

for some constant vector $\mathbf{r}_0$. So $\mathbf{r}(s)$ lies on the same straight line for all s.

(d) False. The trap here is that the curve is in $\mathbb{R}^3$, not $\mathbb{R}^2$. As we saw in Example 1.4.4 of the CLP-4 text, a helix has constant curvature, but does not lie in a plane and so is not part of a circle.

(e) True. The vector field $\mathbf{F} = \boldsymbol{\nabla} f$ is conservative. So, by Theorem 2.4.6.b in the CLP-4 text, the work integral

$$\int_C \boldsymbol{\nabla} f \cdot \mathrm{d}\mathbf{r} = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$$

for any closed curve C, and, in particular, for any circle C.

(f) True. The statement that "the flux out of one hemisphere is equal to the flux into the opposite hemisphere" is equivalent to the statement that "the flux out of the sphere is equal to zero". Since $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$ everywhere, that is true by the divergence theorem.

(g) True. Let S be the boundary of the solid region V. Then, by the divergence theorem (Theorem 4.2.9 of the CLP-4 text),

$$\iint_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S = \iiint_V \boldsymbol{\nabla} \cdot \big(\boldsymbol{\nabla} \times \mathbf{F}\big)\,\mathrm{d}V$$

But $\boldsymbol{\nabla} \cdot \big(\boldsymbol{\nabla} \times \mathbf{F}\big)$ is identically zero, by the screening test vector identity of Theorem 4.1.7.a in the CLP-4 text. So the integral is zero.

S-17: (a) True. Let $\mathbf{F}$ be the vector field. We are assuming that $\nabla \times \mathbf{F} = 0$ on all of $\mathbb{R}^3$. As a result, $\mathbf{F} = \nabla\phi$ for some potential function ϕ. We are also assuming that $0 = \nabla \cdot \mathbf{F} = \nabla \cdot \nabla\phi = \big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\big)\phi$. This is the definition of "ϕ is harmonic".

(b) False. Let $\mathbf{F}$ be the vector field. We are assuming that $\mathbf{F} = \nabla\phi$ for some potential function ϕ. If S is any smooth closed surface, with S being the boundary of the solid V, then, by the divergence theorem, the outward flux of $\mathbf{F}$ through S is

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint_V \nabla\cdot\mathbf{F}\,\mathrm{d}V = \iiint_V \nabla\cdot\nabla\phi\,\mathrm{d}V = \iiint_V \big(\tfrac{\partial^2}{\partial x^2} + \tfrac{\partial^2}{\partial y^2} + \tfrac{\partial^2}{\partial z^2}\big)\phi\,\mathrm{d}V$$

If, for example, $\phi = x^2$, then $\big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\big)\phi = 2$ and the flux of $\mathbf{F}$ through S is twice the volume of V, which is not zero.

S-18: (a) True. The vector field $\boldsymbol{\nabla} f$ is conservative and the work done by a conservative field around any closed curve is zero.

(b) False. By the vector identity Theorem 4.1.7.a in the CLP-4 text, we have

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{F}) = 0$$

for all vector fields $\mathbf{F}$. But $\boldsymbol{\nabla}\cdot(x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}}) = 3$.

S-19: (a) $\int_C \boldsymbol{\nabla} f\cdot\mathrm{d}\mathbf{r} = 0$ is the work done along the curve using the conservative force $\boldsymbol{\nabla} f$. That work is difference between the potential f at the final point minus the potential f at the initial point. If the final and initial points are both on the level surface $f(x, y, z) = 0$, that difference is zero.

(b) The rate of change of the specified vector is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t)\times\mathbf{r}(t) = \mathbf{v}'(t)\times\mathbf{r}(t) + \mathbf{v}(t)\times\mathbf{v}(t)$$

The first term vanishes because $\mathbf{v}'(t) = \mathbf{a}(t) = f(t)\mathbf{r}(t)$ is parallel to $\mathbf{r}(t)$. The second term vanishes because $\mathbf{v}(t) = \mathbf{v}(t)$.

(c) Call the constant vector $\mathbf{v}\times\mathbf{r}$ of part (b) $\mathbf{N}$. This vector is a constant and is perpendicular to both $\mathbf{v}(t)$ and $\mathbf{r}(t)$. In particular

$$\mathbf{N}\cdot\mathbf{r}(t) = 0$$

Assuming that $\mathbf{N}$ is nonzero, this is the equation of the plane through the origin with normal vector $\mathbf{N}$.

(d) Yes, as long as $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$, and $\hat{\mathbf{B}}$ are well-defined, since $\hat{\mathbf{B}} = \hat{\mathbf{T}}\times\hat{\mathbf{N}}$.

(e) No. When the maximum speed occurs $\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = 0$ so that $\mathbf{a} = \kappa(t)\left(\frac{\mathrm{d}s}{\mathrm{d}t}(t)\right)^2\hat{\mathbf{N}}(t)$. If the speed and (constant) curvature are nonzero, the acceleration is nonzero.

S-20: We apply Green's Theorem:

$$\int_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\,dxdy$$

(a) $\dfrac{1}{2}\displaystyle\int_C -y\,dx + x\,dy = \dfrac{1}{2}\iint_R \{1-(-1)\}\,dx\,dy = \text{Area}(R)$

(b) $\dfrac{1}{2}\displaystyle\int_C -x\,dx + y\,dy = \dfrac{1}{2}\iint_R 0\,dx\,dy = 0 \neq \text{Area}(R)$

(c) $\displaystyle\int_C y\,dx = \iint_R \{-1\}\,dx\,dy = -\text{Area}(R) \neq \text{Area}(R)$

(d) $\displaystyle\int_C 3y\,dx + 4x\,dy = \iint_R \{4-3\}\,dx\,dy = \text{Area}(R)$

S-21: (a) *True*. Since $v = |\mathbf{v}| = 1$ is constant, we have

$$\mathbf{a} = \frac{\mathrm{d}v}{\mathrm{d}t}\hat{\mathbf{T}} + v^2\kappa\hat{\mathbf{N}} = 0\hat{\mathbf{T}} + \kappa\hat{\mathbf{N}}.$$

Thus $1 = |\mathbf{a}| = \kappa|\hat{\mathbf{N}}|$, i.e., $\kappa = 1$.

(a) (Again.) Since $\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2 = 1$ for all t, differentiation gives $\mathbf{v}\cdot\mathbf{a} = 0$, i.e., $\mathbf{v} \perp \mathbf{a}$ always. It follows that $|\mathbf{v}\times\mathbf{a}| = |\mathbf{v}|\,|\mathbf{a}|\sin\theta = 1$ always, because the angle θ here is always $\pi/2$. Thus, for all t,

$$\kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{1}{1} = 1.$$

(b) *True*. By the divergence theorem, if V is the solid bounded by S,

$$\iint_S \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint_V \boldsymbol{\nabla}\cdot\big(\boldsymbol{\nabla}\times\mathbf{F}\big)\,\mathrm{d}V = 0$$

since $\boldsymbol{\nabla}\cdot\big(\boldsymbol{\nabla}\times\mathbf{F}\big) = 0$.

(c) *False*. If $\mathbf{F} = 0$ and $\mathbf{G}$ is any nonzero, conservative field, like $\mathbf{G} = 2x\hat{\boldsymbol{\imath}} = \boldsymbol{\nabla}(x^2)$, then

$$\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \oint_C \mathbf{G}\cdot\mathrm{d}\mathbf{r} = 0$$

for every closed curve C.

S-22: (a) Define $\boldsymbol{\Omega}(t) = \mathbf{r}(t)\times\mathbf{v}(t)$. Then by the product rule,

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t} &= \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\times\mathbf{v} + \mathbf{r}\times\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \\ &= \mathbf{v}\times\mathbf{v} + \mathbf{r}\times\big(f(\mathbf{r},\mathbf{v})\mathbf{r}\big). \\ &= \mathbf{0} + f(\mathbf{r},\mathbf{v})\big(\mathbf{r}\times\mathbf{r}\big) = \mathbf{0}.\end{aligned}$$

It follows that $\boldsymbol{\Omega}$ is constant.

(b) By the divergence theorem, where $\mathcal{R}$ is the solid cylinder as described,

$$\iint\limits_{\mathcal{S}} (x\,\hat{\imath} - y\,\hat{\jmath} + z^2\,\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}} \mathrm{d}S = \iiint\limits_{\mathcal{R}} \big(1 - 1 + 2z\big)\,\mathrm{d}V = 2 \iiint\limits_{\mathcal{R}} z\,\mathrm{d}V$$

The solid $\mathcal{R}$ clearly has reflection symmetry across the plane $z = 2$. So the z-coordinate of the centre of mass of $\mathcal{R}$, i.e. the average value of z over $\mathcal{R}$, i.e.

$$\bar{z} = \frac{\iiint_{\mathcal{R}} z\,\mathrm{d}V}{\iiint_{\mathcal{R}} \mathrm{d}V} = \frac{\iiint_{\mathcal{R}} z\,\mathrm{d}V}{\mathrm{Vol}(\mathcal{R})}$$

is 2. Hence

$$\iint\limits_{\mathcal{S}} (x\,\hat{\imath} - y\,\hat{\jmath} + z^2\,\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}} \mathrm{d}S = 2\bar{z}\,\mathrm{Vol}(\mathcal{R}) = 4\,\mathrm{Vol}(\mathcal{R})$$

By basic geometry, $\mathrm{Vol}(\mathcal{R}) = \pi r^2 h = \pi b^2 2$. Hence

$$\iint\limits_{\mathcal{S}} (x\,\hat{\imath} - y\,\hat{\jmath} + z^2\,\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}} \mathrm{d}S = 8\pi b^2$$

(c) By Stokes' theorem (Theorem 4.4.1 in the CLP-4 text),

$$\begin{aligned}
\oint\limits_{\partial D} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint\limits_{D} \mathbf{G} \cdot \hat{\mathbf{n}}\,\mathrm{d}S \implies & \iint\limits_{D} \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S = \iint\limits_{D} \mathbf{G} \cdot \hat{\mathbf{n}}\,\mathrm{d}S \\
\implies & \iint\limits_{D} \big(\boldsymbol{\nabla} \times \mathbf{F} - \mathbf{G}\big) \cdot \hat{\mathbf{n}}\,\mathrm{d}S = 0
\end{aligned}$$

for all disks D. Because this is true for all disks D, the integrand must be zero. To see this, let $\mathbf{H} = \boldsymbol{\nabla} \times \mathbf{F} - \mathbf{G}$. Suppose that $\mathbf{H}(\mathbf{x}_0) \neq 0$. Pick a unit vector $\hat{\mathbf{n}}$ in the direction of $\mathbf{H}(\mathbf{x}_0)$. Let D be a very small flat disk centered on $\mathbf{x}_0$ with normal $\hat{\mathbf{n}}$ (the vector we picked). Then $\mathbf{H}(\mathbf{x}_0) \cdot \hat{\mathbf{n}} > 0$ and, by continuity, $\mathbf{H}(\mathbf{x}) \cdot \hat{\mathbf{n}} > 0$ for all $\mathbf{x}$ on D, if we have picked D small enough. Then $\iint_D \big(\boldsymbol{\nabla} \times \mathbf{F} - \mathbf{G}\big) \cdot \hat{\mathbf{n}}\,\mathrm{d}S > 0$, which is a contradiction. So we conclude that $\boldsymbol{\nabla} \times \mathbf{F} - \mathbf{G} = \mathbf{0}$ and hence $\mathbf{G} = \boldsymbol{\nabla} \times \mathbf{F}$.